ACTUARIAL APPLICATIONS OF TCHEBYSHEF'S INEQUALITY

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Introduction: The purpose of this paper is to introduce some actuarial applications of Tchebyshef's Inequality.

Tchebyshef's Inequality: If $\alpha_1 \geq \alpha_2 \geq \dots \alpha_n \geq 0$ and $\beta_1 \geq \beta_2 \geq \dots \beta_n \geq 0$, then $n \sum_{k=1}^{n} \alpha_k \cdot \beta_k \geq \sum_{k=1}^{n} \alpha_k \cdot \sum_{k=1}^{n} \beta_k$. (1)

A proof of this inequality may be found in [1], page 43

Application to ordinary annuities: Let $\alpha_k = v^k$, $\beta_k = {}_kp_x$ and insert into (1):

$$n \sum_{k=1}^{n} v^k \cdot {}_kp_x \geq \sum_{k=1}^{n} v^k \sum_{k=1}^{n} {}_kp_x$$ or

$$a_{x:\overline{n}|} \geq \frac{a_{\overline{n}|} \cdot e_{x:\overline{n}|}}{n} \qquad (2)$$

This lower bound for $a_{x:\overline{n}|}$ has the interesting property that it is a simple combination of the interest component $a_{\overline{n}|}$, the mortality component $e_{x:\overline{n}|}$ and the term n.

$\frac{a_{x:\overline{n}|}}{e_{x:\overline{n}|}} \geq \frac{a_{\overline{n}|}}{n}$ compares the ratio of the life contingent functions to the certain functions.

In (2), set $n = \omega - x$:

$$a_x \geq \frac{a_{\overline{\omega-x}|} \cdot e_x}{\omega - x} \qquad (3)$$

Since $\omega - x \geq e_x$, $a_{\overline{\omega-x}|} \geq a_{\overline{e_x}|}$. This together with (3) implies:

$$a_x \geq a_{\overline{e_x}|}\left(\frac{e_x}{\omega - x}\right) \qquad (4)$$

Inequality (4) compliments the classical results that $a_{\overline{e_x}|} \geq a_x$; thus,

$$a_{\overline{e_x}|} \geq a_x \geq a_{\overline{e_x}|}\left(\frac{e_x}{\omega - x}\right) \qquad (5)$$

Application to Insurances: Let $\alpha_k = v^k$ and $\beta_k = {}_{k-1|}q_x$ and insert into (1)*:

$$n \sum_{k=1}^{n} v^k \cdot {}_{k-1|}q_x \geq \sum_{k=1}^{n} v^k \cdot \sum_{k=1}^{n} {}_{k-1|}q_x \qquad \text{or}$$

$$A^{\;1}_{x:\overline{n|}} \geq \frac{a_{\overline{n|}} \cdot {}_nq_x}{n} \tag{6}$$

As in the case of annuities, the lower bound is a simple combination of the interest, mortality and term components.

Application to fractional annuities: Let $\alpha_k = v^{k/m}$, $\beta_k = {}_{\frac{k}{m}}p_x$ and apply (1):

$$n \cdot m \sum_{k=1}^{nm} v^{k/m} \cdot {}_{\frac{k}{m}}p_x \geq \sum_{k=1}^{nm} v^{k/m} \sum_{k=1}^{nm} {}_{\frac{k}{m}}p_x \qquad \text{or}$$

$$a^{(m)}_{x:\overline{n|}} \geq \frac{a^{(m)}_{\overline{n|}} \cdot e^{(m)}_{x:\overline{n|}}}{n} \tag{7}$$

This is clearly a generalization of inequality (2).

Let m approach infinity:

$$\bar{a}_{x:\overline{n|}} \geq \frac{\bar{a}_{\overline{n|}} \mathring{e}_{x:\overline{n|}}}{n} \tag{8}$$

Application to decreasing annuities and insurances: Let $\alpha_k = (n-k+1)v^k$, $\beta_k = {}_kp_x$ and apply (1)

$$(Da)_{x:\overline{n|}} \geq (Da)_{\overline{n|}} \cdot \frac{e_{x:\overline{n|}}}{n} \tag{9}$$

If α_k is as before and $\beta_k = {}_{k-1|}q_x$*:

$$(Da)^{\;1}_{x:\overline{n|}} \geq \frac{(Da)_{\overline{n|}} \cdot {}_nq_x}{n} \tag{10}$$

Inequalities for $(Da)^{(m)}_{x:\overline{n|}}$, $(D\bar{a})_{x:\overline{n|}}$, $(DA)^{(m)\;1}_{x:\overline{n|}}$ and $(D\bar{A})^{\;1}_{x:\overline{n|}}$ are also easily derived.

Final Application: For a given $a_{x:\overline{n|}}$, the rate of interest i is to be found by an iterative scheme. The appropriate mortality table is known; an approximate value i_0 is desired to start the iteration. This problem has been solved for an annuity certain in [2]:

* This assumes that d_{x+k} is a decreasing function of k for fixed x.

$$i = \frac{1 - \left(\frac{a_{\overline{n}|}}{n}\right)^2}{a_{\overline{n}|}} \tag{11}$$

If inequality (2) is used as an approximation for $a_{\overline{n}|}$:

$$a_{\overline{n}|} \doteq \frac{n \cdot a_{x:\overline{n}|}}{e_{x:\overline{n}|}} \tag{12}$$

Thus,

$$i_0 = \frac{1 - \left(\frac{a_{x:\overline{n}|}}{e_{x:\overline{n}|}}\right)^2}{n \cdot \left(\frac{a_{x:\overline{n}|}}{e_{x:\overline{n}|}}\right)} \tag{13}$$

For example, if $a_{30} = 22.478$ and the 1958 C.S.O. male tables are used, $\omega = 100$ and $e_{30} \doteq 40.75$. (13) gives $i_0 = 1.8\%$; the actual value of i is 3%.

REFERENCES

[1] Hardy, G.H., Littlewood, J.E., Polya G *Inequalities* Cambridge University Press 1934

[2] Silver, M. *An Approximate Solution For the Unknown Rate of Interest For An Annuity Certain*

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