

ANNUITY CERTAIN SYMBOLS FOR FRACTIONAL PAYMENT PERIODS

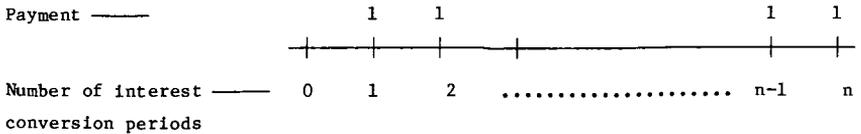
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ABSTRACT

The use of annuity certain symbols provides a convenient short hand for the value, accounting for interest, of a variety of types of sequences of payments at a variety of different points in time. One must, however, be careful with the use of such symbols for fractional payment periods, ie. for  $\overline{j}|i$  where  $j$ , measured in interest conversion periods, represents a fractional number of payment periods. What one might want it to represent and what it does represent, according to standard textbook definitions of the symbol, may be different. This paper discusses the problem, how it develops and some important practical implications. The paper discusses the problem using present value symbols ( $a$ 's) but a similar analysis holds for accumulated value symbols ( $s$ 's).

I. BACKGROUND

Annuity certain symbols have been developed as a short hand for the value, accounting for interest, of a variety of types of sequences of payments at a variety of different points in time, measured most generally in terms of number of interest conversion periods. Thus, the value at time 0 of the payment sequence indicated by the following diagram is represented by  $a_{\overline{n}|i}$ , where  $n$  is the number of interest conversion periods,  $i$  is the effective interest rate per interest conversion period, and the payment period is the same as the interest conversion period.



Using the basic principle of discounting each payment separately, the value at time 0 is  $v + v^2 + \dots + v^n$ . As the sum of a finite geometric progression with common ratio  $v$ , this can be simplified as follows:

$$v + v^2 + \dots + v^n = \frac{v(1-v^n)}{1-v} = \frac{(1-v^n)}{d} = \frac{v(1-v^n)}{iv} = \frac{(1-v^n)}{i}$$

This simplification can be accomplished only if the values are in a geometric progression. But when that is the case, there are then two equivalent mathematical expressions for the annuity certain symbol. For the example above, they are:



(b) provides a payment at  $nt+k$  equal to the discounted value of a payment of 1 at time  $nt+1$ . Then  $a_{\overline{nt+k}|i} = v + v^2 + \dots + v^n + v^{1-k}(v^{nt+k}) = v + v^2 + \dots + v^{nt+1} = a_{\overline{nt+1}|i}$ . The result, of course, is not surprising, given the payment assumed at time  $nt+k$ . In this situation,  $a_{\overline{nt+k}|i}$  is unnecessary, since symbols defined for integral values may be used.

(c) keeps the payments the same and so  $a_{\overline{nt+k}|i} = v + v^2 + \dots + v^n + v^{nt+k}$ . But, again, it cannot be simplified to an expression of the form  $\frac{1-v^{nt+k}}{i}$ , although it can be simplified to  $a_{\overline{nt}|i} + v^{nt+k}$ .

(d) keeps the payments level and the time between payments constant. Thus  $a_{\overline{nt+k}|i} = v^{1+k} + v^{2+k} + \dots + v^{nt+k}$  and thus can be simplified to  $v^{1+k} \frac{(1-v^n)}{1-v} = v^k \frac{(1-v)^n}{i} = v^k a_{\overline{nt}|i}$ . Unfortunately, this is inconsistent with  $v^m a_{\overline{nt}|i} = a_{\overline{m+nt}|i} - a_{\overline{m}|i}$ , for  $m, n$  positive integers, since  $a_{\overline{m+nt}|i} \neq v^m a_{\overline{nt}|i}$ .

(e) does not keep the last payments the same, nor is the time between the last two payments the same. But  $a_{\overline{nt+k}|i} = v + v^2 + \dots + v^n + v^{nt+k} \left[ \frac{(1+i)^k - 1}{i} \right] = \frac{1-v^n}{i} + \frac{v^n - v^{nt+k}}{i} = \frac{1-v^{nt+k}}{i}$ . In fact, (e) is the situation for which  $a_{\overline{nt+k}|i}$  is defined in Kellison's [3] and Donald's [1] texts on interest theory.  $a_{\overline{nt+k}|i}$  is thus generally defined to be consistent with the simplified form (2) above rather than the basic form (1) above. The result is a rather unusual payment amount at time  $nt+k$ . But as is pointed out in most texts the difference between a payment at time  $nt+k$  of  $\frac{(1+i)^k - 1}{i}$  and a payment at time  $nt+k$  of  $k$  is relatively minor. Note, also, that  $\lim_{i \rightarrow 0} \frac{(1+i)^k - 1}{i} = \lim_{i \rightarrow 0} \frac{k(1+i)^{k-1}}{1} = k$ . Thus  $a_{\overline{nt+k}|i}$ , defined by (e), approximates fairly well the situation in (a). Situations (b), (c) and (d) are taken care of by  $a_{\overline{nt+1}|i}$ ,  $a_{\overline{nt}|i} + v^{nt+k}$  and  $v^k a_{\overline{nt}|i}$ , respectively, and thus there is no great loss by not having a single symbol  $a_{\overline{nt+k}|i}$  to represent those situations.

