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A Risk Premium Calculation Principle Based On The Aggregate Deviations Of The Risk Reserve Process

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ABSTRACT

Let $Z(t,\pi)$ be the risk reserve at time t given a constant risk premium rate of π,with $Z(0,\pi)=0$.We introduce a new risk premium calculation principle Φ that produces risk premium rates ,$\pi^{\Phi}(T)$,for contracts over the finite interval (0,T] as follows: Φ is a non-negative real-valued function such that $\Phi[\bar{Z}(T,\pi^{\Phi}(T))]=\pi^{\Phi}(T)$ for π^{Φ}(T) satisfying $E[\bar{Z}(T,\pi^{\Phi}(T))]=0$,where $\bar{Z}(T,\pi)=\int_0^T Z(t,\pi)dt$. If we denote by,$\Psi$,the net risk premium calculation principle,it is proved that that in most practical situations Φ dominates Ψ in the sense that the risk premium rate generated by Φ is not less than that generated by Ψ for all T, $0<T<\infty$.

Key Words: dominance,risk premium calculation principle.

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1. Introduction

Let us consider an insurance portfolio that consisting of policies (risks) insured over the finite time interval (0,T]. Initially the portfolio consists of N identical risks (called policies or contracts) and there are no new entrants into the portfolio in the time interval (0,T],so we can consider the portfolio as a closed group of risks.The number of policies in force at time t,$0 \le t \le T$, is thus a non-increasing function of t.Each policy remains in existence for a random length of time (possibly of infinite duration) before expiring.Prior to or upon expiring ,each risk can generate a random integral number of positive claims. Let us denote by S(t) and P(t),the aggregate claims generated in the interval (0,t] and the aggregate premiums received in (0,t] respectively.The risk reserve at time t is given by Z(t) where $Z(t)=P(t)-S(t)$, with $Z(0)=0$.

A risk premium calculation principle is a function,defined on the real line ,that enables an insurer to quote a risk premium.Traditionally the function acts on the risk reserve process Z(t) only at time T,the end of the contract period. For examples of risk premium calculation principle see Bühlmann (1970,Chapter 4) and (1980), Gerber (1979,Chapter 5) and Haezendock and Goovaerts (1982).

In the sequel we develop a new risk premium calculation principle Φ based on the aggregate deviations of $Z(t,\pi)$ as follows:Let $Z(t,\pi)$ be the risk reserve at time t,given a constant risk premium rate of π.Without loss of generality we assume that $Z(0,\pi)=0$. The risk premium rate generated by Φ is $\pi^{\Phi}(T)$ where

$$\Phi[\bar{Z}(T,\pi^{\Phi}(T))]=\pi^{\Phi}(T) \tag{1.1}$$

such that

$$E[\bar{Z}(T,\pi^{\Phi}(T))]=0 \tag{1.2}$$

where

$$\bar{Z}(T,\pi)=\int_0^T Z(t,\pi)dt. \tag{1.3}$$

The integral in (1.3) is assumed to exist as a Lebesgue stochastic integral.

The motivation for choosing $\bar{Z}(T,\pi)$ instead of $Z(t,\pi)$ is as follows:In the definition of $Z(t,\pi)$ the history of the random walk of the risk reserve process before time T,as is in the case of most risk premium calculation principles,was ignored.$Z(t,\pi)$ is thus often considered to be sufficient for the calculation of the risk premium rates.Such rates will produce premiums that are indifferent to the evolution of the risk reserve process.This might not be a problem if we assume that the epochs of claims form a stationary point process.When this assumption is not appropriate,it is suggested that we,in some way,attempt to incorporate the inherent non-stationarity of the claim occurrence process into our risk premiums. This can be accomplished by taking into account the evolution of the process in such a way that the expected aggregate deviation of the risk reserve process, about the time axis and over the time interval (0,T], is zero. In other words the expected area under the random walk $Z(t,\pi^*(T))$, $0 \le t \le T$, is zero. This is somewhat similar to the requirement, in the graduation theory of death rates, that the expected accumulated deviations of the actual minus expected deaths to be zero. See Benjamin and Haycocks (1970, Chapter 11).

