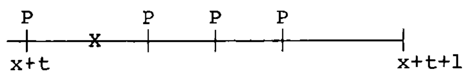
INSTALLMENT PREMIUMS

RICHARD L. LONDON

The installment fractional premium scheme, although not practical for actual use, has some interesting theoretical properties. Jordan states, on page 88 of Life Contingencies, "For example, if the insured dies after paying one quarterly premium for the current year, the three unpaid installments for the balance of the year are deducted from the policy proceeds."



In the above diagram, P represents $\frac{1}{4}\cdot P_x^{[4]}$, and X is the point of death. Jordan suggests that the death benefit paid would be 1-3P. More appropriately, the benefit paid should be

$$1-P\{\ (1+i)^{3/4} + (1+i)^{1/2} + (1+i)^{1/4}\ \},$$

for a benefit paid at year-end, to reflect the "late" collection of the fractional premiums.

If the deduction is administered in this theoretically correct manner, then the value at age x+t of the insurance for that year is

$$v\cdot {}_{\frac{1}{4}}q_{x+t}[1-P\{1+i)^{3/4} + (1+i)^{1/2} + (1+i)^{1/4}\}]$$

$$+\ v\cdot {}_{\frac{1}{4}/\frac{1}{4}}q_{x+t}[1-P\{(1+i)^{1/2} + (1+i)^{1/4}]$$

$$+\ v\cdot {}_{\frac{1}{2}/\frac{1}{4}}q_{x+t}[1-P\{(1+i)^{1/2} + (1+i)^{1/4}\}]$$

$$+\ v\cdot {}_{\frac{3}{4}/\frac{1}{4}}q_{x+t},$$

there being no deduction for death in the final quarter of the year.

The above expression reduces to

$$v \cdot q_{x+t} - P[v^{1/4} \cdot {}_{\frac{1}{4}}q_{x+t} + v^{1/2} \cdot {}_{\frac{1}{2}}q_{x+t} + v^{3/4} \cdot {}_{\frac{3}{4}}q_{x+t}]$$

or $$v \cdot q_{x+t} - P[v^{1/4} + v^{1/2} + v^{3/4} - v^{1/4} \cdot {}_{\frac{1}{4}}p_{x+t} - v^{1/2} \cdot {}_{\frac{1}{2}}p_{x+t} - v^{3/4} \cdot {}_{\frac{3}{4}}p_{x+t} + 1 - 1]$$

or $$v \cdot q_{x+t} - 4P[\ddot{a}^{(4)}_{\overline{1}|} - \ddot{a}^{(4)}_{x+t:\overline{1}|}].$$

Then $v^t \cdot {}_tp_x\{v \cdot q_{x+t} - 4P[\ddot{a}^{(4)}_{\overline{1}|} - \ddot{a}^{(4)}_{x+t:\overline{1}|}]\}$ is the value at age x of the insurance for the year starting at age x+t, and

$$\sum_{t=0}^{\infty} v^t \cdot {}_tp_x\{v \cdot q_{x+t} - 4P[\ddot{a}^{(4)}_{\overline{1}|} - \ddot{a}^{(4)}_{x+t:\overline{1}|}]\}$$

is the total value at age x of the insurance.

This clearly reduces to

$$\sum_{t=0}^{\infty} v^{t+1}\ {}_{t/}q_x - 4P\ddot{a}^{(4)}_{\overline{1}|} \sum_{t=0}^{\infty} v^t\ {}_tp_x + 4P \sum_{t=0}^{\infty} v^t\ {}_tp_x\ddot{a}^{(4)}_{x+t:\overline{1}|}$$

which is $A_x - 4P\,\ddot{a}_{\overline{1}|}^{(4)}\ddot{a}_x + 4P\,\ddot{a}_x^{(4)}$.

To solve for P, we equate this value of insurance at age x to the then value of future premiums, $4P\cdot\ddot{a}_x^{(4)}$:

$$4P\cdot\ddot{a}_x^{(4)} = A_x - 4P\ddot{a}_{\overline{1}|}^{(4)}\ddot{a}_x + 4P\ddot{a}_x^{(4)}, \text{ so } 4P\ddot{a}_{\overline{1}|}^{(4)}\ddot{a}_x = A_x$$

$$\text{and} \qquad 4P = \frac{P_x}{\ddot{a}_{\overline{1}|}^{(4)}} = P_x^{[4]}$$

That $P_x = P_x^{[4]}\ddot{a}_{\overline{1}|}^{(4)}$ is really an intuitive result. Since the installment scheme and the true annual scheme are equivalent in the year of death, then they will be equivalent in years that the insured lives through as long as P_x is the present value at interest only of the installments.

Of course the above result is valid for m in general, i.e., $P_x = P_x^{[m]}\cdot\ddot{a}_{\overline{1}|}^{(m)}$. The example of $m = 4$ merely facilitates the arithmetic.

In the case of benefit paid (hence premiums deducted) at the moment of death, the appropriate deduction is the several unpaid P's <u>discounted</u> <u>back</u> to the point of death to reflect the "early" collection of them.

