A Method For The Calculation of Aggregate Stop-Loss Premiums

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Abstract

Aggregate stop-loss premium rates for group health insurance have usually been based on collective risk theory methods. Unfortunately, these methods yield premium rates that are inadequate, because they ignore systematic and correlated risk. This extra risk, sometimes called parameter uncertainty, results from outside factors or errors. Here another method for calculating the rates, which takes into account the cost for this extra risk, is presented. Some suggestions are made for implementing this calculation on electronic data processing equipment.

I. Introduction

Aggregate stop-loss is an insurance coverage which is usually written on the health claims of an otherwise self insured employer. The coverage provides that the insurer will reimburse the employer for 100% of all health claims incurred exceeding a fixed attachment point set before the insurance period. The period is usually 1 year. The attachment point may be set as a fixed number or, more commonly, as an amount per employee covered. Very often claims over a fixed limit on one person are not included in the aggregate stop loss coverage. This limit is either established to reduce the insurer's risk or because these claims are separately covered under a specific stop-loss coverage. All of the following ideas can also be applied to stop-loss reinsurance, since it is essentially the same coverage.

II. Background

Let Z be the random variable representing the total claims of the group during the insurance period. Its distribution function is F(z). If the attachment point is equal to sU (for a scaling unit or span U and integer s) then the aggregate stop-loss claims $=(Z-sU)^+$,

the net premium $= \Pi_s = E(Z-sU)^+ = \int_{sU}^{\infty}(z-sU)dF(z)$, and

the variance $= V_s = \int_{sU}^{\infty}(z-sU-\Pi_s)^2 dF(z)$. Mereu [4] pointed out that these equations are easier to evaluate if expressed as :

$$\Pi_s = \int_{sU}^{\infty}(z-sU)dF(z) = \int_{sU}^{\infty} zdF(z) - \int_{sU}^{\infty} sUdF(z)$$

$$= \int_0^{\infty} zdF(z) - \int_0^{sU} zdF(z) - \left[\int_0^{\infty} sUdF(z) - \int_0^{sU} sUdF(z)\right]$$

Define: $V(Z) = E[Z-E(Z)]^2 = E(Z^2)-[E(Z)]^2$ and $F_\rho(s) = \int_0^{sU} t^\rho dF(t)$.

$$\Pi_s = E(Z) - \int_0^{sU} z\,dF(z) - sU\left[1-F(sU)\right] = E(Z)-F_1(s)-sU\left[1-F_0(s)\right] \quad (2.1)$$

$$V_s = \int_{sU}^{\infty}(z-sU-\Pi_s)^2 dF(z) = \int_{sU}^{\infty}\left[z^2-2z(sU+\Pi_s)+(s^2U^2+\Pi_s^2+2sU\Pi_s)\right] dF(z)$$

$$= \int_0^{\infty} z^2 dF(z) - F_2(s) - 2(sU+\Pi_s)\left[E(Z)-F_1(s)\right] +(s^2U^2+\Pi_s^2+2sU\Pi_s)\left[1-F_0(s)\right]$$

$$= V(Z) + [E(Z)]^2 - F_2(s) - 2(sU+\Pi_s)\left[E(Z)-F_1(s)\right] + (s^2U^2+\Pi_s^2+2sU\Pi_s)\left[1-F_0(s)\right]$$

Thus all that is needed to calculate Π_s and V_s are $E(Z)$, $V(Z)$, $F_0(s)$, $F_1(s)$, and $F_2(s)$.

The risk theory section of [1] includes a method of calculating Π_s and V_s for one particularly appropriate family of random variables. Let $Z = X = \sum_{j=1}^{N} X_j$ with N a random variable that is equal to the number of claims incurred in the group and has the Poisson distribution: $Pr(N=j) = \frac{e^{-\theta}\theta^j}{j!}$. X_j $(j=1,2,3,\dots)$ is the random variable equal to the amount of the j'th claim. The X_j are independent and identically distributed discrete random variables with $Pr(X_j=iU) = h_i$, $1 \le i \le m$, and $\sum_{i=1}^{m} h_i = 1$. Again U is the scaling unit. The distribution of $X(=Z)$ is called a compound Poisson distribution.

Define: $\mu = E(X_j) = U\sum_{i=1}^{m} ih_i$ and $\sigma^2 = E(X_j-\mu)^2 = U^2\sum_{i=1}^{m} i^2h_i - \mu^2$.

