FORMULAE FOR THE DETERMINATION OF INTEREST RATES
IN AN ANNUITY CERTAIN

by

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Introduction: Often an investor is offered an annuity for a fixed purchase
price. The computation of the yield rate implied by this transaction must
presently be computed by an iteration scheme, usually on a computer. Three
such methods are given, e.g. in [1] pages 60-62. In [2], three closed-form

formulae are introduced. The basic formula $i \doteq \dfrac{1-(\frac{a_{\overline{n}|}}{n})^2}{\frac{a_{\overline{n}|}}{n}}$ gives the yield

rate to within $\frac{2}{3}\%$ for $.01 \le i \le .21$ and $10 \le n \le 60$. The second formula is
the basic formula iterated once, this result agrees with all tabulated values
to within .2%. The third formula is the result of applying a single Newton-
Raphson iteration to the basic formula. This approximation has negligible
errors for $10 \le n \le 60$ and $.01 \le i \le .21$. In this paper, the basic formula
is modified in several ways. While remaining amenable to hand calculation,
the accuracy is substantially increased.

Derivation:

$$\frac{a_{\overline{n}|}}{n} = \frac{v + v^2 + v^3 + \ldots v^n}{n} \ge v^{\frac{n+1}{2}}.$$ This inequality is the classical

arithmetic-geometric inequality. Thus, $(1+i)(\frac{a_{\overline{n}|}}{n})^2 \ge v^n = 1-i \cdot a_{\overline{n}|}$; or

$$i \ge \frac{1-(\frac{a_{\overline{n}|}}{n})^2}{a_{\overline{n}|} + (\frac{a_{\overline{n}|}}{n})^2} \tag{1}$$

Inequality (1) can be very useful; its value is enhanced by the fact that it is
valid for all i and n. For example, an investor is offered a twelve year annual
annuity of $1,000 for $6,200. If a return of at least 11% is required, should

this annuity be purchased? From (1)   . The exact

$$i \geq \frac{1 - (\frac{6.2}{12})^2}{6.2 + (\frac{6.2}{12})^2} = 11.34\%$$

answer is i = 12%. In this case, inequality (1) provides sufficient information.
Had a return of 11½% been required, this method would have failed. The results
of inequality (1) are given in Column I of the table. Except for i = .19 and
n = 10 and 15, the results are within 1% of all exact values. Because inequality
(1) holds for all i and n, it makes an excellent starting value for an iterative
computer program.

The denominator of (1) consists of the sum of two terms which are of
different orders of magnitude; viz, $a_{\overline{n}|}$ and $(\frac{a_{\overline{n}|}}{n})2$. If the smaller term is
neglected, slightly larger values of i will be produced, improving the fit.
Thus,

$$i = \frac{1 - (\frac{a_{\overline{n}|}}{n})2}{a_{\overline{n}|}} \qquad (2)$$

The values of (2) are listed in the table in Column II. All tabulated values
are within $\frac{2}{3}$% of the correct value. For the tabulated values, (2) provides
a better fit than (1). One measure of this is the sum of the absolute values
of all errors for the tabulated values; i.e., $\sum_{i} \sum_{n} |i-i'|$ where i' is the calcu-
lated approximation. This is called the total error and is given at the bottom
of the table under each column. For (1) the total error is .3720 while for (2),
.2559. $(\frac{a_{\overline{n}|}}{n})2$ is a decreasing function of n; neglecting this term for small
values of n introduces an unacceptable error. Only for n ≥ 10 can this term
be reasonably discarded. In [2], this formula is derived in a much more lengthly
fashion with closer attention to the induced errors.

If Column II of the table is observed, there is still a marked ten-
dency towards underestimation; e.g., for 15 ≤ n ≤ 50, only two values of i are
overestimated. Further, the magnitude of the error increases with i. This
suggests a formula of the form:

$$i = C \cdot (\frac{1 - (\frac{a_{\overline{n}|}}{n})2}{a_{\overline{n}|}}) \qquad (3)$$

A computer search for C was carried out by minimizing $E(c) = \sum_i \sum_n |i-i(3)|$, where $i(3)$ is the approximate value given by (3) and the double summation is over all tabulated values. The minimum value was $E(1.03078) = .108205$. Thus,

$$i \doteq 1.03078 \left(\frac{1-(\frac{a_{\overline{n}|}}{n})^2}{a_{\overline{n}|}} \right) \tag{4}$$

The values given by (4) are tabulated in Column III. The total error of equation (4) is less than half the total error of equation (2). For all tabulated values, equation (4) is within .5% of the correct value. For actual calculations, this constant can be set equal to 1.03 without noticeable loss of accuracy.

