EVALUATING OPTIONAL FORMS OF PAYMENT UNDER A PENSION PLAN

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INTRODUCTION

The new Actuarial Mathematics textbook provides methodology from which a pension plan participant can calculate

1) the expected present value of future payments, and
2) the variance of that present value

under any of the plan's optional forms of payment. If the actuarial assumptions (interest and mortality rates) used in these calculations are the same as those used by the plan to calculate the optional benefit amounts, the expected present value will be the same under each form of payment, but the variance amounts will be different. By calculating the variance amounts, a participant can quantify the amount of risk associated with each form of payment.

METHODOLOGY - GENERAL

Generally, the new method treats the participant's time until death as a continuous random variable (T) which has a probability density function given by:

$$f(t) = {}_tp_x\,\mu_{x+t}$$

As long as a given form of payment clearly defines the exact timing and amount of all benefit payments given the participant s exact time of death, the participant can calculate the expected present value and variance of those payments by treating (T) as the random variable.

It should be noted that by treating (T) as the random variable, the calculated variance represents and isolates the variance due to mortality. Investment return is assumed to be fixed and is not a cause of variance. It does, however, influence the level of variance as shown in the Sample Results section of this paper.

METHODOLOGY - SPECIFIC

Assume that a retiring pension plan participant can elect to receive either a) a lump sum distribution, or b) an actuarially equivalent monthly annuity in any of the following forms:

1. L = Life annuity,
2. L + nG = Life annuity with a guaranteed certain period of n years,
3. JL = Joint life annuity,
4. J & LS = Joint and Last survivor annuity, and
5. J & P% S = Joint and P% survivor annuity (reduces to P% of the original amount upon the first death of the annuitant or his beneficiary).

The expected present value of payments is the same under all options. In order to measure the variance under the monthly payment options (the variance under the lump sum option is $0), this paper suggests the following procedure:

1. Develop variance formulas for each of the continuous functions that correspond to the actual monthly forms of payment available, and

2. Express the actual monthly forms of payment in terms of the continuous functions and calculate the variance of each actual form of payment.

Procedures 1 and 2 above are performed in Appendices I and II, respectively, of this paper. All calculations are based on annual annuity factors which are based on the 1984 Unisex Pension Mortality Table, set forward 1 year for males and backward 4 years for females. These factors are shown in Appendix III. Standard approximation formulas found in Jordan are used to approximate continuous factors from non-continuous factors.

SAMPLE RESULTS

Appendix II contains numerical variance calculations for two sample, male participants who are retiring from a pension plan at ages 65 and 55, respectively. Each participant is married to a female of equal age and is offered a choice between a lump sum distribution and six actuarially equivalent monthly annuities. The lump sum and annuity amounts offered to each participant are shown in Table I at the top of the next page and are based on (a) the 1984 Unisex Pension Mortality Table set forward one year for males and backward four years for females, and (b) (i) 5% interest, compounded annually, or (ii) 12% interest, compounded annually:

Table I

		Annual Benefit Amount Offered to a Pension Plan Participant at Retirement			
		5% Interest		12% Interest	
		65	55	65	55
1.	LO	$1,026	$793	$1,556	$1,334
2.	L + 5G	998	786	1,504	1,319
3.	L + 10G	927	766	1,405	1,285
4.	JL	1,241	904	1,767	1,436
5.	J & LS	775	648	1,285	1,190
6.	J & 40% S	1,001	781	1,537	1,326
7.	Lump Sum	$10,000	$10,000	$10,000	$10,000

The expected present value under each form of payment is $10,000 for each participant under either set of assumptions. The variance of the present value for each participant, however, differs according to both the form of payment and the assumptions. Using the variance amounts calculated in Appendix II, and assuming that the present value of payments is normally distributed, the participant can calculate a range for each form of payment that should contain the actual present value of payments 90% of the time. The range will be symmetrical around the expected present value of $10,000 and the size of the range will be:

2 times 1.645

times

The Standard Deviation of the present value of payments under the particular form of payment.

Results of these calculations are shown in Table II below:

Table II

		The Present Value of Payments, Within a 90% Confidence Interval, Equals $10,000 ± $X, where $X =			
		5% Interest		12% Interest	
		65	55	65	55
1.	LO	$7,004	$5,296	$5,370	$3,737
2.	L + 5G	5,994	4,881	3,998	3,116
3.	L + 10G	4,099	3,974	2,200	2,029
4.	JL	8,041	6,159	6,409	4,540
5.	J & LS	3,689	2,477	2,222	1,466
6.	J & 40% S	5,073	3,884	3,942	2,807
7.	Lump Sum	0	0	0	0

Analysis of Table II supports the following general relationships:

* Higher rates of investment return reduce variance due to mortality.

* Adding a guaranteed period to a life annuity, or lengthening an existing guaranteed period, reduces variance due to mortality. The reduction is greater for older participants.

* Joint life annuities are more variable than life only annuities.

* Joint and last survivor annuities are less variable than life only annuities.

When analyzing the results shown in Table II, it is important to remember that these variance figures exclude any amount of variance due to the investment risk.

APPLICATIONS

This paper is written from the perspective of a pension plan participant who wants to know the variance of the present value of payments he will receive from the plan under various forms of payment. Similar variance calculations could also be used to:

1. Quantify the variance due to mortality in the present value of benefits of an ongoing pension plan, or

2. Quantify the variance due to mortality in the present value of future annuity payments to be made a) from a terminated pension plan s trust fund, or b) under the terms of an insurance company s annuity contracts.

By programming the methods presented in Actuarial Mathematics, computers can be used to calculate the variance due to mortality whenever they are used to calculate the expected present value of annuity payments. These variance calculations can provide useful information regarding the liabilities of either a pension plan or an insurance company.

SUPPLEMENT

One form of payment commonly offered under a pension plan is a Joint and P% Contingent Annuity (J & P% C). This type of annuity pays $1 per year to the participant for his lifetime and, if the participant's beneficiary outlives him, $P per year to the beneficiary for her lifetime commencing on the date of the participant's death. This form of payment was not included in the body of this paper because I could not "prove" its variance formula, which was developed in Appendix I-F. In order to prove that variance formula, I must prove that:

$$Cov\left(\bar{a}_{\overline{T}|}, \bar{a}_{\overline{U}|}\right) = \frac{1}{\delta}\left(\bar{a}_{xy} - {}^{2}\bar{a}_{xy}\right) + \frac{1}{\delta}\bar{a}_x \bar{a}_y - \bar{a}_x \bar{a}_{xy}$$

where U = the shortest of S and T,
S = the future lifetime of (y), and
T = the future lifetime of (x).

If the above covariance formula is correct, the following information could be added to Tables I and II:

Table I

Annual Benefit Amount Offered to a Pension Plan Participant at Retirement

	5% Interest		12% Interest	
	65	55	65	55
8. J & 40% C	$909	$728	$1,435	$1,272

Table II

The Present Value of Payments, Within a 90% Confidence Interval, Equals $10,000 ± $X, where $X =

	5% Interest		12% Interest	
	65	55	65	55
8. J & 40% C	4,221	2,970	3,043	1,933

If the above covariance formula is not correct, the numbers in Table II, above, would have to be changed. I would appreciate a proof of the correct covariance formula so that I can add the correct numbers to Tables I and II of this paper.