Panjer [6] has shown $P_i = Pr(X=iU) = \frac{\theta}{i}\sum_{j=1}^{m_i} jh_jP_{i-j}$ and $P_0 = e^{-\theta}$ where $m_i=\min(i,m)$. The P_i can be calculated relatively quickly. Thus Π_s and V_s can be calculated according to (2.1) with: $E(Z) = \sum_{i=0}^{\infty} iUP_i = \theta\mu$, $V(Z) = \theta(\sigma^2+\mu^2)$, and $F_p(s) = \sum_{i=0}^{s}(iU)^{\rho}P_i$.

III. The Extra Risk

There are other sources of variation in the claims on a group besides fluctuation in the size of the claims or the number of claims. These include an unexpected rate of change in the charges for medical care, changes in the course of medical treatment of illnesses, inability to properly assess the expected claims for a group, contagious or new diseases, and errors in rating the group. All of the above sources of fluctuation may not affect the individuals in the group independently.

A way of modeling this extra risk, is to let Z (the total amount of claims for the group in the year)=YX. X is the random variable that appears above, in section 2. Y is another random variable that is not compound Poisson, does not depend on number or size of claims, and is independent of X. For convenience let E(Y)=1. Thus, $E(Z)=E(X)=\theta\mu$.

This method of modeling extra risk has been called parameter uncertainty. Meyer [5] discusses this and in fact gives a convenient method of calculating an approximation to the stop-loss premiums.

IV. The Calculation Problem

As mentioned above, aggregate stop-loss coverage on medical insurance usually is written with a limit on the total amount of claims from each individual in the group that will be included in the coverage. In the model of section III, if the limit were to be applied to the X_j, it would be simple to calculate the effect of the limit. To see this, let LU be the limit on each individual's claims. Then define the random variables:

$$X_{jL}=\min\{X_j,LU\}. \text{ Then } \Pr\{X_{jL}=iU\} = h_{i,L} = \begin{cases} h_i & i<L \\ \sum_{k=i}^{m} h_k & i=L \\ 0 & i>L \end{cases}$$

If $X=\sum_{j=1}^{N} X_{jL}$ and Z=YX then E(Z), V(Z), and $F_r(s)$ (r=0,1,2) are easy to calculate as in section 2.

Actually, the limit should apply to the size of each claim even after it is adjusted by Y. That is $Z=\sum_{j=1}^{N}\min\{YX_j,LU\}$. Neither the method above nor that in [5] can be directly applied to this type of limit.

V. The Calculation Method

Let $Z_{jL} = \min\{YX_j,LU\}$, $Z = \sum_{j=1}^{N} Z_{jL}$ with the X_j and N defined as above, and I(x) = the smallest integer ≥ x. Then the conditional probability:

$$\Pr\{Z_{jL}=iUy|Y=y\} = h_{i,y} \approx \begin{cases} h_i & i < L_y \\ \sum_{k\geq L_y}^{m} h_k & i = I(L_y) \\ 0 & i > I(L_y) \end{cases} \quad (5.1)$$

where $L_y = L/y$.

This expression is approximate because the middle expression in the bracket should refer to $Z_{jL}=LU$. If U is small the approximation will be better. Effectively, Z_{jL} is allowed to get slightly greater than LU. When $i = I(L_y)$, $Z_{jL}=iUy \geq L_y Uy = LU$.

The recursive formula of [6] would still apply to the conditional probabilities in (5.1):

$$P_i(L,y) = P_i = \Pr\{Z=iUy|Y=y\} = \Pr\{X=iU|Y=y\} = \frac{\theta}{i}\sum_{r=1}^{m_{i,y}} r\, h_{r,y}\, P_{i-r} \text{ and } P_0=e^{-\theta}.$$

Where $m_{i,y} = \min\{m,i,\ I(L_y)\}$ and $X = \sum_{j=1}^{N} X_j$. Let $Y\geq 0$, have density function $g(y)$:

$$F_\rho(s) = \sum_{i=0}^{s} (iU)^\rho \Pr\{Z=iU\} \approx \int_0^\infty g(y) \sum_{i=0}^{I(s/y)} \Pr\{Z=iUy|Y=y\}(iUy)^\rho\, dy$$

This expression is approximate because $I(s/y) \geq s/y$

Now we do a lot a rearranging to make this expression easier to calculate:

$$F_\rho(s) \approx \sum_{k=1}^{\alpha} \int_{L/k}^{L/(k-1)} g(y) \sum_{i=0}^{I(s/y)} (iUy)^\rho P_i(L,y)\, dy . \tag{5.2}$$

Let the half open interval $\left[\frac{L}{k} , \frac{L}{k-1} \right]$ be called I_k .

