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POWER SERIES OF ANNUITY COEFFICIENTS

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ABSTRACT

The annuity coefficients $\alpha(m)$, $\beta(m)$ and $\gamma(m)$ are introduced in the new *Actuarial Mathematics* textbook for evaluating the value of a life annuity with payments more frequent than once per year. In this paper we derive the power series expansions of these functions.

1. INTRODUCTION

This paper is motivated by problem 5.44.a of the new *Actuarial Mathematics* textbook [1] : Expand in terms of powers of δ the annuity coefficients $\alpha(m)$ and $\beta(m)$.

Under the assumption of uniform distribution of deaths in each year of age, the following formulas are given in [1, chapter 5]:

$$\ddot{a}_x^{(m)} = \ddot{a}_{\overline{1}|}^{(m)}\ddot{a}_x - \beta(m)A_x . \qquad (1.1)$$

$$\ddot{a}_x^{(m)} = \alpha(m)\ddot{a}_x - \beta(m) . \qquad (1.2)$$

$$a_x^{(m)} = \alpha(m)a_x + \gamma(m) . \qquad (1.3)$$

Let

$$c_j = (j!\cdot m^{j+1})^{-1} \sum_{1\leq k\leq m} [k^j - (k^{j+1}/m)] , \qquad j = 0, 1, 2, 3, \dots$$

In this paper, we shall show that

$$\beta(m) = c_0 + c_1\delta + c_2\delta^2 + c_3\delta^3 + \cdots , \tag{1.4}$$

$$\gamma(m) = c_0 - c_1\delta + c_2\delta^2 - c_3\delta^3 \pm \cdots \tag{1.5}$$

and

$$\begin{aligned} \alpha(m) &= m^{-1} + \beta(m) + \gamma(m) \\ &= 1 + 2(c_2\delta^2 + c_4\delta^4 + c_6\delta^6 + \cdots) . \end{aligned} \tag{1.6}$$

2. CEILING AND FLOOR

For a real number t, let $\lfloor t \rfloor$ denote the *floor* of t, i.e., the greatest integer less than or equal to t, and let $\lceil t \rceil$ denote the *ceiling* of t, i.e., the least integer greater than or equal to t. Observe that, for a real-valued function f defined on [a, b], a and b integers, we have

$$\int_a^b f(\lfloor t \rfloor)\, dt = \sum_{a\leq j<b} f(j)$$

and

$$\int_a^b f(\lceil t \rceil)\, dt = \sum_{a<j\leq b} f(j) .$$

Thus, for a positive integer m and a positive number k, which is divisible by m^{-1}, the following integral representations for annuity-certain functions hold:

$$s_{\overline{k}|}^{(m)} = \int_0^k (1+i)^{\lfloor mt \rfloor/m}\, dt \tag{2.1}$$

$$= \int_0^k (1+i)^{k-\lceil mt \rceil/m}\, dt , \tag{2.2}$$

$$\ddot{s}_{\overline{k}|}^{(m)} = \int_0^k (1+i)^{\lceil mt \rceil/m} \, dt \tag{2.3}$$

$$= \int_0^k (1+i)^{k-\lfloor mt \rfloor/m} \, dt , \tag{2.4}$$

$$a_{\overline{k}|}^{(m)} = \int_0^k v^{\lceil mt \rceil/m} \, dt \tag{2.5}$$

$$= \int_0^k v^{k-\lfloor mt \rfloor/m} \, dt , \tag{2.6}$$

$$\ddot{a}_{\overline{k}|}^{(m)} = \int_0^k v^{\lfloor mt \rfloor/m} \, dt \tag{2.7}$$

$$= \int_0^k v^{k-\lceil mt \rceil/m} \, dt . \tag{2.8}$$

3. POWER SERIES OF $\beta(m)$

The difference between $\ddot{a}_x^{(m)}$ and $\ddot{a}_{\overline{1}|}^{(m)}\ddot{a}_x$ is due to the payments that occur in the year of death. Thus

$$\ddot{a}_x^{(m)} - \ddot{a}_{\overline{1}|}^{(m)}\ddot{a}_x = -E(v^{\lceil T \rceil} \cdot \ddot{s}^{(m)}_{\overline{\lceil T \rceil - \lceil mT \rceil/m}|})$$

$$= -\int_0^\infty v^{\lceil t \rceil} \cdot \ddot{s}^{(m)}_{\overline{\lceil t \rceil - \lceil mt \rceil/m}|} \, d_t q_x . \tag{3.1}$$