APPENDIX I-A

VARIANCE OF A CONTINUOUS LIFE ONLY ANNUITY

1. BENEFIT DESCRIPTION - $\bar{a}_{\overline{T|}}$

A continuous annuity of $1 per year is paid to a participant age (x), for as long as the participant is alive. The annuity value is:

$$\bar{a}_{\overline{T|}}\text{, where } T = \text{the future lifetime of } (x).$$

2. $$E(\bar{a}_{\overline{T|}}) = E\left(\frac{1-v^T}{\delta}\right) = \frac{1}{\delta}\left(1-E(v^T)\right) = \frac{1}{\delta}(1-\bar{A}_x) = \bar{a}_x$$

3. $$E(\bar{a}^2_{\overline{T|}}) = E\left(\frac{1-v^T}{\delta}\right)^2 = \frac{1}{\delta^2}E(1-2v^T+v^{2T})$$

$$= \frac{1}{\delta^2}\left(1-2\bar{A}_x + {}^2\bar{A}_x\right)$$

$$= \frac{1}{\delta^2}\left(1-2(1-\delta\bar{a}_x) + (1-2\delta\,{}^2\bar{a}_x)\right)$$

$$= \frac{1}{\delta^2}\left(2\delta\bar{a}_x - 2\delta\,{}^2\bar{a}_x\right) = \frac{2}{\delta}\left(\bar{a}_x - {}^2\bar{a}_x\right)$$

4. $$Var(\bar{a}_{\overline{T|}}) = E(\bar{a}^2_{\overline{T|}}) - E(\bar{a}_{\overline{T|}})^2$$

$$Var(\bar{a}_{\overline{T|}}) = \frac{2}{\delta}\left(\bar{a}_x - {}^2\bar{a}_x\right) - \bar{a}_x^2$$

APPENDIX I-B

VARIANCE OF A CONTINUOUS LIFE ANNUITY WITH AN N-YEAR GUARANTEED CERTAIN PERIOD

1 BENEFIT DESCRIPTION - $\bar{a}_{\overline{Y}|}$

A continuous annuity of \$1 per year is paid to a participant, age (x), for as long as the participant is alive. Additionally, if the participant dies before receiving payments for n complete years, the continuous annuity of \$1 per year will continue to the participant's beneficiary until payments have been made, either to the participant or his beneficiary for a total of n years. The annuity value is:

$\bar{a}_{\overline{Y}|}$, where Y = the greater of n and T,
n = the guaranteed certain period, and
T = the future lifetime of (x).

2
$$
\begin{aligned}
E(\bar{a}_{\overline{Y}|}) &= \int_0^n {}_tp_x\,\mu_{x+t}\,\bar{a}_{\overline{n}|}\,dt + \int_n^\infty {}_tp_x\,\mu_{x+t}\,\bar{a}_{\overline{t}|}\,dt \\
&= \bar{a}_{\overline{n}|}\int_0^n {}_tp_x\,\mu_{x+t}\,dt + \int_n^\infty {}_tp_x\,\mu_{x+t}\left(\frac{1-v^t}{\delta}\right)dt \\
&= {}_nq_x\,\bar{a}_{\overline{n}|} + \frac{1}{\delta}\int_n^\infty {}_tp_x\,\mu_{x+t}(1-v^t)\,dt \\
&= {}_nq_x\,\bar{a}_{\overline{n}|} + \frac{1}{\delta}\left({}_np_x - (\bar{A}_x - \bar{A}^{1}_{x:\overline{n}|})\right) \\
&= {}_nq_x\,\bar{a}_{\overline{n}|} + \frac{1}{\delta}\left({}_np_x - \left((1-\delta\bar{a}_x) - (1-\delta\bar{a}_{x:\overline{n}|} - v^n\,{}_np_x)\right)\right) \\
&= {}_nq_x\,\bar{a}_{\overline{n}|} + \frac{1}{\delta}\left({}_np_x - 1 + \delta\bar{a}_x + 1 - \delta\bar{a}_{x:\overline{n}|} - v^n\,{}_np_x\right) \\
&= {}_nq_x\,\bar{a}_{\overline{n}|} + \frac{1}{\delta}\left({}_np_x(1-v^n) + \delta\bar{a}_x - \delta\bar{a}_{x:\overline{n}|}\right) \\
&= {}_nq_x\,\bar{a}_{\overline{n}|} + {}_np_x\,\bar{a}_{\overline{n}|} + \bar{a}_x - \bar{a}_{x:\overline{n}|} \\
&= \bar{a}_{\overline{n}|} + \bar{a}_x - \bar{a}_{x:\overline{n}|} \\
&= \bar{a}_{\overline{\overline{x:n}|}}
\end{aligned}
$$

APPENDIX I-B (cont.)

3. $$E(\bar{a}^2_{\overline{T}|}) = \int_0^n {}_tp_x\,\mu_{x+t}\,\bar{a}^2_{\overline{n}|}\,dt + \int_n^\infty {}_tp_x\,\mu_{x+t}\,\bar{a}^2_{\overline{t}|}\,dt$$

$$= \bar{a}^2_{\overline{n}|}\int_0^n {}_tp_x\,\mu_{x+t}\,dt + \int_n^\infty {}_tp_x\,\mu_{x+t}\left(\frac{1-v^t}{\delta}\right)^2 dt$$

$$= {}_nq_x\,\bar{a}^2_{\overline{n}|} + \frac{1}{\delta^2}\int_n^\infty {}_tp_x\,\mu_{x+t}\,(1-2v^t+v^{2t})\,dt$$

$$= {}_nq_x\,\bar{a}^2_{\overline{n}|} + \frac{1}{\delta^2}\left({}_np_x - 2(\bar{A}_x - \bar{A}^1_{x:\overline{n}|}) + ({}^2\bar{A}_x - {}^2\bar{A}^1_{x:\overline{n}|})\right)$$

$$= {}_nq_x\,\bar{a}^2_{\overline{n}|} + \frac{1}{\delta^2}\Big({}_np_x - 2\big((1-\delta\,\bar{a}_x) - (1-\delta\,\bar{a}_{x:\overline{n}|} - v^n\,{}_np_x)\big)$$
$$+ \big((1-2\delta\,{}^2\bar{a}_x) - (1-2\delta\,{}^2\bar{a}_{x:\overline{n}|} - v^{2n}\,{}_np_x)\big)\Big)$$

$$= {}_nq_x\,\bar{a}^2_{\overline{n}|} + \frac{1}{\delta^2}\Big({}_np_x - 2 + 2\delta\,\bar{a}_x + 2 - 2\delta\,\bar{a}_{x:\overline{n}|} - 2v^n\,{}_np_x$$
$$+ 1 - 2\delta\,{}^2\bar{a}_x - 1 + 2\delta\,{}^2\bar{a}_{x:\overline{n}|} + v^{2n}\,{}_np_x\Big)$$

$$= {}_nq_x\,\bar{a}^2_{\overline{n}|} + \frac{1}{\delta^2}\Big({}_np_x(1-2v^n+v^{2n}) + 2\delta\,\bar{a}_x - 2\delta\,\bar{a}_{x:\overline{n}|} - 2\delta\,{}^2\bar{a}_x$$
$$+ 2\delta\,{}^2\bar{a}_{x:\overline{n}|}\Big)$$

$$= {}_nq_x\,\bar{a}^2_{\overline{n}|} + \frac{1}{\delta^2}\Big({}_np_x(1-v^n)^2 + 2\delta\big((\bar{a}_x - \bar{a}_{x:\overline{n}|}) - ({}^2\bar{a}_x - {}^2\bar{a}_{x:\overline{n}|})\big)\Big)$$

$$= {}_nq_x\,\bar{a}^2_{\overline{n}|} + {}_np_x\,\bar{a}^2_{\overline{n}|} + \frac{2}{\delta}\left({}_{n|}\bar{a}_x - {}_{n|}{}^2\bar{a}_x\right)$$

$$= \bar{a}^2_{\overline{n}|} + \frac{2}{\delta}\left({}_{n|}\bar{a}_x - {}_{n|}{}^2\bar{a}_x\right)$$

4. $$Var(\bar{a}_{\overline{T}|}) = E(\bar{a}^2_{\overline{T}|}) - E(\bar{a}_{\overline{T}|})^2$$

$$Var(\bar{a}_{\overline{T}|}) = \bar{a}^2_{\overline{n}|} + \frac{2}{\delta}\left({}_{n|}\bar{a}_x - {}_{n|}{}^2\bar{a}_x\right) - \bar{a}^2_{x:\overline{n}|}$$

