SOME NOTES ON PENSION FUNDING

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I - Valuations with and without pre-retirement mortality.

The purpose of this section is to demonstrate that in a defined benefit plan, valued on the Aggregate method, with a pre-retirement death benefit equal to the present value of the accrued pension based on plan participation, then the choice of the mortality basis in the pre-retirement period is largely immaterial.

Specifically we will show that the normal cost ignoring pre-retirement mortality is only slightly larger than the normal cost where such mortality is included in the assumptions. Naturally in this second situation, the value of the death benefit is included as part of the present value of the benefits.

With this result we can then assert that the seemingly rudimentary pre-ERISA calculations done by non-actuaries on many thousands of pension plans have a sound theoretical base.

Our notation is as follows:

R = Normal Retirement Age

X = Attained age

P = Projected pension at age R. payable monthly for life with n years certain.

Case I - No Pre-Retirement Mortality.

$$\text{Normal Cost} = \frac{P\,\ddot{a}^{(12)}_{\overline{R\,n}|}}{\ddot{s}_{\overline{R-x}|}} \qquad \ldots \quad \underline{I}$$

<u>Case II - Including Pre-Retirement Mortality.</u>

(a) Present Value of Pension = $P \cdot \frac{D_R}{D_x} \, \ddot{a}^{(12)}_{\overline{R:\, n|}}$

(b) Present Value of Death Benefit = $\sum_{t=0}^{R-x-1} P \left(\frac{t+1}{R-x}\right) v^{R-x-t-1} \, \ddot{a}^{(12)}_{\overline{R:\, n|}} \; v^{t+1} \cdot {}_{t|}q_x$

The explanation of this term is as follows:

The accrued pension is $P\left(\frac{t+1}{R-x}\right)$ on a pro-rata basis; the present value as of end of year of death is this amount multiplied by $v^{R-x-t-1} \, \ddot{a}^{(12)}_{\overline{R\cdot n|}}$ - note that since he is now dead there is no mortality between age x+t+1 and age R; allowing for death between x+t and x+t+1 with a discount to age x brings in the factor $v^{t+1} \cdot {}_{t|}q_x$; and we sum on all values of t.

This present value simplifies to:

$$P \cdot \frac{v^{R-x}}{R-x} \cdot \ddot{a}^{(12)}_{\overline{R\, n|}} \sum_{t=0}^{R-x-1} (t+1) \frac{d_{x+t}}{l_x}$$

$$= P \cdot \frac{v^{R-x}}{R-x} \cdot \ddot{a}^{(12)}_{\overline{R\, n|}} \left[\frac{l_x + l_{x+1} + l_{x+2} + \cdots + l_{R-1} - (R-x) l_R}{l_x} \right]$$

$$= P \frac{v^{R-x}}{R-x} \ddot{a}^{(12)}_{\overline{R\, n|}} \left[1 + e_{x:\overline{R-x-1|}} \right] - P \cdot \frac{D_R}{D_x} \cdot \ddot{a}^{(12)}_{\overline{R\, n|}}$$

Total Present Value of Benefits =

$$P \frac{v^{R-x}}{R-x} \ddot{a}^{(12)}_{\overline{R\, n|}} \left[1 + e_{x\,\overline{R-x-1|}} \right]$$

$$\text{Normal Cost} = \frac{P\, v^{R-x}\left[1 + e_{x:\overline{R-x-1}|}\right] \ddot{a}^{(12)}_{R:\overline{n}|}}{(R-x)\quad \ddot{a}_{x:\overline{R-x}|}} \quad \ldots\ldots \quad \text{II}$$

The claim is that I is larger than II <u>but not by very much</u>.

To illustrate this, we have calculated I & II for various ages assuming:
$R = 65$; $P = \$1000$, $\ddot{a}^{(12)}_{R:\overline{n}|} = 10$; $i = .06$ with 1971 G.A.M.

	NORMAL COST	
X	I	II
25	\$ 60.96	\$ 59.49
35	119.33	116.83
45	256.46	252.64
55	715.74	711.58

The difference is less than 2½% in all cases.

Why are the numbers so close? The logical reason is readily seen when we realize that a defined benefit plan is essentially a pure endowment insurance policy for (R-X) years with sum insured $P\ \ddot{a}^{(12)}_{R:\overline{n}|}$

Computing the premium as the sum insured divided by $\ddot{s}_{\overline{R-x}|}$ automatically includes a death benefit equal to the full terminal reserve. (It is as if the policyholder put the premium into a savings account.) In fact the death benefit is a "unit credit" type accrued liability which is well-known to be less than the full terminal reserve. Hence I > II.

The algebra to prove that I > II is instructive.

