THE SIMPLEX ALGORITHM AND THE EXCHANGE METHOD

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The presentation of the simplex algorithm in "A Study Manual For Operations Research" (edited by Eugene A. Narragon) is difficult to follow. The purpose of this note is to show that a better understanding can be obtained by means of a geometric interpretation and the exchange method. The following ideas are not original, see for example E. Stiefel's text "An introduction to numerical mathematics", Academic Press, 1963.

We consider a linear programming problem of the following form:

Maximize $z = c_1x_1+\cdots+c_nx_n$ subject to

(1) $$x_1\geq 0,\cdots,x_n\geq 0,\eta_1\geq 0,\cdots,\eta_m\geq 0 ,$$

where

$\eta_i = a_i - \sum_{j=1}^{n} a_{ij}\, x_j$. It is assumed that $c_j>0$, $a_j>0$, $a_{ij}>0$.

The set of feasible points (i.e., where the inequalities (1) are satisfied) is an intersection of $n+m$ halfspaces and is therefore a convex polyhedron in R^n . Thus the maximum of z will be assumed on one of its vertices.

A <u>combinatorial</u> <u>method</u> to find the solution is as follows. Set n of the variables $x_1,\cdots,x_n,\eta_1,\cdots,\eta_m$ equal to zero; thus we are examining $\binom{n+m}{n}$ points. If one of the other variables has a negative value, the point is not feasible and can be ignored. If all the other variables are nonnegative, the point is a vertex and we compute the z-value. Finally, the optimum is found by comparison.

The simplex algorithm is a more economical method: With it, the optimal vertex can be found without examination of all the other vertixes.

We start with the vertex corresponding to $x_1 = x_2 = \cdots = x_n = 0$. In a first step we exchange one of the x's, say x_j, against one of the η's, say η_i. This means that we proceed to the vertex that is given by $x_\ell = 0$ $(\ell \neq j)$, $\eta_i = 0$. In the second step we exchange one of the remaining x's against one of the remaining η's, etc. At each step, the choice of the pivot (i.e. the decision which x should be exchanged against which η) is governed by two criteria:

(a) The new point should again be feasible, i.e. a vertex.

(b) The z-value should increase.

When it is not possible to perform another step, i.e. to find a pivot such that (a) and (b) are satisfied, we have arrived at the optimal vertex.

The remaining question is how the pivot should be chosen to satisfy (a) and (b). Before we look at it, we consider an example: Maximize

$$z = 100\, x_1 + 200\, x_2 + 50\, x_3$$

subject to

$$\begin{aligned} 5x_1 + 5x_2 + 10x_3 &\leq 1000 \\ 10x_1 + 8x_2 + 5x_3 &\leq 2000 \\ 10x_1 + 5x_2 \quad\quad &\leq 500 \end{aligned}$$

(This is the example discussed in the OR-Manual). We start out with the following table, corresponding to $x_1 = x_2 = x_3 = 0$:

	x_1	x_2	x_3	1
$\eta_1 =$	-5	-5	-10	1000
$\eta_2 =$	-10	-8	-5	2000
$\eta_3 =$	-10	-5	0	500
$z =$	100	200	50	0

Then we exchange x_2 with η_3, and obtain the table corresponding to the vertex $x_1 = x_3 = \eta_3 = 0$;

	x_1	η_3	x_3	1
$\eta_1 =$	5	1	-10	500
$\eta_2 =$	6	1.6	-.2	1200
$x_2 =$	-2	-.2	0	100
$z =$	-300	-40	50	20000

The corresponding z-value is in the lower right-hand-corner and is 20000. Now we exchange x_3 with η_1 and obtain the table corresponding to the vertex $x_1 = \eta_1 = \eta_3 = 0$:

	x_1	η_3	η_1	1
$x_3 =$				50
$\eta_2 =$				950
$x_2 =$				100
$z =$	-275	-35	-5	22500

The last line shows that the maximal value of z is obtained at this vertex and is 22500. The last column shows the values of x_3, η_2, x_2 at the optimal vertex. The remaining entries are not of interest and have not been computed.

Finally we discuss the choice of the pivot. Suppose that after a number of steps we have arrived at the following table:

$$
\begin{array}{r|c|c|}
 & & 1 \\
\hline
 & & a_1 \\
 & a_{ij} & \vdots \\
 & & a_m \\
\hline
z = & c_1 \cdots c_n & c \\
\hline
\end{array}
$$

We are dealing with a vertex, i.e. $a_1 \geq 0, \cdots, a_m \geq 0$, and the corresponding z-value is $c > 0$. If we choose a_{ij} as the new pivot, the coefficients in the last column of the table will change as follows:

$$(\text{i}) \quad a_i \rightarrow -\frac{a_i}{a_{ij}}$$

$$(\text{ii}) \quad a_k \rightarrow a_k - \frac{a_i \, a_{kj}}{a_{ij}}, \quad k \neq i$$

$$(\text{iii}) \quad c \rightarrow c - \frac{a_i \, c_j}{a_{ij}}$$

Since the new value of a_i must be positive, it follows from (i) that the pivot a_{ij} must be negative. Then it follows from postulate (b) and (iii) that c_j must be positive. Now consider the condition that the new value of a_k is nonnegative. This can be written as follows:

$$\frac{a_k}{-a_{kj}} \geq \frac{a_i}{-a_{ij}} \tag{2}$$

Thus the pivot should be chosen according to the following rules:

- Pivot Column: Column j can be chosen iff $c_j>0$ and if at least one of the coefficients in this column is negative.

- Pivot Row: Suppose that column j has been chosen as the pivot column. Now we determine the pivot row i such that (2) holds. Thus whenever a_{kj} is negative, we compute the ratio $a_k/-a_{kj}$. Then i is the value of k for which this ratio is the smallest.

Let us revisit the numerical example. In the first table all c_ℓ's are positive and all $a_{k\ell}$ are negative. Thus all threee columns qualify as pivot column. Quite arbitrarily, we choose $j = 2$. Then we compute

$$a_1/-a_{12} = 1000/5 = 200$$
$$a_2/-a_{22} = 2000/8 = 250$$
$$a_3/-a_{32} = 500/5 = 100$$

Hence $i = 3$. In the second table only the third column qualifies. Hence $j = 3$. We compute

$$a_1/-a_{13} = 500/10 = 50$$

$$a_2/-a_{23} = 1200/.2 = 6000$$

Hence $i = 1$. Finally, in the last table all c_ℓ's are negative: The choice of a new pivot to satisfy (a) and (b) is not possible, and we have arrived at the optimal vertex.