2. Assumptions

Let us define the following random variables:

P(t) = aggregate premiums received in the interval (0,t] from the totality of policy-holders.

S(t) = aggregate claim ammount in the interval (0,t] from the totality of policy-holders.

n(t) = total number of claims that occurred in (0,t].

q(t) = number of policies in force at time t; since there are no new policies issued, then q(t) is a non-increasing function of t with q(0) = N and q(t) = 0,1,2,....,N.

T = duration of each contract issued, $0 \le T < \infty$.

The S(t) process is such that the size of a claim, given the occurrence of a claim, is independent of the time of the occurrence of the claim. The claim sizes are also independently and identically distributed (i.i.d.) random variables. The claims are positive finite first and second moments.

The n(t) process will be assumed to be a self-exciting point process in the sense that the claim intensity at time t, $\nu(t)$ is a function of the number of policies at time t. Thus

$$\nu(t) = f(q(t))$$

where f is a non-negative Lebesgue integrable function satisfying $f(x)=0$ if and only if $x=0$. It is assumed that f is such that the first two moments of n(t) exist $0 \le t < \infty$. For example;

(i) If q(t) is a deterministic function of t, then $f(q(t))$ is also a deterministic function of q(t) and we have that n(t) is a time-dependent Poisson process.

(ii) If q(t) is a stochastic process and $f(q(t)) = \rho(t) q(t)$ for some non-negative function and non-random function $\rho(t)$, then n(t) is a doubly stochastic Poisson process in the sense of Cox and Isham (1980, Chapter 3.3).

Since all of the risks are identical, the risk premium rate, π, charged will be the same for each policy. The amount of premiums recieved from the totality of policy-holders in the small interval $(t, t+\delta t]$ is $\delta P(t) = \pi q(t) \delta t + o(\delta t)$. This leads to the following expression for P(t);

$$P(t) = \pi \int_0^t q(s)\, ds. \tag{2.1}$$

The existence of the integral in (2.1) was proved by Mc Lean and Neuts (1967). Note that q(t) is a bounded non-increasing step function.

Finally we let $Z(t,\pi)$ be the risk reserve at time t, where π is the risk premium rate charged per policy. We thus have

$$Z(t,\pi) = \pi \int_0^t q(s)\, ds - \sum_{i=0}^{n(t)} Y_i \tag{2.2}$$

where Y_i is the size of the i-th claim and $Y_0 = 0$. We assume that $Z(t,\pi)$ is a second order process, i.e. $E[(Z(t,\pi))^2] < \infty$, $0 < t < \infty$ and also that

$$\int_0^T \int_0^T Cov[Z(t,\pi), Z(s,\pi)]\, ds\, dt$$

exists for $0 \le T < \infty$. This guarantees the existence of (1.3). See Loève (1978, Chapter XI, Section 37).

3. The Main Results

Recall the definitions of Φ and $\pi^{*}(T)$ given in equations (1.1) to (1.3). From (2.2) and (1.2) we see that

$$E[\pi^{\Phi}(T)\int_0^T(\int_0^t q(s)ds-\sum_{i=0}^{n(t)}Y_i)dt]=0. \tag{3.1}$$

Write

$$\mu(t)=E[n(t)]=\int_0^t E[\nu(s)]\,ds \tag{3.2}$$

and

$$\lambda(t)=E[q(t)]. \tag{3.3}$$

After taking expectations under the integral signs we get

$$\pi^{\Phi}(T)=E[Y_i]\frac{\int_0^T\mu(t)dt}{\int_0^T\int_0^t\lambda(s)ds\,dt}\quad ,i>0. \tag{3.4}$$

Let Ψ be the traditional net risk premium calculation principle.Then Ψ generates the risk premium π^{*}(T) which satisfies

$$E[Z(T,\pi^{*}(T))]=0. \tag{3.5}$$

From (2.2) we see that this leads to the following

$$\pi^{*}(T)=E[Y_i]\frac{\mu(T)}{\int_0^T\lambda(t)dt}. \tag{3.6}$$

In order to compare Φ and Ψ we introduce the concept of dominance.