APPENDIX I-E

VARIANCE OF A CONTINUOUS JOINT LIFE ANNUITY

1 BENEFIT DESCRIPTION - $\bar{a}_{\overline{u}|}$

A continuous annuity of $1 per year is paid to a participant, age (x), for as long as the participant and his spouse, age (y), are alive. The annuity value is:

$\bar{a}_{\overline{u}|}$, where U = the shortest of S and T,
S = the future lifetime of (y), and
T = the future lifetime of (x).

$$2 \quad E(\bar{a}_{\overline{u}|}) = \int_0^\infty {}_tp_{xy}\,\mu_{x+t:y+t}\,\bar{a}_{\overline{t}|}\,dt = \int_0^\infty {}_tp_{xy}\,\mu_{x+t:y+t}\left(\frac{1-v^t}{\delta}\right)dt$$

$$= \frac{1}{\delta}\int_0^\infty {}_tp_{xy}\,\mu_{x+t:y+t}(1-v^t)\,dt = \frac{1}{\delta}(1-\bar{A}_{xy}) = \bar{a}_{xy}$$

$$3. \quad E(\bar{a}^2_{\overline{u}|}) = \int_0^\infty {}_tp_{xy}\,\mu_{x+t:y+t}\,\bar{a}^2_{\overline{t}|}\,dt = \int_0^\infty {}_tp_{xy}\,\mu_{x+t:y+t}\left(\frac{1-v^t}{\delta}\right)^2 dt$$

$$= \frac{1}{\delta^2}\int_0^\infty {}_tp_{xy}\,\mu_{x+t:y+t}(1-2v^t+v^{2t})\,dt$$

$$= \frac{1}{\delta^2}(1-2\bar{A}_{xy}+{}^2\bar{A}_{xy}) = \frac{1}{\delta^2}(1-2(1-\delta\bar{a}_{xy})+(1-2\delta\,{}^2\bar{a}_{xy}))$$

$$= \frac{1}{\delta^2}(1-2+2\delta\bar{a}_{xy}+1-2\delta\,{}^2\bar{a}_{xy}) = \frac{1}{\delta^2}(2\delta\bar{a}_{xy}-2\delta\,{}^2\bar{a}_{xy})$$

$$= \frac{2}{\delta}(\bar{a}_{xy}-{}^2\bar{a}_{xy})$$

$$4. \quad Var(\bar{a}_{\overline{u}|}) = E(\bar{a}^2_{\overline{u}|}) - E(\bar{a}_{\overline{u}|})^2$$

$$Var(\bar{a}_{\overline{u}|}) = \frac{2}{\delta}(\bar{a}_{xy}-{}^2\bar{a}_{xy}) - \bar{a}^2_{xy}$$

APPENDIX I-D

VARIANCE OF A CONTINUOUS JOINT AND LAST SURVIVOR ANNUITY

1. BENEFIT DESCRIPTION - $\bar{a}_{\overline{z}|}$

A continuous annuity of $1 per year is paid to a participant, age (x), or to the participant s spouse, age (y), for as long as either the participant or his spouse are alive. The annuity value is:

$\bar{a}_{\overline{z}|}$, where z = the longest of S and T,
S = the future lifetime of (y), and
T = the future lifetime of (x).

2 $$E(\bar{a}_{\overline{z}|}) = \int_0^\infty {}_tp_x \mu_{x+t}(1-{}_tp_y)\bar{a}_{\overline{t}|}\,dt + \int_0^\infty {}_tp_y \mu_{y+t}(1-{}_tp_x)\bar{a}_{\overline{t}|}\,dt$$

$$= \int_0^\infty {}_tp_x \mu_{x+t}(1-{}_tp_y)\left(\frac{1-v^t}{\delta}\right)dt + \int_0^\infty {}_tp_y \mu_{y+t}(1-{}_tp_x)\left(\frac{1-v^t}{\delta}\right)dt$$

$$= \frac{1}{\delta}\int_0^\infty {}_tp_x \mu_{x+t}(1-{}_tp_y)(1-v^t)\,dt + \frac{1}{\delta}\int_0^\infty {}_tp_y \mu_{y+t}(1-{}_tp_x)(1-v^t)\,dt$$

$$= \frac{1}{\delta}\int_0^\infty \left({}_tp_x \mu_{x+t} - {}_tp_{xy}\mu_{x+t} + {}_tp_y \mu_{y+t} - {}_tp_{xy}\mu_{y+t}\right)(1-v^t)\,dt$$

$$= \frac{1}{\delta}\int_0^\infty \left({}_tp_x \mu_{x+t} + {}_tp_y \mu_{y+t} - {}_tp_{xy}\mu_{x+t:y+t}\right)(1-v^t)\,dt$$

$$= \frac{1}{\delta}\left(1+1-1-\bar{A}_x - \bar{A}_y + \bar{A}_{xy}\right)$$

$$= \frac{1}{\delta}\left(1-(1-\delta\bar{a}_x) - (1-\delta\bar{a}_y) + (1-\delta\bar{a}_{xy})\right)$$

$$= \frac{1}{\delta}\left(1-1+\delta\bar{a}_x - 1 + \delta\bar{a}_y + 1 - \bar{a}_{xy}\right)$$

$$= \frac{1}{\delta}\left(\delta\bar{a}_x + \delta\bar{a}_y - \bar{a}_{xy}\right)$$

$$= \bar{a}_x + \bar{a}_y - \bar{a}_{xy}$$

$$= \bar{a}_{\overline{xy}}$$

APPENDIX I-D (cont.)

3 $$E\left(\bar{a}^2_{\overline{t|}}\right) = \int_0^\infty {}_tp_x\,\mu_{x+t}\left(1-{}_tp_y\right)\bar{a}^2_{\overline{t|}}\,dt + \int_0^\infty {}_tp_y\,\mu_{y+t}\left(1-{}_tp_x\right)\bar{a}^2_{\overline{t|}}\,dt$$

$$= \int_0^\infty \left({}_tp_x\,\mu_{x+t} - {}_tp_{xy}\,\mu_{x+t} + {}_tp_y\,\mu_{y+t} - {}_tp_{xy}\,\mu_{y+t}\right)\left(\frac{1-2v^t+v^{2t}}{\delta^2}\right)dt$$

$$= \frac{1}{\delta^2}\int_0^\infty \left({}_tp_x\,\mu_{x+t} + {}_tp_y\,\mu_{y+t} - {}_tp_{xy}\,\mu_{x+t:y+t}\right)\left(1-2v^t+v^{2t}\right)dt$$

$$= \frac{1}{\delta^2}\left(1+1-1-2\bar{A}_x-2\bar{A}_y+2\bar{A}_{xy}+{}^2\bar{A}_x+{}^2\bar{A}_y-{}^2\bar{A}_{xy}\right)$$

$$= \frac{1}{\delta^2}\Big(1-2(1-\delta\bar{a}_x)-2(1-\delta\bar{a}_y)+2(1-\delta\bar{a}_{xy})+(1-2\delta\,{}^2\bar{a}_x)$$
$$+(1-2\delta\,{}^2\bar{a}_y)-(1-2\delta\,{}^2\bar{a}_{xy})\Big)$$

$$= \frac{1}{\delta^2}\left(2\delta\bar{a}_x+2\delta\bar{a}_y-2\delta\bar{a}_{xy}-2\delta\,{}^2\bar{a}_x-2\delta\,{}^2\bar{a}_y+2\delta\,{}^2\bar{a}_{xy}\right)$$

$$= \frac{1}{\delta^2}\left(2\delta\,\bar{a}_{\overline{xy}}-2\delta\,{}^2\bar{a}_{\overline{xy}}\right) = \frac{2}{\delta}\left(\bar{a}_{\overline{xy}}-{}^2\bar{a}_{\overline{xy}}\right)$$

