"The Financial Implications of Finite Ruin Theory"

By

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Abstract

An insurance company starts with an initial surplus, collects premium, pays claims to policyholders and pays dividends to stockholders. What remains is the following year's surplus. The process continues. This paper describes how to calculate the distribution of surplus for any given year. With these distributions, one can make the following calculations.

First, one can calculate the pure premium for insolvency insurance. This pure premium will depend upon the initial surplus, the size of the insurance company and the threshold for paying dividends.

Second, one can calculate the expected yield rate of future dividends.

Finite Ruin Theory

1. Introduction

An insurance company starts with an initial surplus. It collects premium, pays claims to policyholders and pays dividends to stockholders. What remains is the following year's surplus. If the following year's surplus is positive, the insurance company can continue writing insurance for another year. The process continues.

The mathematical description of this stochastic process is called ruin theory. The central problem in ruin theory has been to calculate the probability of ruin, that is, the probability that the surplus will become negative. If the time interval under consideration is finite, the description of this stochastic process is called finite ruin theory, otherwise this process is called infinite ruin theory.

Infinite ruin theory has historically been an easier problem to solve. The Society of Actuaries study note on Risk Theory(1983) contains many results on infinite ruin theory. These results demonstrate some principles in sound insurance company management. For example:

1. Increasing the surplus decreases the probability of ruin;
2. Increasing the profit loading decreases the probabilty of ruin; and
3. Lowering the limit of liability on individual claims decreases the probability of ruin.

Finite Ruin Theory

There are a number of criticisms that can be made of this brand of ruin theory.

1. The probability of ruin does not depend the size of the insurer. This is because the size of all insurers who are not ruined over an infinite time horizon is the same, namely infinite. This squarely contradicts the intuitive notion that large insurers are less likely to go insolvent than small insurers.

2. Ruin theory over an infinite time horizon is lacking in its treatment of dividends. If we consider the very sensible procedure of distributing surplus funds over a predetermined amount as dividends, we find that the probability of ruin over an infinite time horizon is one. This dosen't leave us with much to talk about.

3. Ruin theory does not directly answer the financial questions that most concern regulators and investors. What is the expected cost of an insolvency? What is the rate of return that one can expect from investing in an insurance company?

The purpose of this paper is to describe research on finite ruin theory that is currently in progress. First we will describe how to calculate the distribution of surplus at the end of any given year. This will enable us to solve the technical problems described above. With these distributions we can then calculate quantities that are of interest to insurance regulators and investors.

We will then be in position to explore the financial implications of finite ruin theory.

2. The Collective Risk Model

We will use the collective risk model to describe the incurred losses. This model assumes separate claim severity distributions and claim count distributions for each line of insurance written by the insurer. The losses for each claim and each line of insurance are totaled to give X_t. We shall use the version of the model described by Heckman and Meyers(1983) and Meyers and Schenker(1983).

This version of the collective risk model can be described by the following algorithm.

1. Select β at random from an inverse gamma distribution with $E[1/\beta] = 1$ and $Var[1/\beta] = b$.
2. For each line of insurance, i, do the following.
   2.1 Select χ at random from a gamma distribution with $E[\chi] = 1$ and $Var[\chi] = c_i$
   2.2 Select a random number of claims N from a Poisson distribution with mean $\chi \cdot \lambda_i$.
   2.3 Select N claims at random from the claim severity distribution for line of insurance i.
3. Set X_t equal to the sum of all claims selected in Step 2, divided by β.

The parameter c_i, called the contagion parameter, is a measure of the uncertainty in our estimate of the expected claim count, λ_i, for line i. The parameter β, called the mixing parameter, is a measure of our uncertainty of the scale of the claim severity distributions. Note that the random scaling factor, β, acts on all claim severity distributions simultaneously.

The claim severity distributions can be estimated by techniques given by Hogg and Klugman(1984). Methods for estimating the mixing parameter and the contagion parameters are given by Meyers and Schenker(1983). The resulting aggregate loss distribution can be calculated by using the Heckman-Meyers algorithm. The sources of error in this algorithm are of three kinds: (1) the approximation of the claim severity distributions by piecewise linear distributions; (2) discretization error due to numerical integration; and (3) roundoff error. In practice one can obtain a high degree of accuracy by choosing a fine enough partition of points at which the necessary functions are to be evaluated.