The assumption of uniform distribution of deaths in each year of age implies that the differential $d_t q_x$ equals ${}_{\lfloor t \rfloor|}q_x \, dt$. Since the product

$$v^{\lceil t \rceil} \cdot {}_{\lfloor t \rfloor|}q_x$$

is a step function with step size 1 and the function

$$\ddot{s}^{(m)}_{\overline{\lceil t \rceil - \lceil mt \rceil/m}|}$$

is periodic with period 1, we can apply the *Average Value Theorem* ([3], cf. [1, section 3.6]) to factorize (3.1) as the product of

(1) $\int_0^\infty v^{\lceil t \rceil} \cdot {}_{\lfloor t \rfloor|}q_x \, dt$, which is A_x , and

(2) $\int_0^1 \ddot{s}^{(m)}_{\overline{\lceil t \rceil - \lceil mt \rceil/m}|} \, dt$.

Hence, by (1.1)

$$\beta(m) = \int_0^1 \ddot{s}^{(m)}_{\overline{\lceil t\rceil - \lceil mt\rceil/m|}}\, dt$$

$$= \int_0^1 \ddot{s}^{(m)}_{\overline{1 - \lceil mt\rceil/m|}}\, dt \tag{3.2}$$

$$= \int_0^1 [(1 + i)^{1-\lceil mt\rceil/m} - 1]/d^{(m)}\, dt$$

$$= (s^{(m)}_{\overline{1|}} - 1)/d^{(m)} \tag{3.3}$$

$$= (i - i^{(m)})/\, i^{(m)}d^{(m)}. \tag{3.4}$$

Expressions (3.3) and (3.4) are [1, 5.5.5] and [1, 5.5.11], respectively. Neither of these expressions is easy to expand as a power series in δ. On the other hand, by (2.3) the integrand in (3.2) can be expressed as

$$\int_0^{1-\lceil mt\rceil/m} (1 + i)^{\lceil ms\rceil/m}\, ds .$$

Interchanging the order of integration yields

$$\beta(m) = \int_0^1 \left(\int_0^{1-\lceil ms\rceil/m} (1 + i)^{\lceil ms\rceil/m}\, dt\right) ds$$

$$= \int_0^1 (1 - \lceil ms\rceil/m)(1 + i)^{\lceil ms\rceil/m}\, ds$$

$$= \int_0^1 (1 - \lceil ms\rceil/m)e^{\delta\lceil ms\rceil/m}\, ds$$

$$= \int_0^1 (1 - \lceil ms\rceil/m) \sum_{j\geq 0}(\delta\lceil ms\rceil/m)^j/j!\, ds$$

$$= \int_0^1 \sum_{j\geq 0} [(\lceil ms\rceil/m)^j - (\lceil ms\rceil/m)^{j+1}]\, \delta^j/j!\, ds$$

$$= \sum_{j\geq 0} \left(\int_0^1 [(\lceil ms\rceil/m)^j - (\lceil ms\rceil/m)^{j+1}]\, ds\right)\delta^j/j!$$

$$= \sum_{j\geq 0} \{m^{-1}\textstyle\sum_{m\geq k\geq 1} [(k/m)^j - (k/m)^{j+1}]\}\delta^j/j!\, .$$

Hence, the coefficient of δ^j is

$$(j!\cdot m^{j+1})^{-1}\sum_{1\leq k\leq m} [k^j - (k^{j+1}/m)],$$

which we denote by c_j. Thus,

$$c_0 = m^{-1}[m - m(m+1)/(2m)] = (m - 1)/(2m)\,,$$

$$c_1 = m^{-2}[m(m+1)/2 - m(m+1)(2m+1)/(6m)] = (m^2 - 1)/(3!m^2)\,,$$

$$c_2 = (2m^3)^{-1}\{m(m+1)(2m+1)/6 - [m(m+1)/2]^2/m\} = (m^2 - 1)/(4!m^2)\,,$$

$$c_3 = (6m^4)^{-1}\{[m(m+1)/2]^2 - m(m+1)(2m+1)[3m(m+1)-1]/(30m)\}$$

$$= (m^2 - 1)(3m^2 - 2)/(3 \cdot 5! \cdot m^4)\,,$$

$$c_4 = (m^2 - 1)(2m^2 - 3)/(2 \cdot 6! \cdot m^4)\,,$$

$$c_5 = (m^2 - 1)(2m^4 - 5m^2 + 2)/(2 \cdot 7! \cdot m^6)\,,$$

$$c_6 = (m^2 - 1)(3m^4 - 11m^2 + 10)/(3 \cdot 8! \cdot m^6)\,,$$

etc.

