More on a Classic Inequality

by

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Actuarial literature in the past twenty-five years includes discussions of the well known inequality

$$\bar{a}_x < \bar{a}_{\overline{\mathring{e}_x}|} . \qquad (1)$$

Among the recent papers on various aspects of this classic result are those by Berin (1970), Olson (1975) and Sarason (1960).

A standard method for achieving inequality (1) uses Jensen's inequality. This is the method used by Bowers, et al (1983). We will also employ this method in an attempt to learn about

$$D(\delta,x) = \bar{a}_{\overline{\mathring{e}_x}|} - \bar{a}_x . \qquad (2)$$

The Taylor series expansion for $\bar{a}_{\overline{t}|}$ yields

$$\bar{a}_{\overline{t}|} = \bar{a}_{\overline{m}|} + (t-m)e^{-\delta m} - (t-m)^2 \ \delta e^{-\delta\theta(t)}/2 , \qquad (3)$$

where $\theta(t)$ is between t and m. Let t be the random variable T(x), time until death of a life now age x, $m = \mathring{e}_x$, and $\delta > 0$. Taking the expectation of (3), we have

$$\bar{a}_x = \bar{a}_{\overline{\mathring{e}_x}|} - (1/2) \int_0^\infty \delta(t-\mathring{e}_x)^2 e^{-\delta\theta(t)} \, {}_tp_x \mu_{x+t} dt .$$

Replacing $e^{-\delta\theta(t)}$ by its upper-bound of one, we obtain

$$\delta Var(T(x))/2 > \bar{a}_{\overline{\mathring{e}_x}|} - \bar{a}_x = D(\delta,x) \quad (4)$$

Clearly $D(0,x) = 0 = \lim_{\delta\to\infty} D(\delta,x)$ and our bound is not very useful for large values of δ.

Life tables based on recent mortality experience tend to have smaller values of Var(T(x)) than those based on earlier experience. This fact is reflected in the values of $D(x,\delta)$ shown in the following table, the more recent mortality experience produces smaller values of $D(\delta,x)$. In fact, for recent mortality experience, it appears that $D(\delta,x) < 1$.

$$D(\delta,x) = \bar{a}_{\overline{\mathring{e}_x}|} - \bar{a}_x$$

	American Experience	1958 CSO
x	δ = .034401	δ = .034401
5	2.22	.68
20	1.62	.80
40	1.09	.90
60	.64	.74
80	.13	.25

References

1. Berin, B. N. (1971), "Life Contingencies and Compound Interest - The Connecting Link," PCAPP, Vol. 21.

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3. Olson, G. E. (1975), "On Some Actuarial Inequalities," TSA, Vol. 27.

4. Sarason, H. M. (1960), "A Layman's Explanation of the Expectation Annuity," TSA, Vol. 12.