We must show that:

$$P \frac{\ddot{a}^{(12)}_{R:\overline{n}|}}{\ddot{s}_{\overline{R-x}|}} > P \cdot \frac{v^{R-x}\left[1+e_{x:\overline{R-x-1}|}\right]\ddot{a}^{(12)}_{R:\overline{n}|}}{(R-x)\,\ddot{a}_{x:\overline{R-x}|}}$$

i.e. $$\frac{\ddot{a}_{x:\overline{R-x}|}}{\ddot{a}_{\overline{R-x}|}} > \frac{1+e_{x:\overline{R-x-1}|}}{R-x}$$

The left hand side is the weighted average of the ${}_tp_x$'s $t=0,1,2,\ldots,R-x-1$ where the weights are the decreasing numbers $1, v, v^2, \ldots, v^{R-x-1}$ and the right hand side is the arithmetic average of the ${}_tp_x$'s. The two averages are close (the smaller the interest rate the closer they are) and the fact that the weighted average is larger than the ordinary average follows from the result shown in the Appendix.

What does all this mathematics tell us? A couple of basic facts:

1) In valuing a defined benefit plan with a pre-retirement death benefit equal to the present value of the accrued pension based on plan participation it is not because of laziness that pre-retirement mortality is omitted; rather it is theoretically justifiable.

2) In valuing a plan where the vesting schedule is a generous one, we can justify omitting turnover rates completely. (However, if we put them in we must include the cost of anticipated vested benefits.)

II - AGGREGATE vs. UNIT CREDIT.

In costing small defined benefit pension plans the most commonly used funding method is probably the aggregate method - perhaps on an individual basis. Given that the plan does not provide past service benefits, this will certainly provide a larger contribution than the unit credit method in the early years with the reverse holding true in the later years. However, for a new plan, given that each year all assumptions are realized over the life of the plan the unit-credit method will provide, in total, larger contributions. A moment of reflection will confirm this. Since normal costs under unit credit are smaller in the early years than aggregate, the loss of interest in those early years will be compounded and will have to be more than compensated for in the later years. This gives rise to some nice actuarial inequalities.

CASE I NO PRE-RETIREMENT MORTALITY.

Unit Credit Normal Cost at beginning of t'th year = $B\,v^{R-x-t+1}\cdot \ddot{a}^{(12)}_{R:\overline{n|}}$

where B is pension per year of participation.

$$\text{Total Normal Costs} = \sum_{t=1}^{R-x} B\,v^{R-x-t+1}\,\ddot{a}^{(12)}_{R:\overline{n|}}$$

$$= B\,\ddot{a}^{(12)}_{R:\overline{n|}}\left(v+v^2+v^3+\cdots+v^{R-x}\right)$$

$$= B\,\ddot{a}^{(12)}_{R:\overline{n|}}\,a_{\overline{R-x|}}$$

$$\text{Aggregate Normal Cost each year} = \frac{B\,(R-x)\,\ddot{a}^{(12)}_{R:\overline{n|}}}{\ddot{s}_{\overline{R-x|}}}$$

$$\text{Total Normal Costs} = \frac{B\,(R-x)^2\,\ddot{a}^{(12)}_{R:\overline{n|}}}{\ddot{s}_{\overline{R-x|}}}$$

We now assert that: $B\,\ddot{a}^{(12)}_{R:\overline{n|}}\,a_{\overline{R-x|}} > \dfrac{B\,(R-x)^2\,\ddot{a}^{(12)}_{R:\overline{n|}}}{\ddot{s}_{\overline{R-x|}}}$

i.e. $a_{\overline{R-x|}}\;\ddot{s}_{\overline{R-x|}} > (R-x)^2$

This is true by two applications of the arithmetic mean, geometric mean inequality - as follows:

Consider the quantities $\quad v, v^2, v^3, \ldots, v^{R-x}$

Arithmetic Mean = $\dfrac{a_{\overline{R-x}|}}{R-x}$

Geometric Mean = $v^{[1+2+3+\cdots+(R-x)]\left(\frac{1}{R-x}\right)} = v^{\frac{R-x+1}{2}}$

i.e. $\quad a_{\overline{R-x}|} > (R-x)\, v^{\frac{R-x-1}{2}} \quad \ldots\ldots \quad \text{I}$

Now consider the quantities $\quad (1+i), (1+i)^2, \ldots, (1+i)^{R-x}$

Arithmetic Mean = $\dfrac{\ddot{s}_{\overline{R-x}|}}{R-x}$

Geometric Mean = $(1+i)^{[1+2+3+\cdots+(R-x)]\left(\frac{1}{R-x}\right)} = (1+i)^{\frac{R-x+1}{2}}$

i.e. $\quad \ddot{s}_{\overline{R-x}|} > (R-x)\,(1+i)^{\frac{R-x+1}{2}} \quad \ldots\ldots \quad \text{II}$

Now multiply I and II together and the result follows.