## II. PRACTICAL IMPLICATIONS

Two general types of problems may potentially involve use of  $a_{\overline{n+k}|i}$ ,  $n$  integral,  $0 < k < 1$ , as defined above. The first type is to determine the present value of a sequence of payments over a term that involves a fractional number of payment periods. The second type is to determine the term of an annuity, given a lump sum value and a level of payments desired.

For the first type,  $a_{\overline{n+k}|i}$ , as defined above, is generally not appropriate since it represents a unique irregular payment of  $\frac{(1+i)^k - 1}{i}$  per unit of regular payment, at the end of  $n+k$  payment periods. A given sequence of payments would usually not have the pattern of payments implied by the above definition of  $a_{\overline{n+k}|i}$ . If the interest conversion period and the payment period are not the same, it is not uncommon to have a fractional number for the term of the annuity measured in interest conversion periods. For example, the ~~50~~<sup>50</sup> monthly installments on a loan with a term of 18 months and an effective rate of interest of 12%, have present value represented by  $600 a_{\overline{18}|.12}^{(12)}$ . However, the number of payment periods is integral and the basic summation is:  $50v^k + 50v^{2k} + \dots + 50v^{18k} = 50v^k \frac{(1 - (v^k)^{18})}{1 - v^k} = 50 \cdot 12 \frac{(1 - v^{\frac{18}{12}})}{.12} = 600 a_{\overline{18}|.12}^{(12)}$ .

This is not a problem involving an annuity symbol with a fractional term as measured in payment periods. The irregular payment of the form  $\frac{(1+i)^k - 1}{i}$ , arises when the term, as measured in payment periods, is fractional. In that case, the present value is best determined by use of basic principles for the payments over the fractional period, and annuity symbols for an integral number of payment periods for payments over the period involving an integral number of payment periods.

For the second type of problem, although  $a_{\overline{n+k}|i}$ , as defined above, is appropriate, it is generally not used since its use results in an irregular payment at a fractional payment period. It is preferable to use annuity symbols for an integral number of payment periods, and determine an irregular payment to be made at the same time as the last regular payment, or one payment period after the last regular payment. Example 3.5 from Kellison's text is illustrative.

If this definition of  $a_{\overline{n+k}|i}$  is such that  $a_{\overline{n+k}|i}$  is, for all practical purpose, hardly ever used, what is the value of such a definition? It does provide

a more complete theory from a mathematical standpoint. It is helpful to have the function  $a_{\overline{n}|i}$  defined for all positive values of  $n$  (note that the function can be defined for negative  $n$  in a manner consistent with the definition for fractional  $n$ ; i.e.  $a_{\overline{-n}|i} = \frac{1-v^{-n}}{i}$ ); to my knowledge, that definition has even less practical value).

Unfortunately, the primary motivation for this paper is the practical misinterpretation of  $a_{\overline{n+k}|i}$  as defined above. Two instances of possible misinterpretation have come to my attention.

First, it was brought to my attention that a computer software package which calculated values of annuity certain used a form analagous to  $a_{\overline{n+k}|i}$ . This is not a problem in itself; however, the individuals using it were using it to determine the present value of a sequence of payments different from that sequence implied by use of  $a_{\overline{n+k}|i} = \frac{1-v^{n+k}}{i}$ . The values they obtained from the computer were not what they thought they were obtaining. Fortunately, they discovered their error. This situation also points out a critical problem in use of software packages. Such packages are very valuable and efficient, but those who use them must be sure they understand what it is that the package calculates.

The second case of possible misinterpretation involves the incorrect use of a deferred annuity certain for the expectation of life in determining the value of future retirement benefits for purposes of divorce settlements. There are many reasons why such usage is incorrect; among those reasons (although not as important as some other reasons) is the fact that such a term is usually for a fractional number of payment periods. Use of a deferred form of  $a_{\overline{n+k}|i}$ , as defined above, implies a payment pattern of regular payments ( $P$ ) for an integral number of payment periods and a payment of the form  $P \cdot \left( \frac{(1+i)^k - 1}{i} \right)$  at  $n+k$ . This is not the pattern of payments that pension plans use in making payments.

Two alternative approaches to problem 48 from Chapter 4 of Kellison's text further illustrate the potential for misinterpretation of the symbol.