4 $$Var\left(\bar{a}_{\overline{t|}}\right) = E\left(\bar{a}^2_{\overline{t|}}\right) - E\left(\bar{a}_{\overline{t|}}\right)^2$$

$$Var\left(\bar{a}_{\overline{t|}}\right) = \frac{2}{\delta}\left(\bar{a}_{\overline{xy}}-{}^2\bar{a}_{\overline{xy}}\right)-\bar{a}^2_{\overline{xy}}$$

APPENDIX 1-E

VARIANCE OF A CONTINUOUS JOINT AND P% SURVIVOR ANNUITY

1. BENEFIT DESCRIPTION - $P\,\bar{a}_{\overline{z}|} + (1-P)\,\bar{a}_{\overline{u}|}$

A continuous annuity of \$1 per year is paid to a participant, age (x), for as long as the participant and his spouse, age (y), are alive. Additionally, a continuous annuity of \$P per year is paid to the surviving member after the first death of the participant or the spouse for as long as the surviving member lives. The annuity value is:

$$P\,\bar{a}_{\overline{z}|} + (1-P)\,\bar{a}_{\overline{u}|}, \text{ where}$$

z = the longest of S and T,
u = the shortest of S and T,
S = the future lifetime of (y), and
T = the future lifetime of (x).

2 $$E\left(P\,\bar{a}_{\overline{z}|} + (1-P)\,\bar{a}_{\overline{u}|}\right) = P\,E\left(\bar{a}_{\overline{z}|}\right) + (1-P)\,E\left(\bar{a}_{\overline{u}|}\right)$$

$$= P\,\bar{a}_{\overline{xy}} + (1-P)\,\bar{a}_{xy}$$

3. $$Cov\left(\bar{a}_{\overline{z}|}, \bar{a}_{\overline{u}|}\right) = E\left(\bar{a}_{\overline{z}|}\,\bar{a}_{\overline{u}|}\right) - E\left(\bar{a}_{\overline{z}|}\right)E\left(\bar{a}_{\overline{u}|}\right)$$

$$= E\left(\left(\frac{1-v^z}{\delta}\right)\left(\frac{1-v^u}{\delta}\right)\right) - \bar{a}_{\overline{xy}}\,\bar{a}_{xy}$$

$$= \frac{1}{\delta^2}E\left(1 - v^z - v^u + v^z v^u\right) - \bar{a}_{\overline{xy}}\,\bar{a}_{xy}$$

$$= \frac{1}{\delta^2}\left(1 - \bar{A}_{\overline{xy}} - \bar{A}_{xy} + \bar{A}_x\,\bar{A}_y\right) - \bar{a}_{xy}\,\bar{a}_{\overline{xy}}$$

$$= \frac{1}{\delta^2}\left(1 - (1-\delta\,\bar{a}_{\overline{xy}}) - (1-\delta\,\bar{a}_{xy}) + (1-\delta\,\bar{a}_x)(1-\delta\,\bar{a}_y)\right) - \bar{a}_{xy}\,\bar{a}_{\overline{xy}}$$

$$= \frac{1}{\delta^2}\left(1 - 1 + \delta\,\bar{a}_{\overline{xy}} - 1 + \delta\,\bar{a}_{xy} + 1 - \delta\,\bar{a}_x - \delta\,\bar{a}_y + \delta^2\,\bar{a}_x\,\bar{a}_y\right) - \bar{a}_{\overline{xy}}\,\bar{a}_{xy}$$

APPENDIX I-E (cont)

$$Cov(\overline{a}_{\overline{Z}|}, \overline{a}_{\overline{U}|}) = \frac{1}{\delta^2}\left(\delta \overline{a}_{\overline{xy}} + \delta \overline{a}_{xy} - \delta \overline{a}_x - \delta \overline{a}_y + \delta^2 \overline{a}_x \overline{a}_y\right) - \overline{a}_{xy}\, \overline{a}_{\overline{xy}}$$

$$= \frac{1}{\delta^2}\left(\delta \overline{a}_{\overline{xy}} - \delta \overline{a}_{\overline{xy}} + \delta^2 \overline{a}_x \overline{a}_y\right) - \overline{a}_{xy}\, \overline{a}_{\overline{xy}}$$

$$= \overline{a}_x \overline{a}_y - \overline{a}_{xy}\left(\overline{a}_x + \overline{a}_y - \overline{a}_{xy}\right)$$

$$= \overline{a}_x \overline{a}_y - \overline{a}_x \overline{a}_{xy} - \overline{a}_y \overline{a}_{xy} + \overline{a}_{xy}^2$$

$$= (\overline{a}_x - \overline{a}_{xy})(\overline{a}_y - \overline{a}_{xy})$$

4. $$Var\left(P \overline{a}_{\overline{Z}|} + (1-P)\overline{a}_{\overline{U}|}\right) = P^2 Var(\overline{a}_{\overline{Z}|}) + (1-P)^2 Var(\overline{a}_{\overline{U}|}) + 2P(1-P)\, Cov(\overline{a}_{\overline{Z}|}, \overline{a}_{\overline{U}|})$$

$$= P^2\left[\frac{2}{\delta}\left(\overline{a}_{\overline{xy}} - {}^2\overline{a}_{\overline{xy}}\right) - \overline{a}_{\overline{xy}}^2\right] + (1-P)^2\left[\frac{2}{\delta}\left(\overline{a}_{xy} - {}^2\overline{a}_{xy}\right) - \overline{a}_{xy}^2\right] + 2P(1-P)\left[(\overline{a}_x - \overline{a}_{xy})(\overline{a}_y - \overline{a}_{xy})\right]$$

APPENDIX I-F

VARIANCE OF A CONTINUOUS JOINT AND P% CONTINGENT ANNUITY

1. BENEFIT DESCRIPTION - $\bar{a}_{\overline{T|}} + P\,\bar{a}_{\overline{S|}} - P\,\bar{a}_{\overline{U|}}$

A continuous annuity of $1 per year is paid to a participant age (x), for as long as the participant is alive. Additionally, upon the participant s death if he is survived by his spouse, age (y), a continuous annuity of $P per year is paid to the surviving spouse for her remaining lifetime. The annuity value is:

$$\bar{a}_{\overline{T|}} + P\,\bar{a}_{\overline{S|}} - P\,\bar{a}_{\overline{U|}}, \quad \text{where}$$

U = the shortest of S and T,
S = the future lifetime of (y), and
T = the future lifetime of (x).

2. $$E\left(\bar{a}_{\overline{T|}} + P\,\bar{a}_{\overline{S|}} - P\,\bar{a}_{\overline{U|}}\right) = E(\bar{a}_{\overline{T|}}) + P\,E(\bar{a}_{\overline{S|}}) - P\,E(\bar{a}_{\overline{U|}})$$
$$= \bar{a}_x + P\,\bar{a}_y - P\,\bar{a}_{xy}$$

3. $$Cov\left(\bar{a}_{\overline{T|}}, \bar{a}_{\overline{U|}}\right) = \frac{1}{\delta}\left(\bar{a}_{xy} - {}^2\bar{a}_{xy}\right) + \frac{1}{\delta}\bar{a}_x\,\bar{a}_y - \bar{a}_x\,\bar{a}_{xy}$$

APPENDIX I-F (cont)