CASE II WITH PRE-RETIREMENT MORTALITY.

Now consider m participants all age x.

The unit credit normal cost at beginning of t'th year = $B \cdot m \cdot \dfrac{l_{x+t-1}}{l_x} \dfrac{D_R}{D_{x+t-1}} \ddot{a}^{(12)}_{\overline{R:n}|}$

Note: $\quad m \cdot \dfrac{l_{x+t-1}}{l_x}$ is number expected to be in plan at beginning of t'th year.

i.e., normal cost $\quad = B\, m \dfrac{D_R}{D_x} (1+i)^{t-1}\, \ddot{a}^{(12)}_{\overline{R:n}|}$

Total normal costs = $\displaystyle\sum_{t=1}^{R-x} B\,m \cdot \frac{D_R}{D_x} (1+i)^{t-1}\, \ddot{a}^{(12)}_{R:\overline{n}|}$

$$= B\,m.\ \ddot{a}^{(12)}_{R:\overline{n}|} \cdot \frac{D_R}{D_x}\, s_{\overline{R-x}|}$$

Under Aggregate normal cost per life each year will be $\dfrac{B(R-x).\ddot{a}^{(12)}_{R:\overline{n}|}}{\ddot{s}_{x:\overline{R-x}|}}$

and total normal costs = $\dfrac{B.(R-x)\,\ddot{a}^{(12)}_{R:\overline{n}|}}{\ddot{s}_{x.\overline{R-x}|}}\left[m + m\dfrac{l_{x+1}}{l_x} + m\dfrac{l_{x+2}}{l_x} + \cdots + m\dfrac{l_{R-1}}{l_x}\right]$

$$= \frac{B\,m\,(R-x).\ \ddot{a}^{(12)}_{R:\overline{n}|}}{\ddot{s}_{x:\overline{R-x}|}}\left[1 + e_{x:\overline{R-x-1}|}\right]$$

and now we claim that

$$B\,m \cdot \frac{D_R}{D_x}\,\ddot{a}^{(12)}_{R:\overline{n}|}\ s_{\overline{R-x}|} > \frac{B\,m\,(R-x)\,\ddot{a}^{(12)}_{R:\overline{n}|}}{\ddot{s}_{x:\overline{R-x}|}}\left[1 + e_{x:\overline{R-x-1}|}\right]$$

or

$$\frac{s_{\overline{R-x}|}}{R-x} > \frac{1 + e_{x:\overline{R-x-1}|}}{\ddot{a}_{x:\overline{R-x}|}}$$

Call the left hand side A and right hand side B. Note that B is the reciprocal of the weighted average of v^t, $t = 0,1,2,\ldots, R-x-1$ with weights ${}_tp_x$, $t = 0,1,2\cdots$. Because the v^t's decrease and the weights decrease it follows from the result in the Appendix that this weighted average is greater than the ordinary average.

i.e. $\dfrac{1 + v\,p_x + v^2\,{}_2p_x + \cdots + v^{R-x-1}\,{}_{R-x-1}p_x}{1 + p_x + {}_2p_x + \cdots + {}_{R-x-1}p_x} > \dfrac{1 + v + v^2 + \cdots + v^{R-x-1}}{R-x} = \dfrac{\ddot{a}_{\overline{R-x}|}}{R-x}$

But since $$a_{\overline{R-x}|}\,\ddot{s}_{\overline{R-x}|} = \ddot{a}_{\overline{R-x}|}\,s_{\overline{R-x}|} > (R-x)^2$$

$$\frac{R-x}{\ddot{a}_{\overline{R-x}|}} < \frac{s_{\overline{R-x}|}}{R-x} = A.$$

This proves $B < A$.

(Note: new entrants have been ignored in this analysis.)

APPENDIX

We quoted a result concerning weighted averages being larger than arithmetic averages. Formally: consider n_1 quantities all equal to a_1

n_2 " " " " a_2

$\vdots$

n_r " " " " a_r

where $n_1 > n_2 > \cdots > n_r > 0$ and $a_1 > a_2 > \cdots > a_r > 0$

Then:

$$\frac{n_1 a_1 + n_2 a_2 + \cdots + n_r a_r}{n_1 + n_2 + \cdots + n_r} > \frac{a_1 + a_2 + \cdots + a_r}{r}$$

To prove this we show that

$$r\left[n_1 a_1 + n_2 a_2 + \cdots + n_r a_r\right] - \left[n_1 + n_2 + \cdots + n_r\right]\left[a_1 + a_2 + \cdots + a_r\right] = \sum_{i=1}^{r-1} \sum_{j=i+1}^{r} (a_i - a_j)(n_i - n_j)$$

and since all products are positive, the result follows.

Note: After independently arriving at this result, I learned from a previous paper in ARCH that this is called CHEBYCHEV'S INEQUALITY.