AN OVERVIEW OF ACTUARIAL DECREMENT RATE ESTIMATION

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ABSTRACT

To estimate a rate of decrement prevailing in a population from a sample drawn from that population, one must select an estimator procedure. In many practical cases one must also make an assumption concerning the distribution of the decrement over the interval on which the rate to be estimated is defined. The type of rate to be estimated must also be chosen. Traditionally actuaries have chosen to estimate the effective rate, q, using the moment estimation procedure, and the Balducci distribution assumption.

This paper attempts to broaden the actuary's view of decrement rate estimation beyond this traditional combination. Various estimation results are stated without proof.

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I. INTRODUCTION

This paper is intended mainly for actuarial students, educators and practitioners, especially those whose familiarity with this subject is limited to that provided by the syllabus for the Society of Actuaries examinations. The syllabus tends to promote a very restricted understanding of this topic, and some misconceptions of it as well.

This paper attempts to broaden the actuary's understanding of decrement rate estimation, and to clear up some of the more common misconceptions which have been promoted. Some recent papers in the actuarial literature have made a nice contribution toward this broadening; others have reinforced some of the misconceptions.

Many observations and results are stated in this paper without proof, as befits the role of this paper as an introduction and overview of the topic.

Throughout the paper it is assumed that our objective is to obtain an estimate of the effective annual rate of some decrement. Of greatest interest to actuaries are age-specific decrement rates, such as mortality or disability. Thus we are consistently dealing with an age interval $[x, x+1]$, where x is not necessarily an integer, and we are interested in an estimate, $\hat{q}$, of the decrement for this interval.

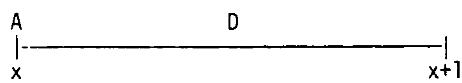
II. SAMPLE RATES

The basic objective is to estimate a rate of decrement prevailing in a population, and this is approached by looking at a sample of cases drawn from that population. The data of the sample, along with an estimation procedure and suitable assumptions (when needed), allow us to obtain such an estimate <u>for the population</u> directly. A popular misconception is that we have obtained the rate of decrement experienced <u>by the sample</u>, and we then use this rate as our estimate of the population rate.

This two-step view is not generally correct. In many cases the concept of the "rate experienced by the sample", hereinafter called the "sample rate" as distinguished from the "population rate", does not even exist; it is not obtainable.

When does the sample rate exist, and when does it not? The answer is that the sample rate does exist whenever there is no migration within the unit age interval on which the rate to be estimated is defined. Migration is present whenever some cases in the sample come under observation at an age within the internal (entrants) <u>or</u> whenever some cases leave the sample within the age interval for a reason other than the decrement being estimated (exits). This latter situation is frequently referred to as censoring. It should be noted that migration at either interval boundary is the same as no migration at all; the sample rate then exists.

This no migration case can be represented by the following diagram:



A units come under observation at the beginning of the estimation interval, of which D become decrements during the interval. We may or may not have information as to the distribution of the D decrements over the interval.

The sample is rate is $q^s = D/A$. This is true by the definition of an effective rate, which is independent of the distribution of the sample decrements. Furthermore, this sample rate is a very reasonable estimate of the population rate. The same rate is produced by both popular estimation procedures of maximum likelihood and the method of moments.

III. DISTRIBUTION ASSUMPTIONS

Similarly, when we make a decrement distribution assumption, it is an assumption regarding decrement distribution prevailing on the population, not the sample. The sample decrements have an actual distribution over the age interval, whether known or not, and would only by coincidence follow the assumed distribution to which the population is subject.

Actuaries are familiar with the popular distribution assumptions of Balducci, uniform, and constant force. Traditionally, the Balducci assumption has been the most popular, to the extent of almost excluding the others from consideration.

It has been shown by several writers that the numerical effect on the estimated rate, by various distribution assumptions, is trivial. Thus it is reasonable to use the most mathematically convenient assumption, which is generally the Balducci. However, from purely theoretical considerations, the Balducci is the least reasonable of the three, since it implies a decreasing force of decrement over the age interval. For the decrement of mortality, at least, this is not logical, except at juvenile ages. Furthermore, with computerized calculations, the Balducci loses its advantage of numerical convenience. It would appear, therefore, that the stage is being set for the demise of the Balducci as the automatic, or "natural" distribution assumption.

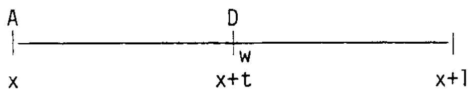
The uniform distribution assumption, theoretically more logical than the Balducci, has some nice properties. In various situations it will produce the same estimate using either the maximum likelihood or the method of moments estimation procedures. The constant force assumption also has nice properties in some cases.