Using the same technique, a computer search was instituted over the coefficients of

$$i \doteq \frac{A-B(\frac{a_{\overline{n}|}}{n})^2}{a_{\overline{n}|}} \tag{5}$$

The coefficients which minimize the error function are: $A = 1.0337$ and $B = 1.0667$.

$$i = \frac{1.0337 - 1.0667(\frac{a_{\overline{n}|}}{n})^2}{a_{\overline{n}|}} \tag{6}$$

The values given by equation (6) are tabulated in Column IV. The total error is .0997 and the formula is within .0042 for all tabulated values. Except for very high interest rates (17% and 19%) at the end of the table ($n = 40$ to 50) the formula is substantially more accurate than .0042. Excluding these cases, the maximum error does not exceed .27%. A slight modification of equation (b) which is easier to remember and yields almost the same accuracy is:

$$i = \frac{1.03333 - 1.06666(\frac{a_{\overline{n}|}}{n})^2}{a_{\overline{n}|}} \tag{7}$$

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An analysis of the error terms associated with equation (6) suggests a small correction term which is an increasing function of both i and n. A computer search was carried out over the coefficients of:

$$i \doteq \frac{A - B\left(\dfrac{a_{\overline{n}|}}{n}\right)^2 - C\left(\dfrac{n}{a_{\overline{n}|}}\right)^2}{a_{\overline{n}|}} \qquad (8)$$

The coefficients which minimized the error function $E(A,B,C) = \sum_i \sum_n |i - i'|$ were found by an approximate gradient method.

$$i \doteq \frac{1.0495 - 1.105\left(\dfrac{a_{\overline{n}|}}{n}\right)^2 - .000546\left(\dfrac{n}{a_{\overline{n}|}}\right)^2}{a_{\overline{n}|}} \qquad (9)$$

These values are tabulated in Column V of the table. The total error is .0578 and the maximum error for any tabulated value is less than $\frac{1}{3}$%. The worst fit clearly occurs for n = 10. In the interval $15 \le n \le 50$, all tabulated values are with .1% (except the very last value in table n = 50, i = .19). The additional term $\left(\dfrac{n}{a_{\overline{n}|}}\right)^2$ is easily calculated by inverting $\left(\dfrac{a_{\overline{n}|}}{n}\right)^2$. In order to facilitate remembering the coefficients, the formula can be used with only slight loss of accuracy.

$$i \doteq \frac{1.05 - 1.1\left(\dfrac{a_{\overline{n}|}}{n}\right)^2 - .0005\left(\dfrac{n}{a_{\overline{n}|}}\right)^2}{a_{\overline{n}|}} \qquad (10)$$

Finally, extreme accuracy can be obtained with a desk calculator by applying a single application of the Newton–Raphson Method to the result obtained from any of the methods discussed. It is an easy exercise to show that the Newton–Raphson Formula becomes:

$$i_2 = \frac{(1+i_1)^{n+1} - i_1(n+1) - 1}{a_{\overline{n}|}(1+i_1)^{n+1} - n} \qquad (11)$$

The result of applying formula (6) to (11) is shown in Column VI. For all
tabulated values, the error is less than 10^{-4}.

Conclusion: When $.01 \leq i \leq .19$ and $10 < n \leq 50$, the problem of determining
the interest rate from n and $a_{\overline{n}|}$ can be readily solved with a desk calculator
without the use of an iterative scheme. If an error of $\frac{1}{2}\%$ can be tolerated,
equation (4) is recommended. If i and n are not near the ends of the tabulated
values, equation (6) or (7) usually have an error of less than $\frac{1}{4}\%$. If accuracy
to with $\frac{1}{10}\%$ is required $(15 \leq n \leq 50)$, equation (9) or (10) is indicated. A
single application of the Newton-Raphson Method (equation (11)) to the result
of equation (9) or (4) gives the first four decimal places without error.

Additional Remarks:

1. Although not the subject treated here, the results of this paper indicate
   that the most efficient algorithm for a computer program to solve this
   problem for all i and n would be to use approximation(1) as a starting value for
   a Newton-Raphson iteration. This method converges quickly for all values
   tried and with far fewer iterations than a simple iteration scheme.

2. All computations were carried out on a TRS-80 Model I Level II home computer
   using BASIC.

References

[1] Kellison, S. G. The Theory of Interest
           Richard D. Irwin, Inc., 1970

[2] Silver, M. An Approximate Solution For The Unknown Rate Of Interest For An
           Annuity Certain
           The Journal of Risk and Insurance Vol. XLVIII No. 1 March 1981