The problem is stated as follows:

"The last payment under an annuity represented by  $a_{\overline{n+1/12}|}^{(4)}$  is taken as  $1/12$ . Show that the error involved in this assumption is

$$\frac{1}{12} \left[ 1 - \frac{i^{(12)}}{i^{(4)}} \right] "$$

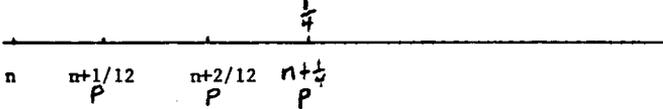
The first approach proceeds in a manner similar to the development of the definition of  $a_{\overline{n+k}|}^{(4)}$ ,  $n$  integral,  $0 < k < 1$ :

$$a_{\overline{n+1/12}|}^{(4)} = \frac{1 - v^{n+1/12}}{i^{(4)}} = \frac{1 - v^n + v^n - v^{n+1/12}}{i^{(4)}} = a_{\overline{n}|}^{(4)} + v^{n+1/12} \left( \frac{(1+i)^{1/12} - 1}{i^{(4)}} \right)$$

Thus, the definition of  $a_{\overline{n+1/12}|}^{(4)}$  implies a payment of  $\frac{(1+i)^{1/12} - 1}{i^{(4)}}$  at  $n + 1/12$ .

Using  $1/12$  implies an error of  $\frac{1}{12} - \frac{(1+i)^{1/12} - 1}{i^{(4)}} = \frac{1}{12} \left[ 1 - \frac{12 \left( \frac{(1+i)^{1/12} - 1}{i^{(4)}} \right)}{i^{(4)}} \right] = \frac{1}{12} \left[ 1 - \frac{i^{(12)}}{i^{(4)}} \right]$

An alternative approach uses the diagram below:



The reasoning is as follows:

$a_{\overline{n}|}^{(4)}$  implies payments of  $\frac{1}{4}$  every quarter of an interest conversion period.

Thus  $a_{\overline{n+1/12}|}^{(4)}$  should represent payments of  $\frac{1}{4}$  for  $n$  interest conversion periods and a payment at  $n+1/12$  such that the value at  $n+1/4$  of that payment and

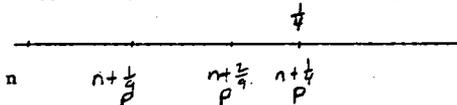
payments at  $n+1/12$  and  $n+1/4$  should equal  $\frac{1}{4}$ ; that is,  $P \cdot \left[ (1+i)^{1/12} + (1+i)^{1/4} + 1 \right] = \frac{1}{4}$

$$\text{or } P = \frac{1}{4} \left( \frac{1}{(1+i)^{12} + (1+i)^{12} + 1} \right) = \frac{1}{4} \left( \frac{1}{\frac{(1+i)^{12} - 1}{(1+i)^{12} - 1}} \right) = \frac{(1+i)^{12} - 1}{4((1+i)^{12} - 1)} = \frac{(1+i)^{12} - 1}{i^{(12)}}$$

Using this approach, the result for the payment at  $n+1/12$  is the same. One may be tempted to conclude then that this alternate approach provides an equivalent definition for the annuity certain symbol for a term involving a fractional number of payment periods. However, to see that the two approaches do not provide

equivalent definitions, consider the same problem as above, except using  $a_{\overline{n+\frac{1}{9}}|i}^{(4)}$ . The first approach yields  $a_{\overline{n+\frac{1}{9}}|i}^{(4)} = \frac{1 - v^{n+\frac{1}{9}}}{i^{(4)}} = \frac{1 - v^n + v^n - v^{n+\frac{1}{9}}}{i^{(4)}}$   
 $= a_{\overline{n}|i}^{(4)} + v^{n+\frac{1}{9}} \frac{((1+i)^{\frac{1}{9}} - 1)}{i^{(4)}}$  implying a payment at  $n+1/9$  of  $\frac{(1+i)^{\frac{1}{9}} - 1}{i^{(4)}}$ .

The second approach yields the following diagram:



with a payment at  $n+1/9$  of  $P$ , developed as follows:

$$P \left[ (1+i)^{\frac{1}{9}-\frac{1}{9}} + (1+i)^{\frac{1}{9}-\frac{2}{9}} + 1 \right] = \frac{1}{4}$$

or  $P = \frac{1}{4} \left( \frac{1}{(1+i)^{\frac{2}{9}} + (1+i)^{\frac{1}{9}} + 1} \right)$  which does not equal  $\frac{(1+i)^{\frac{1}{9}} - 1}{i^{(4)}}$ .

In this case, then, use of an annuity symbol with a fractional number of payment periods, leads to two reasonable and apparently equivalent interpretations, which in fact are not equivalent.

### III. CONCLUSION

Perhaps the main lesson in this paper is to not lose sight of the basic principles of discounting or accumulating individual payments. Annuity symbols provide convenient short hand in many instances, but it may be better not to even attempt to define an annuity symbol for a term involving a fractional number of payment periods. It has little use and its availability may mislead some into using it when they should not.

REFERENCES

1. Donald, D. W. A., Compound Interest and Annuities-Certain, Cambridge; University Press, 1956.
2. Jordan, Chester W., Life Contingencies, Chicago; Society of Actuaries, 1967.
3. Kellison, Stephen G., The Theory of Interest, Homewood, Illinois; Richard D. Irwin, 1970.