THE CLASSICAL DEFINITION OF COMPOUND INTEREST IS ADEQUATE

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Introduction:

Suppose one dollar is invested at the compound rate of interest i. It has been known for centuries that the accumulation function is given by $a(n) = (1+i)^n$ for n an integer. What can be said about a(t) when t is not an integer? Most authors assume that $a(t) = (1+i)^t$ for all t. For example, [1] page 7: "Strictly speaking, the accumulation function for compound interest has been defined only for integral values of t. However, it is natural to assume that interest is accruing continuously and, therefore, to extend the definition to non-integral values of t." This assumption seems typical of the current literature. In [2], the literature is more thoroughly examined with the same negative result. This leads to the suspicion that the function $(1+i)^t$ does not, in fact, uniquely represent compound interest and that other functions are at least mathematically possible. In [2], infinitely many other functions are given which agree with $(1+i)^n$ for all integers. Based on this analysis, Professor Chouinard concludes in [2] that in order to be completely rigorous compound interest must be redefined in terms of a constant force of interest. It is the purpose of this paper to demonstrate in a mathematically rigorous manner that compound interest as usually understood is uniquely given by $(1+i)^t$ for all real t.

The Paradox:

Let $\delta(t)$ be the force of interest at time t; then, $a(1) = e^{\int_0^1 \delta(t)dt}$. According to most authors, any function $\delta(t)$ will yield compound interest at the annual rate of i as long as $\int_0^1 \delta(t)dt = \ln(1+i)$. Let $\delta(t) = \frac{\ln(1+i)}{2\sqrt{t}}$, so that $\int_0^1 \delta(t)dt = \int_0^1 \frac{\ln(1+i)}{2\sqrt{t}}\,dt = \ln(1+i)$ and $a(1) = 1+i$. If one dollar is invested at this force of interest, then at time $t = \varepsilon$, $a(\varepsilon) = e^{\int_0^{\varepsilon} \frac{\ln(1+i)}{2\sqrt{t}} dt} = e^{\sqrt{\varepsilon}\cdot\ln(1+i)} = (1+i)^{\sqrt{\varepsilon}}$. Suppose the investment is withdrawn at time $t = \varepsilon$ and instantly redeposited. This process can be repeated $\frac{1}{\varepsilon}$ times over the entire year. Under these circumstances, $a(1) = e^{\frac{1}{\varepsilon}\int_0^1 \frac{\ln(1+i)}{2\sqrt{t}} dt} = (1+i)^{\frac{1}{\sqrt{\varepsilon}}}$. Clearly as $\varepsilon \to 0$, $a(1) \to \infty$. Thus, an investor can invest one dollar for a one year period at the compound rate of interest i and obtain as much interest as desired! Just as absurd, but at the other end of the spectrum, a borrower could offer $\delta(t) = 2t\ln(1+i)$ and claim to be paying compound interest at the rate of i because $a(1) = e^{\int_0^1 2t\ln(1+i)dt} = 1+i$. Should the borrower repay and instantly reborrow the money $\frac{1}{\varepsilon}$ times during the year, $a(1) = e^{\frac{1}{\varepsilon}\int_0^{\varepsilon} 2t\ln(1+i)dt} = (1+i)^{\varepsilon}$. As $\varepsilon \to 0$, $a(1) = 1$! Thus, the borrower could claim to have paid compound interest over the entire year at the annual rate i and yet have to pay no interest! While this paradox demonstrates that there is indeed a problem, it also points to the resolution.

The Solution:

In our view, the two forces of interest discussed in the previous section do not represent compound interest at the rate i because they violate the following axiom:

Axiom: One dollar invested for the entire period of n years at the compound rate of i must accumulate to $(1+i)^n$ at the end of n years. With this axiom, the following theorem follows:

Theorem: Under compound interest at the rate i, the force of interest is unique and equals the constant $\ln(1+i)$.

Proof: One dollar invested at the force of interest $\delta(t)$ for a period of ε years where $0 \le \varepsilon \le 1$, yields $a(\varepsilon) = e^{\int_0^{\varepsilon}\delta(t)dt}$. If this amount is withdrawn and instantly reinvested $\frac{1}{\varepsilon}$ times during the first year, $a(1) = e^{\frac{1}{\varepsilon}\int_0^\varepsilon \delta(t)dt}$. By the axiom, $\frac{1}{\varepsilon}\int_0^\varepsilon \delta(t)dt = \ln(1+i)$ or $\int_0^\varepsilon \delta(t)dt = \varepsilon \ln(1+i)$. Differentiate both sides with respect to ε and apply the fundamental theorem of calculus to the left side: $\delta(\varepsilon) = \ln(1+i)$. Since this is true for any ε, the theorem follows.

Corollary: Let t be any nonnegative real number. Under compound interest at the rate of i, one dollar accumulates to $(1+i)^t$ at the end of the t years.

Proof: $a(t) = e^{\int_0^t \ln(1+i)dt} = (1+i)^t$.

Thus, the old textbook treatment of compound interest intuitively found the correct results and has now been placed on a mathematically rigorous foundation.

Finally, it is interesting to observe that simple interest obeys its analog to the axiom given here. Let one dollar be invested for the period ε, where $0 \leq \varepsilon \leq 1$ and then instantly withdrawn and redeposited $\frac{1}{\varepsilon}$ times during the first year. At the end of the first ε period, there is accumulated one dollar in principal and εi dollars in interest. This distinction is important because under simple interest, the interest may not be converted into principal until year's end. At the end of 2ε, there is still one dollar in principal and $2\varepsilon i$ dollars of interest. At the end of the year, there will be one dollar of principal and $(\frac{1}{\varepsilon})\varepsilon \cdot i = i$ dollars of interest. Therefore, under simple interest $a(1) = 1+i$, as required. Even if unequal periods were chosen, it is easy to show that one dollar would still accumulate to 1+i at the end of one year.

REFERENCES

[1] Kellison, S.G., The Theory of Interest
Richard D. Irwin, Inc. 1970

[2] Chouinard, P., Should The Definition of Compound Interest Be Modified?
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