Finite Ruin Theory

We shall be illustrating our results with a single example. The following table gives the parameters.

Table 1

	Line #1	Line #2	Line #3
Claim Severity Distribution	Lognormal $\mu=6, \sigma=2$	Pareto b=10000, q=1.5	Weibull b=100, c=.25
Per Claim Retention	1000000	1000000	1000000
Expected Claim Count	2000	300	4000
Contagion Parameter	.025	.040	.015

Mixing Parameter	=	.010
Expected Loss	=	20804660
Risk Premium	=	21532823
Initial Surplus	=	10000000
Maximum Surplus	=	10000000

The cumulative distribution functions for the Pareto and Weibull distributions are $F(x)=1-(b/(x+b))^q$ and $F(x) = 1 - \exp(-(x/b)^c)$ respectively. We say that a random variable S has a lognormal distribution with parameters μ and σ if log(S) has a normal distribution with mean μ and standard deviation σ.

3. Finite Ruin Theory

The financial status of an insurance company is usually measured at year end. Accordingly a discrete treatment of financial results is assumed, i.e. the state of the insurance company's finances will be calculated at time points t = 0,1,2,... where t is in years.

Let:

P = Risk Premium (assumed constant for all years);

X_t = incurred loss during year t;

D_t = stockholder dividends paid at the end fo year t; and

U_t = surplus at end of year t.

We will assume the insurer has a maximum surplus A. Any funds available to the insurer totaling more than A are distributed to the stockholders in the form of dividends.

Thus we assume the following formula for dividends.

$$D_t = \text{maximum}[U_{t-1}+P-X_t-A,\ 0]$$

Given an initial surplus, U_0, we define the random variables:

$$V_1 = U_0 + P - X_1; \text{ and}$$

$$U_1 = V_1 - D_1.$$

Finite Ruin Theory

If $U_1 < 0$ the insurer is insolvent and ceases operation. If $U_1 \geq 0$, we assume the insurer continues operation, and we define the random variables:

$$V_2 = U_1 + P - X_2; \text{ and}$$

$$U_2 = V_2 - D_2.$$

In general, if $U_{t-1} \geq 0$, define:

$$V_t = U_{t-1} + P - X_t; \text{ and}$$

$$U_t = V_t - D_t.$$

Let $F_t(v)$ be the cumulative distribution function (c.d.f) for V_t. Note that $F_t(v) = 1$ for $v \geq A + P$.

Define: $$p_t = \int_0^{A+P} dF_t(v).$$

p_t represents the probability that the insurer is solvent at the end of year t given solvency at the end of year t-1.

Define: $s_t = p_1 \cdot p_2 \cdots p_t$.

s_t represents the probability that the insurer is solvent at the end of t years.

Traditionally, the central problem of ruin theory was to find the probability of ruin (i.e. the probability that the surplus becomes negative) within t years. In our terminology this probability is given by $1 - s_t$.

Finite Ruin Theory

We now define some additional quantities. Let:

$$r_t = -\int_{-\infty}^{0} v \cdot dF_t(v)$$

$$i_t = \int_{0}^{A} v \cdot dF_t(v) + A \cdot (1 - F_t(A))$$

$$d_t = \int_{A}^{A+P} (v-A) \cdot dF_t(v)$$

r_t represents the pure premium for insolvency insurance for year t, given that the insurer is still solvent at the end of t-1 years. The unconditional pure premium for insolvency insurance for this period is $s_{t-1} \cdot r_t$.

i_t/p_t and d_t/p_t respectively represent the expected surplus and the expected dividend for those insurers who are still solvent at the end of t years.

The results of these calculations for the sample insured are given in Exhibit I. Meyers(1986a) gives a detailed description of these calculations. Generally speaking, the method involves sucessive applications of the Heckman-Meyers algorithm. As mentioned above, the method relies on numerical analysis and can, with sufficient effort, give accurate results.