4. $$\mathrm{Var}(\bar{a}_{\overline{T}} + \rho\bar{a}_{\overline{S}} - \rho\bar{a}_{\overline{U}}) = \mathrm{Var}(\bar{a}_{\overline{T}}) + \rho^2\mathrm{Var}(\bar{a}_{\overline{S}})$$
$$+\rho^2\mathrm{Var}(\bar{a}_{\overline{U}}) + 2\rho\,\mathrm{Cov}(\bar{a}_{\overline{T}}, \bar{a}_{\overline{S}})$$
$$-2\rho\,\mathrm{Cov}(\bar{a}_{\overline{T}}, \bar{a}_{\overline{U}}) - 2\rho^2\mathrm{Cov}(\bar{a}_{\overline{S}}, \bar{a}_{\overline{U}})$$

$$= \left[\tfrac{2}{\delta}(\bar{a}_x - \overset{2}{\bar{a}}_x) - \bar{a}_x^2\right] + \rho^2\left[\tfrac{2}{\delta}(\bar{a}_y - \overset{2}{\bar{a}}_y) - \bar{a}_y^2\right]$$
$$+\rho^2\left[\tfrac{2}{\delta}(\bar{a}_{xy} - \overset{2}{\bar{a}}_{xy}) - \bar{a}_{xy}^2\right] + 2\rho\left[0\right]$$
$$-2\rho\left[\tfrac{1}{\delta}(\bar{a}_{xy} - \overset{2}{\bar{a}}_{xy}) + \tfrac{1}{2}\bar{a}_x\bar{a}_y - \bar{a}_x\bar{a}_{xy}\right]$$
$$-2\rho^2\left[\tfrac{1}{\delta}(\bar{a}_{xy} - \overset{2}{\bar{a}}_{xy}) + \tfrac{1}{2}\bar{a}_x\bar{a}_y - \bar{a}_y\bar{a}_{xy}\right]$$

$$= \left[\tfrac{2}{\delta}(\bar{a}_x - \overset{2}{\bar{a}}_x) - \bar{a}_x^2\right] + \rho^2\left[\tfrac{2}{\delta}(\bar{a}_y - \overset{2}{\bar{a}}_y) - \bar{a}_y^2 - \bar{a}_{xy}^2 - \bar{a}_x\bar{a}_y + 2\bar{a}_y\bar{a}_{xy}\right]$$
$$-2\rho\left[\tfrac{1}{\delta}(\bar{a}_{xy} - \overset{2}{\bar{a}}_{xy}) + \tfrac{1}{2}\bar{a}_x\bar{a}_y - \bar{a}_x\bar{a}_{xy}\right]$$

$$= \left[\tfrac{2}{\delta}(\bar{a}_x - \overset{2}{\bar{a}}_x) - \bar{a}_x^2\right] + \rho^2\left[\tfrac{2}{\delta}(\bar{a}_y - \overset{2}{\bar{a}}_y) - (\bar{a}_y - \bar{a}_{xy})^2 - \bar{a}_x\bar{a}_y\right]$$
$$-\rho\left[\tfrac{2}{\delta}(\bar{a}_{xy} - \overset{2}{\bar{a}}_{xy}) + \bar{a}_x\bar{a}_y - 2\bar{a}_x\bar{a}_{xy}\right]$$

APPENDIX II-A

VARIANCE OF A MONTHLY LIFE ONLY ANNUITY

1. BENEFIT DESCRIPTION – $\ddot{a}^{(12)}_{\overline{T}|}$

A monthly annuity of \$1/12 per month is paid to a participant, age (x), on the first day of each month that the participant is alive. The annuity value is:

$\ddot{a}^{(12)}_{\overline{T}|}$, where T = the future lifetime of (x)

2 Variance Formula

$$Var\left(\ddot{a}^{(12)}_{\overline{T}|}\right) = Var\left(\frac{1-v^T}{12(1-v^{1/12})}\right) = Var\left(\frac{\delta}{12(1-v^{1/12})}\cdot\frac{1-v^T}{\delta}\right)$$

$$= \left(\frac{\delta}{12(1-v^{1/12})}\right)^2 Var\left(\bar{a}_{\overline{T}|}\right)$$

$$= \left(\frac{\delta}{12(1-v^{1/12})}\right)^2 \left[\frac{2}{\delta}\left(\bar{a}_x - {}^2\bar{a}_x\right) - \bar{a}_x^2\right]$$

3 Sample Calculations

(a) $x = 65$, $i = 5\%$

$$Var\left(\ddot{a}^{(12)}_{\overline{T}|}\right) = \left(\frac{.04879}{(12)\left(1-\left(\frac{1}{1.05}\right)^{1/12}\right)}\right)^2 \left[\frac{2}{.04879}(9.694-6.983) - 9.694^2\right]$$

$$= 17.226; \quad SD = 4.150$$

(b) $x = 55$, $i = 5\%$

$$Var\left(\ddot{a}^{(12)}_{\overline{T}|}\right) = \left(\frac{.04879}{12\left(1-\left(\frac{1}{1.05}\right)^{1/12}\right)}\right)^2 \left[\frac{2}{.04879}(12.557-8.310) - 12.557^2\right]$$

$$= 16.481; \quad SD = 4.060$$

APPENDIX II-A (cont)

(c) $x=65$, $i=12\%$

$$Var\left(\ddot{a}^{(12)}_{\overline{x}|}\right)=\left(\frac{.113329}{12\left(1-\left(\frac{1}{1.12}\right)^{1/12}\right)}\right)^2\left[\frac{2}{.113329}(6.375-3.825)-6.375^2\right]$$

$= 4.403$; $SD = 2.098$

(d) $x=55$, $i=12\%$

$$Var\left(\ddot{a}^{(12)}_{\overline{x}|}\right)=\left(\frac{.113329}{12\left(1-\left(\frac{1}{1.12}\right)^{1/12}\right)}\right)^2\left[\frac{2}{.113329}(7.442-4.141)-7.442^2\right]$$

$= 2.899$; $SD = 1.703$

APPENDIX II-B

VARIANCE OF A MONTHLY LIFE ANNUITY WITH AN N-YEAR GUARANTEED CERTAIN PERIOD

1. BENEFIT DESCRIPTION - $\ddot{a}^{(12)}_{\overline{Y}|}$

A monthly annuity of \$1/12 per month is paid to a participant, age (x), at the beginning of each month that the participant is alive. Additionally, if the participant dies before receiving (12 x n) monthly payments, monthly payments of \$1/12 will continue to the participant's beneficiary until a total of (12 x n) monthly payments have been received by either the participant or his beneficiary. The annuity value is:

$\ddot{a}^{(12)}_{\overline{Y}|}$, where Y = the greater of n and T,
n = the guaranteed certain period, and
T = the future lifetime of (x)

2 Variance Formula

$$Var\left(\ddot{a}^{(12)}_{\overline{Y}|}\right) = Var\left(\frac{1-v^{Y}}{12(1-v^{1/12})}\right) = Var\left(\frac{\delta}{12(1-v^{1/12})}\ \frac{1-v^{Y}}{\delta}\right)$$

$$= \left(\frac{\delta}{12(1-v^{1/12})}\right)^2 Var\left(\bar{a}_{\overline{Y}|}\right)$$

$$= \left(\frac{\delta}{12(1-v^{1/12})}\right)^2 \left[\bar{a}^2_{\overline{n}|} + \frac{2}{\delta}\left({}_{n|}\bar{a}_x - {}_{n|}{}^{2}\bar{a}_x\right) - \bar{a}^2_{x\overline{n}|}\right]$$

3 Sample Calculations

(a) $x=65,\ i=5\%,\ n=5$

$$Var\left(\ddot{a}^{(12)}_{\overline{Y}|}\right) = \left(\frac{.04879}{12\left(1-\left(\frac{1}{1.05}\right)^{1/12}\right)}\right)^2\left[4.437^2 + \frac{2}{.04879}(5.54-3.268) - 9.977^2\right]$$

$$= 13.333,\quad SD = 3.651$$

(b) $x=65,\ i=5\%,\ n=10$

$$Var\left(\ddot{a}^{(12)}_{\overline{Y}|}\right) = \left(\frac{.04879}{12\left(1-\left(\frac{1}{1.05}\right)^{1/12}\right)}\right)^2\left[7.913^2 + \frac{2}{.04879}(2.835-1.369) - 10.748^2\right]$$

$$= 7.223;\quad SD = 2.688$$

(c) $x=55,\ i=5\%,\ n=5$

$$Var(\ddot{a}^{(12)}_{\overline{x}|}) = \left(\frac{.04879}{12(1-(\frac{1}{1.05})^{1/12})}\right)^2\left[4.437^2 + \frac{2}{.04879}(8.236-4.452) - 12.673^2\right]$$