Now define $h_{i,k} = h_{i,y}$ as in (5.1) for $Y\in I_k$. This is meaningful since $Y\in I_k$ implies that $k \geq L_y > k-1$.

$$\text{Then define: } P_{i,k}(L) = P_{i,k} = \Pr\left\{X=iU|\ Y\in I_k\right\}. = \frac{\theta}{i}\sum_{r=1}^{m_{i,k}} r\, h_{r,k} P_{i-r,k} \tag{5.3}$$

$m_{i,k} = \min \left\{ m,i,I(L/k) \right\}$. Since $Y\in I_k$ also implies that

$\frac{sk}{L} \geq \frac{s}{Y} > \frac{s(k-1)}{L}$, we have from (5.2):

$$F_\rho(s) \approx \sum_{k=1}^{\infty} \int_{L/k}^{L/(k-1)} g(y) \sum_{i=0}^{I(sk/L)} (iUy)^\rho P_{i,k}\, dy.$$ This is also approximate

since $I(sk/L) \geq I(s/y) \geq s/y$.

$$F_\rho(s) \approx U^\rho \sum_{k=1}^{\infty} \sum_{i=0}^{I(sk/L)} i^\rho P_{i,k} \int_{L/k}^{L/(k-1)} g(y) y^\rho dy$$

$$= U^\rho \sum_{k=1}^{m-1} \sum_{i=0}^{I(sk/L)} i^\rho P_{i,k} \int_{L/k}^{L/(k-1)} g(y) y^\rho dy + U^\rho \sum_{k=m}^{\infty} \sum_{i=0}^{I(sk/L)} i^\rho P_{i,k} \int_{L/k}^{L/(k-1)} g(y) y^\rho dy$$

$$= U^\rho \sum_{k=1}^{m-1} \int_{L/k}^{L/(k-1)} g(y) y^\rho dy \sum_{i=0}^{I(sk/L)} i^\rho P_{i,k} + U^\rho \sum_{i=1}^{\infty} i^\rho P_i \sum_{k_i}^{\infty} \int_{L/k}^{L/(k-1)} g(y) y^\rho dy$$

$$= U^\rho \sum_{k=1}^{m-1} \int_{L/k}^{L/(k-1)} g(y) y^\rho dy \sum_{i=0}^{I(sk/L)} i^\rho P_{i,k} + U^\rho \sum_{i=1}^{\infty} i^\rho P_i \int_{0}^{L/(k_i-1)} g(y) y^\rho dy \qquad (5.4)$$

where $k_i = \max\left\{ I\left[\frac{iL}{s} \right], m \right\}$. If $g(y)$ does not take too long to integrate, the sums in (5.4) can be evaluated in a reasonable time on electronic data processing equipment.

VI The Distribution of Y

The premiums calculated by this method are very sensitive to the Var(Y) but not very sensitive to the shape of the density g. I have used both the normal and the gamma densities. The results did not differ substantially. The normal density is convenient because some software has built-in subroutines to evaluate it. Unfortunately, the normal density extends into the negative numbers. The gamma density has positive values and has a shape similar to the normal for higher values of the gamma's parameters.

The gamma density is: $g(y) = \frac{\beta^\alpha y^{\alpha-1} e^{-\beta y}}{\Gamma(\alpha)}$; $y \geq 0$. If Y is Gamma then $E(Y)=\alpha/\beta$ and $V(Y)=\alpha/\beta^2$. If E(Y) is set to 1, as before, then $\alpha=\beta$ and $Var(Y)=1/\beta$. Now note that if $\alpha=\beta$ is an integer, repeated integration by parts gives the integral of g as a finite sum. This can be programmed with a relatively low execution time. The restriction to integers is not too big a problem. If the $Var(Y)=1/\beta$ is in the neighborhood of 2%, the difference in Var(Y) between $\beta=50$ and $\beta=51$ is only $1/50-1/51 \approx 0.0004$.