It is interesting to observe in Exhibit I that the distribution of U_t converges to a limiting distribution. The quantites p_t, r_t, i_t/p_t and d_t/p_t are identical for $t \geq 26$.

This fact enables us to extend the columns of this exhibit indefinitely without resorting to the Heckman-Meyers algorithm.

At this time we do not have a mathematical proof that the distribution of U_t converges to a limiting distribution.

There are two totals of interest. They are the total pure premium for insolvency insurance and the expected total dividends paid to the stockholders.

The total pure premium for insolvency insurance should reflect both the probability of insolvency and the time value of money. We assume that funds to pay for insolvency during year t are available at the beginning of year t. Accordingly, this total is given by the following formula.

$$\sum_{t=1}^{\infty} r_t \cdot s_{t-1} \cdot v_r^{t-1},$$

where v_r is the discount factor. In this paper we assume an interest rate of 6% for discounting the pure premium for insolvency insurance.

Finite Ruin Theory

The expected total dividends paid to stockholders should refect both the probability that dividends will be paid (note that no dividends will be paid after insolvency) and the time value of money. We assume that dividends for year t are paid at the end of year t. Accordingly, this total is given by the following formula.

$$\sum_{t=1}^{\infty} d_t \cdot s_{t-1} \cdot v_d^t,$$

where v_d is the discount factor.

This total will depend, in part, on the investment made by the stockholders. It represents the return on their investment.

Below, it is proposed that the pure premium for insolvency insurance be considered as part of the investment made by the stockholders. Accordingly, we define the rate of return on the investment as the interest rate for which

$$\sum_{t=1}^{\infty} d_t \cdot s_{t-1} \cdot v_d^t$$

is equal to the sum of the initial surplus and the pure premium for insolvency insurance.

4. Financial Implications

4.1 Insolvency Insurance

Many states set up guaranty funds to pay amounts due to policyholders of insolvent insurance companies. The money for the guaranty funds comes from assessments to licensed insurance companies in proportion to premium written.

There are a number of actions a prudent insurance company can take to prevent insolvency. These include:

1. Maintaining an adequate surplus;
2. Maintaining an adequate security loading;
3. Purchasing excess of loss reinsurance; and
4. Increasing the size of the company to gain the benefits of pooling.

The present method of guaranty fund assessments does not reward these actions. Using finite ruin theory one can properly account for these actions, and thus provide financial incentives for prudent management of an insurance company.

The author is proposing that contributions to guaranty funds be based on the pure premium for insolvency insurance as defined above. These ideas will be discussed in greater detail in a paper by Meyers(1984b).

Finite Ruin Theory

4.2 The Rate of Return on the Stockholder's Investment

In evaluating investment opportunities, one considers such factors as the expected yield on one's investment, and how the actual yield may vary from the expected yield. Finite ruin theory provides the information neccessary to answer these questions.

The yield for the insurer described in Table 1 is 8.48%. Exhibit I provides the expected dividends and the probability of receiving dividends in future years. If the stockholders decide to reinvest profits by choosing $12,000,000 as the maximum surplus, the yield drops to 7.43%, but as Exhibit II shows, the stockholders are likely to be receiving dividends for a longer period of time.

The amount one chooses to invest in an insurance company depends upon the.yield of alternative investments available. One would expect the amount of surplus to approach the amount required to yield the same rate as the alternative investments. This assumes no adjustment in reinsurance arrangements or security loadings, and that the investor is risk neutral.

Suppose, for example, that alternative investments yield 12.5%. Finite ruin theory predicts that the surplus supporting the insurance company described by Table 1 will shrink to something in the neighborhood of $6,000,000. But as Exhibit III shows, this investment will be riskier than an investment of $10,000,000 in the same company. Thus the results may vary depending on the investor's attitude toward risk.

5. Miscelleneous Remarks

The role of dividends in finite ruin theory was originally discussed by de Finetti(1957). A more recent account is given by Buhlmann(1970).