$= 14.252;\quad SD = 3.775$

(d) $x=55,\ i=5\%,\ n=10$

$$Var(\ddot{a}^{(12)}_{\overline{x}|}) = \left(\frac{.04879}{12(1-(\frac{1}{1.05})^{1/12})}\right)^2\left[7.913^2 + \frac{2}{.04879}(5.095-2.253) - 13.008^2\right]$$

$= 9.950,\quad SD = 3.154$

(e) $x=65,\ i=12\%,\ n=5$

$$Var(\ddot{a}^{(12)}_{\overline{x}|}) = \left(\frac{.113329}{12(1-(\frac{1}{1.12})^{1/12})}\right)^2\left[3.817^2 + \frac{2}{.113329}(2.788-.995) - 6.605^2\right]$$

$= 2.610;\quad SD = 1.616$

(f) $x=65,\ i=12\%,\ n=10$

$$Var(\ddot{a}^{(12)}_{\overline{x}|}) = \left(\frac{.113329}{12(1-(\frac{1}{1.12})^{1/12})}\right)^2\left[5.983^2 + \frac{2}{.113329}(1.092-.233) - 7.075^2\right]$$

$= .907;\quad SD = .952$

(g) $x=55,\ i=12\%,\ n=5$

$$Var(\ddot{a}^{(12)}_{\overline{x}|}) = \left(\frac{.113329}{12(1-(\frac{1}{1.12})^{1/12})}\right)^2\left[3.817^2 + \frac{2}{.113329}(3.719-1.215) - 7.531^2\right]$$

$= 2.063,\quad SD = 1.436$

(h) $x=55,\ i=12\%,\ n=10$

$$Var(\ddot{a}^{(12)}_{\overline{x}|}) = \left(\frac{.113329}{12(1-(\frac{1}{1.12})^{1/12})}\right)^2\left[5.983^2 + \frac{2}{.113329}(1.758-.339) - 7.741^2\right]$$

$= .922\qquad SD = .960$

APPENDIX II-C

VARIANCE OF A MONTHLY JOINT LIFE ANNUITY

1. BENEFIT DESCRIPTION - $\ddot{a}^{(12)}_{\overline{u|}}$

A monthly annuity of $1/12 per month is paid to a participant, age (x), at the beginning of each month that the participant and his spouse, age (y), are alive. The annuity value is:

$\ddot{a}^{(12)}_{\overline{u|}}$, where u = the shortest of S and T,
S = the future lifetime of (y), and
T = the future lifetime of (x).

2 Variance Formula

$$Var(\ddot{a}^{(12)}_{\overline{u|}}) = Var\left(\frac{1-v^u}{12(1-v^{1/12})}\right) = Var\left(\frac{\delta}{12(1-v^{1/12})} \quad \frac{1-v^u}{\delta}\right)$$

$$= \left(\frac{\delta}{12(1-v^{1/12})}\right)^2 Var(\bar{a}_{\overline{u|}})$$

$$= \left(\frac{\delta}{12(1-v^{1/12})}\right)^2 \left[\frac{2}{\delta}(\bar{a}_{xy} - {}^2\bar{a}_{xy}) - \bar{a}_{xy}^2\right]$$

3. Sample Calculations

(a) $x=65,\ y=65,\ i=5\%$

$$Var(\ddot{a}^{(12)}_{\overline{u|}}) = \left(\frac{.04879}{12(1-(\frac{1}{1.05})^{1/12})}\right)^2 \left[\frac{2}{.04879}(8.007-6.066) - 8.007^2\right]$$

$$= 15.516\ ; \quad SD = 3.939$$

(b) $x=55,\ y=55,\ i=5\%$

$$Var(\ddot{a}^{(12)}_{\overline{u|}}) = \left(\frac{.04879}{12(1-(\frac{1}{1.05})^{1/12})}\right)^2 \left[\frac{2}{.04879}(11.012-7.637) - 11.012^2\right]$$

$$= 17.153\ , \quad SD = 4.142$$

APPENDIX II - C (cont)

(c) $x = 65,\ y = 65,\ i = 12\%$

$$Var\left(\ddot{a}^{(12)}_{\overline{xy}}\right) = \left(\frac{.113329}{12\left(1-\left(\frac{1}{1.12}\right)^{1/12}\right)}\right)^2 \left[\frac{2}{.113329}(5\ 605 - 3.552) - 5\ 605^2\right]$$

$$= 4.861\ ;\ SD = 2\ 205$$

(d) $x = 55,\ y = 55,\ i = 12\%$

$$Var\left(\ddot{a}^{(12)}_{\overline{xy}}\right) = \left(\frac{113329}{12\left(1-\left(\frac{1}{1.12}\right)^{1/12}\right)}\right)^2 \left[\frac{2}{.113329}(6\ 91 - 3\ 997) - 6\ 91^2\right]$$

$$= 3\ 695\ ;\ SD = 1\ 922$$

APPENDIX II-D

VARIANCE OF A MONTHLY JOINT
AND LAST SURVIVOR ANNUITY

1. BENEFIT DESCRIPTION - $\ddot{a}^{(12)}_{\overline{z}|}$

A monthly annuity of $1/12 per month is paid to a participant, age (x), or to the participant's spouse, age (y), at the beginning of each month that either the participant or his spouse are alive. The annuity value is:

$\ddot{a}^{(12)}_{\overline{z}|}$, where z = the longest of S and T,
S = the future lifetime of (y), and
T = the future lifetime of (x)

2 Variance Formula

$$Var(\ddot{a}^{(12)}_{\overline{z}|}) = Var\left(\frac{1-v^z}{12(1-v^{1/12})}\right) = Var\left(\frac{\delta}{12(1-v^{1/12})} \cdot \frac{1-v^z}{\delta}\right)$$

$$= \left(\frac{\delta}{12(1-v^{1/12})}\right)^2 Var(\bar{a}_{\overline{z}|})$$

$$= \left(\frac{\delta}{12(1-v^{1/12})}\right)^2 \left[\frac{2}{\delta}\left(\bar{a}_{\overline{xy}} - {}^2\bar{a}_{\overline{xy}}\right) - \bar{a}^2_{\overline{xy}}\right]$$

3 Sample Calculations

(a) $x=65$, $y=65$, $i=5\%$

$$Var(\ddot{a}^{(12)}_{\overline{z}|}) = \left(\frac{.04879}{12(1-(\frac{1}{1.05})^{1/12})}\right)^2 \left[\frac{2}{.04879}(12.849-8.612) - 12.849^2\right]$$

$$= 8.374;\ SD = 2.894$$

(b) $x=55$, $y=55$, $i=5\%$

$$Var(\ddot{a}^{(12)}_{\overline{z}|}) = \left(\frac{.04879}{12(1-(\frac{1}{1.05})^{1/12})}\right)^2 \left[\frac{2}{.04879}(15.379-9.478) - 15.379^2\right]$$

$$= 5.401;\ SD = 2.324$$

APPENDIX II-D (cont)

(c) $x=65,\ y=65,\ i=12\%$

$$Var\left(\ddot{a}_{\overline{27}}^{(12)}\right) = \left(\frac{.113329}{12\left(1-\left(\frac{1}{1.12}\right)^{1/12}\right)}\right)^2 \left[\frac{2}{.113329}(7.729-4.282)-7.729^2\right]$$