4. POWER SERIES OF $\gamma(m)$ AND $\alpha(m)$

The power series in δ of $\gamma(m)$ may be derived by a method parallel to the one in the last section. However, we can easily show that the coefficient of δ^j in the power series expansion of $\gamma(m)$ is simply $(-1)^j c_j$.

The formula for $\gamma(m)$ that corresponds to (3.4) is

$$\gamma(m) = (d^{(m)} - d)/i^{(m)}d^{(m)}\,. \tag{4.1}$$

Recall that

$$i^{(m)} = m(e^{\delta/m} - 1) \tag{4.2}$$

and

$$d^{(m)} = m(1 - e^{-\delta/m})\,. \tag{4.3}$$

Considering $\beta(m)$ as a function of δ, write

$$f(\delta) = \beta(m)\,.$$

It easily follows from (3.4), (4.1), (4.2) and (4.3) that

$$f(-\delta) = \gamma(m)\,.$$

Thus, expansion (1.5) holds.

As pointed out in problem 5.19 of [1], subtracting equation (1.3) from equation (1.2) yields

$$m^{-1} = \alpha(m) - \beta(m) - \gamma(m) .$$

Thus, expansion (1.6) holds.

5. DERIVATION OF (1.2), (1.3) & (4.1)

To make this paper complete, we now prove (1.2), (1.3) and (4.1). Recall that

$$\ddot{a}_x^{(m)} = E(\ddot{a}^{(m)}_{\overline{\lceil mT \rceil /m|}}) \tag{5.1}$$

and

$$a_x^{(m)} = E(a^{(m)}_{\overline{\lfloor mT \rfloor /m|}}) \ . \tag{5.2}$$

Now,

$$\begin{aligned}
\ddot{a}^{(m)}_{\overline{\lceil mT \rceil /m|}} &= (1 - v^{m^{-1}\lceil mT \rceil})/d^{(m)} \\
&= (1 - v^{\lceil T \rceil}(1+i)^{\lceil T \rceil - m^{-1}\lceil mT \rceil})/d^{(m)} \\
&= [d(1+i)^{\lceil T \rceil - m^{-1}\lceil mT \rceil}\ddot{a}_{\overline{\lceil T \rceil|}} + 1 - (1+i)^{\lceil T \rceil - m^{-1}\lceil mT \rceil}]/d^{(m)}.
\end{aligned}$$

Taking expected values and applying the Uniform Distribution of Deaths Assumption and the Average Value Theorem yield

$$\begin{aligned}
\ddot{a}_x^{(m)} &= (d/d^{(m)})E((1+i)^{\lceil T \rceil - m^{-1}\lceil mT \rceil})\ddot{a}_x + E([1 - (1+i)^{\lceil T \rceil - m^{-1}\lceil mT \rceil}]/d^{(m)}) \\
&= (d/d^{(m)})E((1+i)^{\lceil T \rceil - m^{-1}\lceil mT \rceil})\ddot{a}_x - E([(1+i)^{\lceil T \rceil - m^{-1}\lceil mT \rceil} - 1]/d^{(m)}) \\
&= (d/d^{(m)})(i/i^{(m)})\ddot{a}_x - [(i/i^{(m)} - 1)]/d^{(m)} \\
&= \alpha(m)\ddot{a}_x - \beta(m) .
\end{aligned}$$

Similarly,

$$a^{(m)}_{\overline{\lceil mT\rceil/m|}} = (1 - v^{m^{-1}\lfloor mT\rfloor})/i^{(m)}$$

$$= (1 - v^{\lfloor T\rfloor}v^{m^{-1}\lfloor mT\rfloor - \lfloor T\rfloor})/i^{(m)}$$

$$= [iv^{m^{-1}\lfloor mT\rfloor - \lfloor T\rfloor}a_{\overline{\lfloor T\rfloor|}} + 1 - v^{m^{-1}\lfloor mT\rfloor - \lfloor T\rfloor}]/i^{(m)}.$$