There are a number of factors that remain to be explored. These include the following:

1. the effect of investment income;
2. the effect of uncertain loss reserves; and
3. the effect of the underwriting cycle.

This list is by no means exhaustive.

Exhibit I

INSURER SOLVENCY MODEL

INITIAL SURPLUS = 10000000
CAP ON SURPLUS = 10000000

PREMIUM PAYMENT EACH PERIOD = 21532823
EXPECTED LOSS EACH PERIOD = 20804660

YEAR	PT	ST	RT	RT*ST-1	IT/PT	DT/PT
1	.99729	.99729	3870	3870	9050676	1710563
2	.98875	.98607	20889	20832	8553995	1357246
3	.98112	.96745	36909	36395	8280232	1218146
4	.97623	.94446	47404	45861	8127466	1148814
5	.97337	.91930	53593	50617	8041619	1111374
6	.97173	.89331	57136	52526	7993169	1090537
7	.97080	.86723	59149	52838	7965757	1078805
8	.97028	.84145	60290	52285	7950225	1072168
9	.96998	.81619	60937	51275	7941417	1068407
10	.96981	.79155	61304	50036	7936419	1066273
11	.96971	.76757	61512	48690	7933583	1065063
12	.96966	.74428	61630	47306	7931974	1064375
13	.96963	.72168	61697	45920	7931060	1063985
14	.96961	.69974	61735	44553	7930541	1063764
15	.96960	.67847	61757	43214	7930247	1063638
16	.96959	.65784	61769	41909	7930079	1063567
17	.96959	.63784	61776	40639	7929984	1063526
18	.96959	.61844	61780	39406	7929931	1063503
19	.96959	.59963	61783	38209	7929900	1063490
20	.96959	.58139	61784	37047	7929883	1063483
21	.96959	.56371	61785	35921	7929873	1063479
22	.96959	.54657	61785	34829	7929867	1063476
23	.96959	.52994	61785	33770	7929864	1063475
24	.96959	.51383	61785	32743	7929862	1063474
25	.96959	.49820	61785	31747	7929861	1063474
26	.96959	.48305	61785	30781	7929861	1063473
27	.96959	.46835	61785	29845	7929860	1063473
28	.96959	.45411	61785	28937	7929860	1063473
29	.96959	.44030	61785	28057	7929860	1063473
30	.96959	.42691	61785	27204	7929860	1063473
31	.96959	.41392	61785	26376	7929860	1063473
32	.96959	.40133	61785	25574	7929860	1063473
33	.96959	.38913	61785	24796	7929860	1063473
34	.96959	.37729	61785	24042	7929860	1063473
35	.96959	.36582	61785	23311	7929860	1063473