Thus, for the situation illustrated in the diagram, the benefit paid is

$$1 - P\{v^{1/4 - s} + v^{1/2 - s} + v^{3/4 - s}\}$$

for death at time s, $0<s<1/4$, within the first quarter.

Here it is necessary to recognize the point of death (for appropriate discounting), as well as the quarter of death (for the number of P's to be discounted and deducted).

The value at age x+t of the insurance for that year is thus

$$\int_0^{1/4} v^s\, {}_sp_{x+t}\mu_{x+t+s}[1 - P\{v^{1/4-s} + v^{1/2-s} + v^{3/4-s}\}]\, ds$$

$$+ v^{1/4}\, {}_{\frac{1}{4}}p_{x+t} \int_0^{1/4} v^s\, {}_sp_{x+t+\frac{1}{4}}\mu_{x+t+\frac{1}{4}+s}[1 - P\{v^{1/4-s} + v^{1/2-s}\}]\, ds$$

$$+ v^{1/2}\, {}_{\frac{1}{2}}p_{x+t} \int_0^{1/4} v^s\, {}_sp_{x+t+\frac{1}{2}}\mu_{x+t+\frac{1}{2}+s}[1 - P \cdot v^{1/4-s}]ds$$

$$+ v^{3/4}\, {}_{\frac{3}{4}}p_{x+t} \int_0^{1/4} v^s\, {}_sp_{x+t+\frac{3}{4}}\mu_{x+t+\frac{3}{4}+s} ds,$$

there being no deduction for death in the final quarter of the year.

This expression reduces to

$$\bar{A}^{\,1}_{x+t:\overline{1}|} - P[v^{1/4}\, {}_{\frac{1}{4}}q_{x+t} + v^{1/2}\, {}_{\frac{1}{2}}q_{x+t} + v^{3/4} \cdot {}_{\frac{3}{4}}q_{x+t}] \qquad \text{or}$$

$$\bar{A}^{\,1}_{x+t:\overline{1}|} - 4P[\ddot{a}^{(4)}_{\overline{1}|} - \ddot{a}^{(4)}_{x+t:\overline{1}|}], \quad \text{as before.}$$

Then the value at age x of the entire insurance coverage is $\bar{A}_x - 4P\, \ddot{a}^{(4)}_{\overline{1}|}\ddot{a}_x + 4P\, \ddot{a}^{(4)}_x$, which, when equated to the value at issue of all future premiums, yields

$$4P = \frac{P(\bar{A}_x)}{\ddot{a}^{(4)}_{\overline{1}|}}$$

In this case, of course, P stands for $\frac{1}{4} \cdot P^{[4]}\ (\bar{A}_x)$.

In like manner one can consider the case where premium is paid continuously, on the installment scheme. That is, unpaid continuous premium, due beyond the point of death, is deducted from the face amount with interest added (for year-end benefits) and discounted (for moment-of-death benefits). No standard symbol exists for continuous payment installment, so we use here $\lim_{m\to\infty} P_x^{[m]}$ and $\lim_{m\to\infty} P^{[m]}\ (\bar{A}_x)$ for these two cases.
Then the four cases herein developed result in the following:

$$P_x^{[m]} \cdot \ddot{a}_{\overline{1}|}^{(m)} = P_x$$

$$P^{[m]}(\bar{A}_x) \cdot \ddot{a}_{\overline{1}|}^{(m)} = P(\bar{A}_x)$$

$$\lim_{m\to\infty} P_x^{[m]} \cdot \bar{a}_{\overline{1}|} = P_x$$

$$\lim_{m\to\infty} P^{[m]}(\bar{A}_x) \cdot \bar{a}_{\overline{1}|} = P(\bar{A}_x)$$

A natural consequence of these interest-only relationships among the premiums is that policy year terminal reserves for any of the four installment schemes are identically equal to the corresponding annual premium case. For example

$${}_tV_x^{[m]} = {}_tV_x$$

and

$$ {}_tV^{[m]}(\bar{A}_x) = {}_tV(\bar{A}_x) \ . $$

Illustrating the first case only, the value of future benefits, at age x+t, is

$A_{x+t} - mP\ \ddot{a}^{(m)}_{\overline{1}|}\ \ddot{a}_{x+t} + mP\ \ddot{a}^{(m)}_{x+t}$ $(P = \frac{1}{m} \cdot p^{[m]}_x)$ and the value of future premiums is $mP\ \ddot{a}^{(m)}_{x+t}$. Then

$$ {}_tV^{[m]}_x = PVFB - PVFP = A_{x+t} - mP\ \ddot{a}^{(m)}_{\overline{1}|}\ \ddot{a}_{x+t} = A_{x+t} - P_x\ddot{a}_{x+t} = {}_tV_x \ . $$

The demonstrations for the other three cases are correspondingly simple.

Jordan shows, on page 109 of Life Contingencies, that ${}_tV^{[m]}_x \doteq {}_tV_x$. The approximate nature of the equality follows from his failure to consider interest in the premium deduction. When the deduction is appropriately administered, the equality of the reserves is exact.

REFERENCE

Jordan, C.W. Life Contingencies, Society of Actuaries, Chicago, 1967.