Hence, under the Uniform Distribution of Deaths Assumption,

$$a_x^{(m)} = (i/i^{(m)})E(v^{m^{-1}\lfloor mT\rfloor - \lfloor T\rfloor})a_x + E([1 - v^{m^{-1}\lfloor mT\rfloor - \lfloor T\rfloor}]/i^{(m)})$$

$$= (i/i^{(m)})(d/d^{(m)})a_x + [1 - (d/d^{(m)})]/i^{(m)}$$

$$= \alpha(m)a_x + \gamma(m).$$

6. REMARKS

(i) Observe that

$$\beta(m) = \int_z^{z+1} \ddot{s}^{(m)}_{\overline{\lceil t\rceil - \lceil mt\rceil/m|}}\, dt, \text{ z an arbitrary real number.}$$

$$= \int_0^1 \ddot{s}^{(m)}_{\overline{\lceil t\rceil - \lceil mt\rceil/m|}}\, dt$$

$$= \int_0^1 \ddot{s}^{(m)}_{\overline{1 - \lceil mt\rceil/m|}}\, dt$$

$$= \int_0^{1-m^{-1}} \ddot{s}^{(m)}_{\overline{1 - \lceil mt\rceil/m|}}\, dt$$

$$= (j^{(m)}\ddot{s})^{(m)}_{\overline{1 - 1/m|}}.$$

Similarly,

$$\gamma(m) = \int_z^{z+1} a^{(m)}_{\overline{m^{-1}\lfloor mt\rfloor - \lfloor t\rfloor|}}\, dt, \text{ z arbitrary.}$$

$$= \int_0^1 a^{(m)}_{\overline{m^{-1}\lfloor mt\rfloor - \lfloor t\rfloor|}}\, dt$$

$$= \int_0^1 a^{(m)}_{\overline{m^{-1}\lfloor mt \rfloor}|}\, dt$$

$$= \int_{1/m}^1 a^{(m)}_{\overline{m^{-1}\lfloor mt \rfloor}|}\, dt$$

$$= (D^{(m)}a)^{(m)}_{\overline{1-1/m}|}.$$

Hence,

$$\alpha(m) = m^{-1} + \beta(m) + \gamma(m)$$

$$= (I^{(m)}s)^{(m)}_{\overline{1}|} + (D^{(m)}a)^{(m)}_{\overline{1-1/m}|}$$

$$= (I^{(m)}\ddot{s})^{(m)}_{\overline{1-1/m}|} + (D^{(m)}\ddot{a})^{(m)}_{\overline{1}|}.$$

This result should be compared with [2, page 86, example 4.8].

(ii) For the remainder of this paper, we assume $j \geq 1$. By the Euler-Maclaurin formula,

$$\sum_{0 \leq k \leq m-1} k^j = m^{j+1}/(j+1) - m^j/2 + jm^{j-1}/12 - j(j-1)(j-2)m^{j-3}/720$$
$$+ j(j-1)(j-2)(j-3)(j-4)m^{j-5}/30240 \pm \cdots$$

This series, in descending powers of m, ends with the term in the first power of m if $j = 1, 2, 4, 6, \ldots$, and with the term in the second power of m if $j = 3, 5, 7, 9, \ldots$. In deriving the next formula it is important to note that, for j odd, $j > 1$, the ending term of the series above is not a constant term but an m^2 term. Now,

$$\sum_{1 \leq k \leq m} [k^j - (k^{j+1}/m)]$$

$$= \sum_{0 \leq k \leq m} [k^j - (k^{j+1}/m)]$$

$$= \sum_{0 \leq k \leq m-1} [k^j - (k^{j+1}/m)]$$

$$= m^{j+1}/(j+1)(j+2) - m^{j-1}/12 + j(j-1)m^{j-3}/240$$
$$- j(j-1)(j-2)(j-3)m^{j-5}/6048 \pm \cdots$$

Consequently,

$$c_j = (j!\cdot m^{j+1})^{-1} \sum_{1\le k\le m} [k^j - (k^{j+1}/m)]$$

$$= \{1 - [(j+1)(j+2)/12]m^{-2} \pm \cdots \pm \zeta m^{-2[j/2]}\}/(j+2)! \ .$$

Because of the observation just made, there are two expressions for the last coefficient ζ in this series, depending on whether j is odd or even.

Since for m = 1

$$\sum_{1\le k\le m} [k^j - (k^{j+1}/m)] = 0$$

and c_j is a polynomial in m^{-2}, the term

$$1 - m^{-2}$$

is a factor of c_j. Hence,

$$c_j = (1 - m^{-2})\{1 - [(j^2+3j-10)/12]m^{-2} \pm \cdots \mp \zeta m^{2-2[j/2]}\}/(j+2)! \ .$$

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