Definition: We say a risk premium calculation principle Θ_1 dominates a risk premium calculation principle Θ_2 if for any finite time interval (0,T],the risk premium generated by Θ_1 is greater than or equal to risk premium rate generated by Θ_2.

Comparing (3.4) and (3.6),it is obvious that

$$\pi^{\Phi}(T)\geq\pi^{*}(T)\ \text{ if and } only \text{ if }\ \frac{M(T)}{A(T)}\geq\frac{\frac{d}{dT}M(T)}{\frac{d}{dT}A(T)}, \tag{3.7}$$

where

$$M(T)=\int_0^T\mu(t)\,dt \tag{3.8}$$

and

$$A(T)=\int_0^T\int_0^t\lambda(s)dsdt. \tag{3.9}$$

Rearranging the right hand side of (3.7),we get

$$\pi^{\Phi}(T) \geq \pi^{*}(T) \text{ if and only if } \frac{d}{dT} ln(A(T)) \geq \frac{d}{dT} ln(M(T)), \tag{3.10}$$

which proves the following theorem:

Theorem 1:

(i) Φ dominates Ψ if and only if $\frac{d}{dt}[ln(\frac{A(t)}{M(t)})] \geq 0$, for *all* $t \geq 0$.

(ii) $\Phi = \Psi$ if and only if $\frac{d}{dt}[ln(\frac{A(t)}{M(t)})] = 0$, for *all* $t \geq 0$.

(iii) Ψ dominates Φ if and only if $\frac{d}{dt}[ln(\frac{A(t)}{M(t)})] \leq 0$, for *all* $t \geq 0$.

The condition (iii) in Theorem 1 deserves special mention.Since Ψ is the net risk premium principle we will not want to charge a risk premium rate that is less than $\pi^{*}(t)$.So we cannot use Φ in that case.But in practice we will invariably find that for each t, $\mu(t) \leq \int_0^t \lambda(s)ds$, a result analogous to the statement that expected number of deaths is less than the total expected exposure of cohort of lives.So condition (iii) is not expected to be satisfied in practice.

It is instructive to compare Φ and Ψ for the case where the n(t) process is a time homogeneous Poisson process.This is accomplished by the following theorem:

Theorem 2:

In the classical Lundberg model of collective risk theory which is based on the following assumption that the aggregate claim process is a time homogeneous Poisson process and the aggregate premium,P(t),is proportional t,we have $\Phi = \Psi$.

Proof: Obvious.

If in assumption (i) of Theorem 2 we replace the constant ρ by a non-constant function of time,$\rho(t)$,a non-homogeneous Poisson process results for n(t).Under these conditions,we loose the identity relationship between Φ and Ψ.However this process can be transformed into a time homogeneous Poisson process by changing to operational time $\tau(t)$ given by

$$\tau(t) = \int_0^t \rho(s)ds.$$

This transformation,however,does not retain the identity $\Phi = \Psi$ because $\frac{d}{d\tau}[\frac{A(\tau)}{M(\tau)}]$ is not identical to 0 under this new operational time.

4. Conclusions

The risk premium calculation principle Φ introduced in this paper is designed to take into account the history of the risk reserve process. It is only one of many possible methods of including the history of the random walk of risk reserves,$Z(t,\pi)$.That this approach leads to a simple expression for the risk premium is,in the opinion of the author,a very desirable feature of risk premium calculation principles Φ. Notice that in keeping with the traditions of risk theory,the premium calculation principle Φ produces a flat premium rate,$\pi^{\Phi}(T)$,for the case where the claims form a non-stationary Poisson process.

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