IV. ESTIMATION PROCEDURES

Perhaps the most significant area in which the actuary's view of this topic has been limited by the Syllabus is that of estimation procedure. The method of moments, that is, equating actual to expected decrements over the unit age interval, has been urged to the almost complete exclusion of others, such as maximum likelihood and the product-moment method (also called the Kaplan-Meier method). Both of these latter methods have been shown to possess excellent statistical properties.

Although the moment method is certainly a valid estimation procedure, it appears to have potential for misuse. Several results using this procedure have appeared in the literature in which the correctness of the expression for expected decrements is questionable. One example of this is given here and a second example is given in the next section of the paper.

Consider the familiar situation of withdrawals within the unit age interval illustrated by the following diagram:



All actuaries have learned to write the moment equation

$$A \cdot q_x - w \cdot {}_{1-t}q_{x+t} = D ,$$

then to adopt the Balducci distribution assumption, and finally to conclude that

$$\hat{q} = \frac{D}{A-(1-t)w} .$$

This is such a time-honoured result that it must surely be correct. But it has recently been challenged by Jan Hoem in ARCH 1980.1*.

Hoem argues that, although none of the w cases did, in fact, become decrements over the interval $(x,x+t)$, some of them could have. Hence the left side of the moment equation is not a correct statement of expected decrements. He is saying that $w \cdot {}_tq_x$ decrements are expected over $(x,x+t)$, notwithstanding the fact that none did occur, and $(A-w) \cdot q_x$ additional decrements are expected over $(x,x+1)$. Thus he argues, the correct moment equation is

$$(A-w) \cdot q_x + w \cdot {}_tq_x = D,$$

which, incidentally, solves more conveniently under the uniform distribution assumption than under the Balducci, but then still results in

$$\hat{q} = \frac{D}{A-(1-t)w} \quad .$$

We have reached the familiar time-honoured result, but via a different route.

This paper neither agrees nor disagrees with Hoem's argument. It is presented here to illustrate a possible fallacy in actuarial conventional wisdom, and to suggest an area for further study.

* Hoem, J.M. "Exposed-to-Risk Considerations Based on the Balducci Assumption and Other Assumptions in the Analysis of Mortality". ARCH 1980.1, Society of Actuaries, 47-51.

V. CONCEPT OF EXPOSURE

Closely allied to the discussion of estimation procedures is the actuarial tradition of expressing estimated rates as the ratio of observed decrements to a measure of exposure. This convenient explanation, or intuitive interpretation, of the estimated rate is present under the moment method and Balducci distribution assumption, and is no doubt one reason why that combination has been so commonly used.

This is another example of the restricted view which actuaries have of rate estimation. Although this form is intuitively appealing, that reason alone should not be sufficient to exclude other procedures and assumptions from consideration.

Various writers have suggested that, in some cases, a similar ratio form, with its measure of exposure, is obtained under the uniform distribution assumption. These results appear to be obtained from a moment procedure, but it can be argued that the moment equations used are not theoretically correct. The results can not be called "wrong", since they only claim to be estimates of the prevailing population rate, but the procedure should be recognized as *ad hoc* involving certain simplifying assumptions.

VI. AN ALTERNATE "STANDARD" APPROACH

It has been stated that the moment procedure, Balducci assumption combination has been the "standard" actuarial approach to rate estimation for quite some time, and reasons for that choice have been seen.

Several writers, including Hoem in the aforementioned paper and Donald Jones in a letter to The Actuary in February, 1979, have suggested the alternative combination of the maximum likelihood procedure, constant force assumption. Under this combination, an estimate of the population constant force of decrement over the unit age interval, $\hat{\mu}$, is conveniently obtained. From this estimate, one obtains an estimate of the effective rate by

$$\hat{q} = 1 - e^{-\hat{\mu}} ,$$

due to the invariance property of the maximum likelihood procedure.

Furthermore, the estimate $\hat{\mu}$ is easily found to be identical to the sample central rate. It is expressible in the ratio form of observed decrements to exact total exposure to the decrement in the unit age interval. This denominator is analogous to the life-table symbol L_x . Proof of this result is given in Hoem's paper, and is omitted here.

As Hoem points out, this use of the maximum likelihood procedure avoids the alleged incorrectness in the moment procedure, cited earlier. The use of constant force, in place of either the Balducci or uniform assumptions, allows a much more convenient solution of the maximum likelihood equation.

VII. CONCLUSION

This paper will hopefully have the effect of broadening the actuary's understanding of the subject of decrement rate estimation beyond that obtained in the Society's syllabus, and suggests that a new direction in the Society's education program for this topic would be appropriate. This suggestion was also made by Jones in his 1979 letter cited earlier. This writer acknowledges the contribution of Dr. Jones in motivating much of the material in this paper. In addition, the assistance of Dr. Harry Panjer is also gratefully acknowledged.