i	I	II	III	IV	V	VI
n = 10						
.03	.0294	.0319	.0329	.0302	.0287	.0300
.05	.0485	.0523	.0538	.0515	.0505	.0500
.07	.0674	.0721	.0744	.0723	.0717	.0700
.09	.0861	.0916	.0945	.0926	.0924	.0900
.11	.1047	.1109	.1143	.1127	.1129	.1100
.13	.1233	.1300	.1340	.1326	.1331	.1300
.15	.1419	.1491	.1537	.1524	.1532	.1500
.17	.1606	.1681	.1732	.1722	.1733	.1700
.19	.1793	.1871	.1928	.1920	.1933	.1900
n = 15						
.03	.0292	.0307	.0317	.0300	.0292	.0300
.05	.0480	.0502	.0518	.0504	.0500	.0500
.07	.0666	.0693	.0714	.0703	.0703	.0700
.09	.0852	.0882	.0909	.0900	.0904	.0900
.11	.1038	.1071	.1104	.1097	.1103	.1100
.13	.1255	.1260	.1299	.1293	.1302	.1300
.15	.1414	.1450	.1495	.1491	.1502	.1500
.17	.1604	.1642	.1692	.1689	.1702	.1700
.19	.1795	.1834	.1891	.1889	.1902	.1900
n = 20						
.03	.0289	.0300	.0309	.0298	.0294	.0300
.05	.0476	.0491	.0506	.0497	.0497	.0500
.07	.0662	.0679	.0700	.0693	.0696	.0700
.09	.0848	.0867	.0894	.0889	.0895	.0900
.11	.1036	.1057	.1089	.1086	.1094	.1100
.13	.1226	.1248	.1286	.1284	.1294	.1300
.15	.1419	.1441	.1485	.1485	.1495	.1500
.17	.1614	.1636	.1687	.1687	.1697	.1700
.19	.1810	.1833	.1889	.1890	.1900	.1900
n = 25						
.03	.0288	.0296	.0305	.0296	.0294	.0300
.05	.0473	.0484	.0499	.0493	.0494	.0500
.07	.0659	.0672	.0692	.0688	.0692	.0700
.09	.0848	.0861	.0887	.0885	.0891	.0900
.11	.1039	.1053	.1085	.1084	.1092	.1100
.13	.1233	.1247	.1285	.1285	.1294	.1300
.15	.1429	.1444	.1488	.1489	.1497	.1500
.17	.1627	.1642	.1693	.1694	.1700	.1700
.19	.1827	.1842	.1898	.1901	.1904	.1900

i	I	II	III	IV	V	VI
n = 30						
.03	.0286	.0292	.0301	.0295	.0294	.0300
.05	.0472	.0480	.0494	.0490	.0493	.0500
.07	.0659	.0668	.0689	.0686	.0691	.0700
.09	.0850	.0859	.0886	.0884	.0891	.0900
.11	.1044	.1054	.1086	.1086	.1093	.1100
.13	.1240	.1251	.1289	.1290	.1296	.1300
.15	.1440	.1450	.1495	.1497	.1500	.1500
.17	.1640	.1651	.1701	.1704	.1704	.1700
.19	.1841	.1852	.1909	.1913	.1906	.1900
n = 35						
.03	.0285	.0290	.0299	.0294	.0294	.0300
.05	.0471	.0477	.0492	.0489	.0492	.0500
.07	.0660	.0667	.0687	.0686	.0691	.0700
.09	.0853	.0860	.0887	.0886	.0892	.0900
.11	.1049	.1059	.1090	.1090	.1096	.1100
.13	.1249	.1256	.1295	.1297	.1300	.1300
.15	.1450	.1457	.1502	.1505	.1503	.1500
.17	.1651	.1659	.1710	.1714	.1705	.1700
.19	.1854	.1861	.1919	.1923	.1905	.1900
n = 40						
.03	.0284	.0288	.0297	.0293	.0294	.0300
.05	.0470	.0476	.0490	.0488	.0491	.0500
.07	.0661	.0667	.0687	.0686	.0691	.0700
.09	.0857	.0862	.0889	.0889	.0894	.0900
.11	.1055	.1061	.1094	.1095	.1098	.1100
.13	.1256	.1262	.1301	.1303	.1302	.1300
.15	.1458	.1464	.1509	.1512	.1504	.1500
.17	.1660	.1666	.1718	.1721	.1704	.1700
.19	.1863	.1869	.1926	.1931	.1900	.1900
n = 45						
.03	.0283	.0287	.0296	.0292	.0293	.0300
.05	.0471	.0475	.0489	.0488	.0492	.0500
.07	.0663	.0668	.0688	.0688	.0693	.0700
.09	.0861	.0865	.0892	.0893	.0897	.0900
.11	.1061	.1066	.1098	.1100	.1101	.1100
.13	.1263	.1268	.1307	.1309	.1304	.1300
.15	.1465	.1470	.1515	.1518	.1503	.1500
.17	.1668	.1672	.1724	.1728	.1699	.1700
.19	.1870	.1875	.1932	.1937	.1890	.1900

i	I	II	III	IV	V	VI
n = 50						
.03	.0283	.0286	.0295	.0292	.0293	.0300
.05	.0471	.0475	.0489	.0488	.0492	.0500
.07	.0666	.0669	.0690	.0690	.0694	.0700
.09	.0865	.0868	.0895	.0896	.0899	.0900
.11	.1066	.1070	.1103	.1105	.1102	.1100
.13	.1268	.1272	.1311	.1314	.1303	.1300
.15	.1471	.1475	.1520	.1524	.1500	.1500
.17	.1673	.1677	.1729	.1733	.1692	.1700
.19	.1875	.1879	.1937	.1942	.1877	.1900
Total Error	.3720	.2559	.1082	.0997	.0578	