VII. Parameter Estimation

The parameter θ can be estimated by determining a claim rate per covered individual from a claim study. For a particular group, set θ equal to this claim rate multiplied by the number of members in the group. Of course, different claim rates by the age, sex, occupation, etc. of the members is a desirable refinement. Many insurers already have adjustments for these factors available.

The h_i's can be set equal to the empirical distribution of the size of claims (severity) of the insurer's entire group medical line of business. These need to be adjusted, for medical cost inflation trend, to the appropriate level for the time period of the proposed stop-loss insurance. Further refinements, such as different severity tables for adults, children, or various areas may be desirable. These should be combined into a single table for the group which is to be rated.

The empirical distribution will have a span of 1¢. In order to make the calculation possible in a reasonable length of time it is necessary to use a span of at least $1,000. Two methods are described by Gerber and Jones [3] to calculate the h_i's for a larger span. They show that these methods put a lower or upper bound on the stop-loss premium.

Var(Y) (or $1/\beta$ if the gamma density is being used) can be estimated as follows: Select a sample of groups that have been in force for a reasonable length of time such as 5 years, with very little change in coverage. In each year calculate their loss ratios by dividing their incurred claims by tabular premiums which adjust for changes in the membership of the groups, changes in benefits, trend in medical costs, etc. Calculate the normalized variance of these loss ratios over the period for each group. That is, if the loss ratio for the group in year i is z_i, i=1,2,..n, then the normalized variance $\hat{R} = \dfrac{\frac{1}{n}\sum_{i=1}^{n}(z_i - \bar{z})^2}{\bar{z}^2}$, where $\bar{z} = \frac{1}{n}\sum_{i=1}^{n} z_i$. Then set $\hat{V}(Y) = \dfrac{\hat{R} - R(X)}{1+R(X)}$, where: $R(X)=V(X)/[E(X)]^2 = \dfrac{\sigma^2 + \mu^2}{\theta\mu^2}$. $|\hat{V}(Y)|$ can then be averaged over all the sample groups. I have not evaluated the properties of this estimator. In one study the value was about 2%.

Another possibility would be to use the value of $\hat{V}(Y)$ from the group being rated, or a value in between its own and the sample average. Credibility for 2nd moments of claims has been discussed in, for example, [2].

VIII. Programming Notes and Results

I wrote a program in FORTRAN to evaluate (5.4). The method is to calculate each $P_{i,k}$ using (5.3), accumulate to $\sum_{i<(sk/L)} i^\rho P_{i,k}$, multiply by $\int_A^{L/(k-1)} g(y)y^\rho dy$, and sum all of these products from k=1 to m. This is the first sum of (5.4). The second sum of (5.4) is similarly calculated, this time determining $i^{\rho}P_i$ for $i>(sm/L)$, multiplying each by $\int_0^{B_i} g(y)y^{\rho}dy$, and summing the products until they are negligible. When calculating $P_{i,k}$ it is not necessary to save all of the prior values of $P_{i,k}$. Only $P_{i-m,k}$ through $P_{i-1,k}$ are needed to calculate $P_{i,k}$.

The values of $P_{i,k}$, for small i and reasonably large θ are smaller than the smallest number that can be handled by the floating point arithmetic of most machines. One solution to this problem is to save the logarithms of the numbers. Unfortunately, the repeated use of even the built-in log and antilog functions makes the program's run time impracticably long. A better solution is to store the prior values as a ratio to the current. That is, save $P_{i-m',k}/P_{i-1,k}$ through $P_{i-2,k}/P_{i-1,k}$ for calculating $P_{i,k}$.

The program calculated premiums that appeared to be both reasonable as compared to the size of the risk and competitive. They varied appropriately with the limit L. As a test of the approximation in section 6 above, values were run assuming that $Var(Y)=0$. The approximation was found to introduce only extremely small errors.

For a group of 100 members the execution time on a large mainframe computer was about 10 minutes. Unfortunately, the time tends to increase quickly as the size of group increases. Thus, it probably would be impractical to use this method for stop-loss reinsurance. If the span U were increased, the run times would be reduced.

Execution time on a microcomputer, such as an IBM-AT or its clone, would be about 50 times as long. Some of the new mathematics coprocessors for PC's might reduce this time.

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