$= 1.105$; $SD = 1.051$

(d) $x=55,\ y=55,\ i=12\%$

$$Var\left(\ddot{a}_{\overline{27}}^{(12)}\right) = \left(\frac{.113329}{12\left(1-\left(\frac{1}{1.12}\right)^{1/12}\right)}\right)^2 \left[\frac{2}{.113329}(8.355-4.368)-8.355^2\right]$$

$= .561$; $SD = .749$

APPENDIX II-E

VARIANCE OF A MONTHLY JOINT AND P% SURVIVOR ANNUITY

1. BENEFIT DESCRIPTION – $P\ddot{a}_{\overline{z}|}^{(12)} + (1-P)\ddot{a}_{\overline{u}|}^{(12)}$

A monthly annuity of \$1/12 per month is paid to a participant, age (x), at the beginning of each month that he and his spouse, age (y), are alive. Additionally, a monthly annuity of \$P/12 per month is paid to the surviving member at the beginning of each month after the first death of the participant or spouse for as long as the surviving member is alive. The annuity value is:

$$P\ddot{a}_{\overline{z}|}^{(12)} + (1-P)\ddot{a}_{\overline{u}|}^{(12)}, \text{ where}$$

z = the longest of S and T,
u = the shortest of S and T,
S = the future lifetime of (y), and
T = the future lifetime of (x)

2 Variance Formula

$$Var\left(P\ddot{a}_{\overline{z}|}^{(12)} + (1-P)\ddot{a}_{\overline{u}|}^{(12)}\right) = Var\left(P \cdot \frac{1-v^{z}}{12(1-v^{1/12})} + (1-P)\frac{1-v^{u}}{12(1-v^{1/12})}\right)$$

$$= Var\left(P\frac{\delta}{12(1-v^{1/12})} \cdot \frac{1-v^{z}}{\delta} + (1-P)\frac{\delta}{12(1-v^{1/12})} \quad \frac{1-v^{u}}{\delta}\right)$$

$$= \left(\frac{\delta}{12(1-v^{1/12})}\right)^2 \left\{P^2 Var(\bar{a}_{\overline{z}|}) + (1-P)^2 Var(\bar{a}_{\overline{u}|}) + 2P(1-P)Cov(\bar{a}_{\overline{z}|}, \bar{a}_{\overline{u}|})\right\}$$

$$= \left(\frac{\delta}{12(1-v^{1/12})}\right)^2 \left\{P^2\left[\frac{2}{\delta}(\bar{a}_{\overline{xy}} - {}^2\bar{a}_{\overline{xy}}) - \bar{a}_{\overline{xy}}^2\right] + (1-P)^2\left[\frac{2}{\delta}(\bar{a}_{xy} - {}^2\bar{a}_{xy}) - \bar{a}_{xy}^2\right]\right.$$
$$\left. + 2P(1-P)\left[(\bar{a}_x - \bar{a}_{xy})(\bar{a}_y - \bar{a}_{xy})\right]\right\}$$

3 Sample Calculations

(a) $x=65,\ y=65,\ i=5\%,\ p=.4$

$$Var\left(P\ddot{a}_{\overline{z}|}^{(12)} + (1-P)\ddot{a}_{\overline{u}|}^{(12)}\right) = \left(\frac{.04879}{12\left(1-\left(\frac{1}{1.05}\right)^{1/12}\right)}\right)^2 \left\{.4^2\left[\frac{2}{.04879}(12\,849 - 8\,618)\right.\right.$$
$$\left. - 12\,849^2\right] + (1-.4)^2\left[\frac{2}{.04879}(8\,007 - 6\,066) - 8\,007^2\right]$$
$$\left. + 2(.4)(1-.4)\left[(9.694 - 8.007)(11\,162 - 8\,007)\right]\right\}$$

$$= 9\,491; \qquad SD = 3.081$$

APPENDIX II-E (cont)

(b) $x=55,\ y=55,\ i=5\%,\ p=.4$

$$Var\left(P\ddot{a}^{(12)}_{\overline{xy}}+(1-P)\,\ddot{a}^{(12)}_{xy}\right)=\left(\frac{.04879}{12\left(1-\left(\frac{1}{1.05}\right)^{1/12}\right)}\right)^2\left\{4^2\left[\frac{2}{.04879}(15.379-9.478)\right.\right.$$

$$\left.-15.379^2\right]+(1-.4)^2\left[\frac{2}{.04879}(11.012-7.637)-11.012^2\right]$$

$$\left.+2(.4)(1-.4)\Big[(12.557-11.012)(13.834-11.012)\Big]\right\}$$

$$=9.141;\quad SD=3.023$$

(c) $x=65,\ y=65,\ i=12\%,\ p=.4$

$$Var\left(P\ddot{a}^{(12)}_{\overline{xy}}+(1-P)\,\ddot{a}^{(12)}_{xy}\right)=\left(\frac{.113329}{12\left(1-\left(\frac{1}{1.12}\right)^{1/12}\right)}\right)^2\left\{.4^2\left[\frac{2}{.113329}(7.729-4.282)\right.\right.$$

$$\left.-7.729^2\right]+(1-.4)^2\left[\frac{2}{.113329}(5.605-3.552)-5.605^2\right]$$

$$\left.+2(.4)(1-.4)\Big[(6.375-5.605)(6.959-5.605)\Big]\right\}$$

$$=2.432;\quad SD=1.559$$

(d) $x=55,\ y=55,\ i=12\%,\ p=.4$

$$Var\left(P\ddot{a}^{(12)}_{\overline{xy}}+(1-P)\,\ddot{a}^{(12)}_{xy}\right)=\left(\frac{.113329}{12\left(1-\left(\frac{1}{1.12}\right)^{1/12}\right)}\right)^2\left\{4^2\left[\frac{2}{.113329}(8.355-4.368)\right.\right.$$

$$\left.-8.355^2\right]+(1-.4)^2\left[\frac{2}{.113329}(6.91-3.997)-6.91^2\right]$$

$$\left.+2(.4)(1-.4)\Big[(7.442-6.91)(7.823-6.91)\Big]\right\}$$

$$=1.655;\quad SD=1.287$$

APPENDIX II-F

VARIANCE OF A MONTHLY JOINT AND P% CONTINGENT ANNUITY

1. BENEFIT DESCRIPTION – $\ddot{a}^{(12)}_{\overline{T}|} + P\,\ddot{a}^{(12)}_{\overline{S}|} - P\,\ddot{a}^{(12)}_{\overline{U}|}$

A monthly annuity of $1/12 per month is paid to a participant, age (x), for as long as the participant is alive. Additionally, upon the participant's death if he is survived by his spouse, age (y), a monthly annuity of $P/12 per month is paid to the surviving spouse at the beginning of each month that the spouse remains alive. The annuity value is:

$$\ddot{a}^{(12)}_{\overline{T}|} + P\,\ddot{a}^{(12)}_{\overline{S}|} - P\,\ddot{a}^{(12)}_{\overline{U}|}, \text{ where}$$

U = the shorter of S and T,
S = the future lifetime of (y), and
T = the future lifetime of (x)