PURE PREMIUM FOR INSOLVENCY INSURANCE 627570
DISCOUNTED AT INTEREST RATE 6.00%

RATE OF RETURN TO STOCKHOLDERS 8.48%

Exhibit II

INSURER SOLVENCY MODEL

INITIAL SURPLUS = 10000000
CAP ON SURPLUS = 12000000

PREMIUM PAYMENT EACH PERIOD = 21532823
EXPECTED LOSS EACH PERIOD = 20804660

YEAR	PT	ST	RT	RT*ST-1	IT/PT	DT/PT
1	.99723	.99723	3870	3870	10043721	718103
2	.98983	.98709	19259	19206	9869960	1032032
3	.98501	.97229	29790	29406	9745680	1044010
4	.98260	.95537	35064	34093	9665464	1029585
5	.98131	.93752	37836	36147	9614082	1016007
6	.98057	.91931	39427	36964	9581131	1006220
7	.98012	.90103	40396	37136	9559969	999645
8	.97984	.88286	41003	36945	9546368	995339
9	.97965	.86490	41389	36541	9537621	992547
10	.97954	.84720	41636	36011	9531994	990745
11	.97946	.82980	41795	35408	9528373	989584
12	.97942	.81272	41897	34766	9526043	988836
13	.97938	.79597	41962	34104	9524544	988355
14	.97936	.77954	42005	33434	9523578	988045
15	.97935	.76344	42032	32765	9522957	987845
16	.97934	.74767	42049	32102	9522557	987717
17	.97934	.73223	42061	31447	9522300	987634
18	.97933	.71709	42068	30803	9522134	987581
19	.97933	.70227	42072	30170	9522028	987547
20	.97933	.68776	42075	29548	9521959	987525
21	.97933	.67354	42077	28939	9521915	987511
22	.97933	.65962	42079	28342	9521887	987502
23	.97933	.64599	42079	27756	9521868	987496
24	.97933	.63263	42080	27183	9521856	987492
25	.97933	.61956	42080	26621	9521849	987489
26	.97933	.60675	42081	26071	9521844	987488
27	.97933	.59421	42081	25532	9521841	987487
28	.97933	.58192	42081	25004	9521839	987486
29	.97933	.56989	42081	24488	9521838	987486
30	.97933	.55811	42081	23981	9521837	987486
31	.97933	.54658	42081	23486	9521836	987485
32	.97933	.53528	42081	23000	9521836	987485
33	.97933	.52421	42081	22525	9521836	987485
34	.97933	.51338	42081	22059	9521835	987485
35	.97933	.50277	42081	21603	9521835	987485

PURE PREMIUM FOR INSOLVENCY INSURANCE 489359
DISCOUNTED AT INTEREST RATE 6.00%

RATE OF RETURN TO STOCKHOLDERS 7.43%

Exhibit III

INSURER SOLVENCY MODEL

INITIAL SURPLUS = 6000000
CAP ON SURPLUS = 6000000

PREMIUM PAYMENT EACH PERIOD = 21532823
EXPECTED LOSS EACH PERIOD = 20804660

YEAR	PT	ST	RT	RT*ST-1	IT/PT	DT/PT
1	.97048	.97048	50618	50618	5227147	1757804
2	.94296	.91513	112146	108836	4972395	1462092
3	.93238	.85325	136602	125008	4885519	1374967
4	.92870	.79241	145168	123864	4855491	1345513
5	.92742	.73489	148140	117387	4845059	1335314
6	.92697	.68123	149174	109626	4841429	1331767
7	.92682	.63138	149533	101866	4840164	1330531
8	.92677	.58514	149658	94490	4839724	1330101
9	.92675	.54227	149702	87596	4839570	1329951
10	.92674	.50255	149717	81188	4839517	1329899
11	.92674	.46573	149722	75242	4839498	1329881
12	.92674	.43161	149724	69731	4839492	1329874
13	.92674	.39999	149725	64623	4839490	1329872
14	.92674	.37069	149725	59888	4839489	1329872
15	.92674	.34353	149725	55501	4839489	1329871
16	.92674	.31836	149725	51435	4839488	1329871
17	.92674	.29504	149725	47666	4839488	1329871
18	.92674	.27342	149725	44174	4839488	1329871
19	.92674	.25339	149725	40938	4839488	1329871
20	.92674	.23483	149725	37939	4839488	1329871
21	.92674	.21762	149725	35159	4839488	1329871
22	.92674	.20168	149725	32583	4839488	1329871
23	.92674	.18690	149725	30196	4839488	1329871
24	.92674	.17321	149725	27984	4839488	1329871
25	.92674	.16052	149725	25934	4839488	1329871
26	.92674	.14876	149725	24034	4839488	1329871
27	.92674	.13786	149725	22273	4839488	1329871
28	.92674	.12776	149725	20641	4839488	1329871
29	.92674	.11840	149725	19129	4839488	1329871
30	.92674	.10973	149725	17727	4839488	1329871
31	.92674	.10169	149725	16428	4839488	1329871
32	.92674	.09424	149725	15225	4839488	1329871
33	.92674	.08733	149725	14109	4839488	1329871
34	.92674	.08094	149725	13076	4839488	1329871
35	.92674	.07501	149725	12118	4839488	1329871

PURE PREMIUM FOR INSOLVENCY INSURANCE 1115323
DISCOUNTED AT INTEREST RATE 6.00%

RATE OF RETURN TO STOCKHOLDERS 12.59%

Finite Ruin Theory

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