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PERCENTILE OF A DEFERRED INSURANCE

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ABSTRACT

We present a method to calculate the percentile of the distribution of the present value of the death benefit for a continuous deferred whole life insurance.

Example 4.3.c of the new *Actuarial Mathematics* textbook [1] discusses how one calculates the median of the distribution of the present value of the benefit payment for a deferred whole life insurance payable at the moment of death of (x). We find that many of our students have difficulties following the solution given in [1] because this is the first time they have encountered a random variable which is partly continuous and partly discrete. This note presents a treatment utilizing the concept of an indicator function.

Let $I(T > m)$ denote the random variable which takes the value 1 if $T > m$ is true and the value 0 if $T > m$ is false. Thus the random variable of interest is

$$Z = I(T > m) \cdot v^T .$$

Clearly, $E(Z) = {}_{m|}\bar{A}_x$. The problem we wish to solve is: Given a number $f \in (0, 1)$, find the nonnegative number ξ such that $\Pr(Z \leq \xi) = f$.

First, note that

$$\begin{aligned} \Pr(Z = 0) &= \Pr[I(T > m) = 0] \\ &= \Pr(T \leq m) \\ &= {}_mq_x . \end{aligned}$$

Thus, if ${}_mq_x \geq f$, then $\xi = 0$. Hence, we shall consider only the case where ${}_mq_x < f$.

For simplicity, write $1(T > m)$ as I. Then

$$\Pr(Z \le \xi) = \Pr[(Z \le \xi) \cap (I = 0)] + \Pr[(Z \le \xi) \cap (I = 1)] .$$

Let us now evaluate the two terms on the right side of the equation above.

$$\begin{aligned} \Pr[(Z \le \xi) \cap (I = 0)] &= \Pr(Z \le \xi \mid I = 0) \cdot \Pr(I = 0) \\ &= \Pr(0 \cdot v^T \le \xi \mid I = 0) \cdot \Pr(I = 0) \\ &= \Pr(0 \le \xi \mid I = 0) \cdot \Pr(T \le m) \\ &= 1 \cdot \Pr(T \le m) \\ &= {}_m q_x . \end{aligned}$$

$$\begin{aligned} \Pr[(Z \le \xi) \cap (I = 1)] &= \Pr[(1 \cdot v^T \le \xi) \cap (I = 1)] \\ &= \Pr[(v^T \le \xi) \cap (I = 1)] \\ &= \Pr[(v^T \le \xi) \cap (T > m)] \\ &= \Pr[(T \ge h) \cap (T > m)] , \end{aligned}$$

where

$$h = \log_v \xi = (-1/\delta)(\ln \xi) .$$

Put

$$k = \max(h, m),$$

then

$$\begin{aligned} \Pr[(Z \le \xi) \cap (I = 1)] &= \Pr(T \ge k) \\ &= {}_k p_x . \end{aligned}$$

Hence

$$f = {}_m q_x + {}_k p_x .$$

Since

$$1 = {}_m q_x + {}_m p_x$$

and

$$1 > f ,$$

we have

$$k = h .$$

Thus

$$f = {}_m q_x + {}_h p_x , \qquad (1)$$

from which we calculate ξ.

There is a slightly easier way to derive (1). Instead of solving

$$\Pr(Z \le \xi) = f$$

directly, consider

$$\Pr(Z > \xi) = 1 - f .$$

As before,

$$\Pr(Z > \xi) = \Pr[(Z > \xi) \cap (I = 0)] + \Pr[(Z > \xi) \cap (I = 1)] .$$

Now,

$$\begin{aligned}\Pr[(Z > \xi) \cap (I = 0)] &= \Pr(0 \cdot v^T > \xi \mid I = 0) \cdot \Pr(I = 0) \\ &= 0,\end{aligned}$$

since ξ is a nonnegative number.

$$\begin{aligned}\Pr[(Z > \xi) \cap (I = 1)] &= \Pr[(v^T > \xi) \cap (I = 1)] \\ &= \Pr[(v^T > \xi) \cap (T > m)] \\ &= \Pr[(T < h) \cap (T > m)] \\ &= \Pr(m < T < h) \\ &= {}_mp_x - {}_hp_x .\end{aligned}$$

Hence,

$$1 - f = {}_mp_x - {}_hp_x ,$$

which is equation (1).

REMARKS

(i) A consequence of the development above is that, for $y \in [0, v^m]$,

$$\Pr(Z \leq y) = {}_mq_x + {}_{(-1/\delta)(\ln y)}p_x .$$

For an illustration of the probability density function of Z, see [1, Figure 4.4].

(ii) A good exercise for an actuarial student is to repeat the derivation above with the random variable

$$Z = I(m < T \leq m+n) \cdot v^{\lceil T \rceil} ,$$

where m and n are positive integers and $\lceil T \rceil$ is the least integer greater than or equal to T (see [2]). In this case,

$$E(Z) = {}_{m|n}A_x .$$

REFERENCES

1. N.L. Bowers, Jr., H.U. Gerber, J.C. Hickman, D.A. Jones and C.J. Nesbitt, *Actuarial Mathematics* (40-11-84). Itasca, Illinois: Society of Actuaries, 1984.
2. E.S.W. Shiu, "Integer Functions and Life Contigencies," *TSA*, XXXIV (1982), 571-590; Discussion 591-600.