2 Variance Formula

$$\mathrm{Var}\left(\ddot{a}^{(12)}_{\overline{T}|} + P\ddot{a}^{(12)}_{\overline{S}|} - P\ddot{a}^{(12)}_{\overline{U}|}\right) = \mathrm{Var}\left(\frac{1-v^T}{12(1-v^{1/12})} + P\frac{1-v^S}{12(1-v^{1/12})} - P\frac{1-v^U}{12(1-v^{1/12})}\right)$$

$$= \mathrm{Var}\left(\frac{\delta}{12(1-v^{1/12})}\left[\frac{1-v^T}{\delta} + P\,\frac{1-v^S}{\delta} - P\,\frac{1-v^U}{\delta}\right]\right)$$

$$= \left(\frac{\delta}{12(1-v^{1/12})}\right)^2 \mathrm{Var}\left(\bar{a}_{\overline{T}|} + P\bar{a}_{\overline{S}|} - P\bar{a}_{\overline{U}|}\right)$$

$$= \left(\frac{\delta}{12(1-v^{1/12})}\right)^2 \left\{\left[\frac{2}{\delta}(\bar{a}_x - {}^2\bar{a}_x) - \bar{a}_x^2\right] + P^2\left[\frac{2}{\delta}(\bar{a}_y - {}^2\bar{a}_y) - (\bar{a}_y - \bar{a}_{xy})^2 - \bar{a}_x\,\bar{a}_y\right]\right.$$
$$\left. - P\left[\frac{2}{\delta}(\bar{a}_{xy} - {}^2\bar{a}_{xy}) + \bar{a}_x\,\bar{a}_y - 2\bar{a}_x\,\bar{a}_{xy}\right]\right\}$$

3 Sample Calculations

(a) $x = 65,\ y = 65,\ i = 5\%,\ P = .4$

$$\mathrm{Var}\left(\ddot{a}^{(12)}_{\overline{T}|} + P\ddot{a}^{(12)}_{\overline{S}|} - P\ddot{a}^{(12)}_{\overline{U}|}\right) = \left(\frac{.04879}{12\left(1-\left(\frac{1}{1.05}\right)^{1/12}\right)}\right)^2 \left\{\frac{2}{.04879}(9.694 - 6.983) - 9.69\right.$$
$$+ .4^2\left[\frac{2}{.04879}(11.162 - 7.701) - (11.162 - 8.007)^2 - (9.694)(11.162)\right]$$
$$- .4\left[\frac{2}{.04879}(8.007 - 6.066) + (9.694)(11.162) - 2(9.694)(8.0\right.$$

$$= 7970; \quad SD = 2.823$$

(b) $x=55,\ y=55,\ i=5\%,\ \rho=.4$

$$Var\left(\ddot{a}^{(12)}_{\overline{xy}|}+\rho\,\ddot{a}^{(12)}_{x}-\rho\,\ddot{a}^{(12)}_{y}\right)=\left(\frac{.04879}{12\left(1-\left(\frac{1}{1.05}\right)^{1/12}\right)}\right)^{2}\Big\{\Big[\frac{2}{.04879}(12.557-8.310)-12.557^{2}\Big]+.4^{2}\Big[\frac{2}{.04879}(13.834-8.805)-(13.834-11.012)^{2}-(12.557)\cdot(13.834)\Big]-.4\Big[\frac{2}{.04879}(11.012-7.637)+(12.557)(13.834)-2(12.557)\cdot(11.012)\Big]\Big\}=6.152\ ;\ SD=2.480$$

(c) $x=65,\ y=65,\ i=12\%,\ \rho=.4$

$$Var\left(\ddot{a}^{(12)}_{\overline{xy}|}+\rho\,\ddot{a}^{(12)}_{x}-\rho\,\ddot{a}^{(12)}_{y}\right)=\left(\frac{.113329}{12\left(1-\left(\frac{1}{1.12}\right)^{1/12}\right)}\right)^{2}\Big\{\Big[\frac{2}{.113329}(6.375-3.825)-6.375^{2}\Big]+.4^{2}\Big[\frac{2}{.113329}(6.959-4.009)-(6.959-5.605)^{2}-(6.375)(6.959)\Big]-.4\Big[\frac{2}{.113329}(5.605-3.552)+(6.375)(6.959)-2(6.375)(5.605)\Big]\Big\}$$

$$=1.663\ ;\ SD=1.289$$

(d) $x=55,\ y=55,\ i=12\%,\ \rho=.4$

$$Var\left(\ddot{a}^{(12)}_{\overline{xy}|}+\rho\,\ddot{a}^{(12)}_{x}-\rho\,\ddot{a}^{(12)}_{y}\right)=\left(\frac{.113329}{12\left(1-\left(\frac{1}{1.12}\right)^{1/12}\right)}\right)^{2}\Big\{\Big[\frac{2}{.113329}(7.442-4.141)-7.442^{2}\Big]+.4^{2}\Big[\frac{2}{.113329}(7.823-4.234)-(7.823-6.91)^{2}-(7.442)(7.823)\Big]-.4\Big[\frac{2}{.113329}(6.91-3.997)+(7.442)(7.823)-2(7.442)(6.91)\Big]\Big\}$$

$$=.854\ ;\ SD=.924$$

APPENDIX III

APPROXIMATION FORMULAS AND ANNUITY FACTORS

A APPROXIMATION FORMULAS

The following approximation factors were taken from Jordan

1. $\mu_x \cong \dfrac{1 - {}_2p_{x-1}}{2 \cdot p_{x-1}}$

2. $\bar{a}_x \cong \ddot{a}_x - \frac{1}{2} - \frac{1}{12}(\mu_x + \delta)$

3. ${}_{n|}\bar{a}_x \cong {}_{n|}\ddot{a}_x - v^n\,{}_np_x \left(\frac{1}{2} + \frac{1}{12}(\mu_{x+n} + \delta)\right)$

B ANNUITY FACTORS

1 Monthly Factors

		Interest Rate	
		5%	12%
a)	$\ddot{a}^{(12)}_{65m}$	9 742	6.428
b)	$\ddot{a}^{(12)}_{65F}$	11.209	7.012
c)	$\ddot{a}^{(12)}_{65m:65F}$	8 056	5 660
d)	$\ddot{a}^{(12)}_{65m:\overline{5}|}$	10 020	6 649
e)	$\ddot{a}^{(12)}_{65m:\overline{10}|}$	10 786	7 115
f)	$\ddot{a}^{(12)}_{55m}$	12 604	7.494
g)	$\ddot{a}^{(12)}_{55F}$	13.881	7 875
h)	$\ddot{a}^{(12)}_{55m:55F}$	11 059	6 963
i)	$\ddot{a}^{(12)}_{55m:\overline{5}|}$	12 717	7.582
j)	$\ddot{a}^{(12)}_{55m:\overline{10}|}$	13 050	7 723

APPENDIX III (cont)

2 Continuous Factors

	Interest Rate 5%	Interest Rate 12%		Interest Rate 5%	Interest Rate 12%
$\bar{a}_{\overline{5\rceil}}$	4.437	3 817			
$\bar{a}_{\overline{10\rceil}}$	7 913	5983			
$\bar{a}_{65m}$	9.694	6.375	$\bar{a}_{55m}$	12.557	7 442
${}^2\bar{a}_{65m}$	6.983	3.825	${}^2\bar{a}_{55m}$	8 310	4 141
$\bar{a}_{65F}$	11.162	6.959	$\bar{a}_{55F}$	13.834	7 823
${}^2\bar{a}_{65F}$	7.701	4.009	${}^2\bar{a}_{55F}$	8.805	4 234
$\bar{a}_{65m:65F}$	8.007	5 605	$\bar{a}_{55m:55F}$	11.012	6 910
${}^2\bar{a}_{65m:65F}$	6.066	3.552	${}^2\bar{a}_{55m:55F}$	7.637	3 997
$_{5\mid}\bar{a}_{65m}$	5.540	2.788	$_{5\mid}\bar{a}_{55m}$	8 236	3.719
$_{10\mid}\bar{a}_{65m}$	2835	1.092	$_{10\mid}\bar{a}_{55m}$	5.095	1.758
$_{5\mid}{}^2\bar{a}_{65m}$	3.268	995	$_{5\mid}{}^2\bar{a}_{55m}$	4.452	1.215
$_{10\mid}{}^2\bar{a}_{65m}$	1.369	.233	$_{10\mid}{}^2\bar{a}_{55m}$	2253	339