On The Moments Of Compound Interest Functions When Interest Varies As An AR(2) Process

by

Colin M. Ramsay
Department of Statistical and Actuarial Sciences
University of Western Ontario
London, Ontario, Canada N6A 5B9

ABSTRACT

It is assumed that the force of interest varies as an auto-regressive process of order 2, AR(2). The moments of the accumulated value of 1 paid at the beginning of each period, for n periods, are developed using moment generating functions.

Keywords: auto-regressive process, moment generating function, stochastic interest.

1. Let us assume that the force of interest over the time period $(t-1,t]$ is δ_t, $t = 1,2,\ldots$. Like Westcott (1981) we model δ_t as an auto-regressive process of order 2 (AR(2)). For fixed and known constants δ_0 and δ_{-1} we have

$$(\delta_t-\bar{\delta}) = a(\delta_{t-1}-\bar{\delta}) - b(\delta_{t-2}-\bar{\delta}) + e_t, \quad t = 1,2,\ldots \tag{1.1}$$

where $\bar{\delta}$ is a constant about which the force of interest is expected to fluctuate. The sequence $\{e_t\}$ is a sequence of mutually independently and identically distributed (i.i.d.) normal random variables with zero mean and finite variance σ^2. This model of δ_t ensures that the force of interest generates stochastic interest rates. Finally we assume that a and b are real-valued constants, and, unlike Westcott (1981), a and b satisfy $a^2 > 4b$. This ensures that the roots of the characteristic equation of (1.1) are real. Panjer and Bellhouse (1980, Table 1) demonstrated that in most situations the roots of this characteristic equation are real.

Following Westcott (1981, Section 2), (1.1) can be written as

$$u_t = au_{t-1} - bu_{t-2} + e_t \tag{1.2}$$

where $u_t = \delta_t - \bar{\delta}$. Let λ_1 and λ_2 be the real roots of the equation

$$\lambda^2 - a\lambda + b = 0$$

i.e.

$$\lambda_1,\lambda_2 = \frac{a\pm\sqrt{a^2-4b}}{2} .$$

This leads to the following expression for u_t satisfying (1.2)

$$u_t = a_1\lambda_1^t + a_2\lambda_2^t + \sum_{j=0}^{t-1} (b_1\lambda_1^j+b_2\lambda_2^j)e_{t-j}, \quad t = 1,2,\ldots \tag{1.3}$$

where, for given initial conditions u_o and u_{-1},

$$b_1 = \frac{\lambda_1}{\lambda_1-\lambda_2}, \quad b_2 = \frac{-\lambda_2}{\lambda_1-\lambda_2}$$

$$a_1 = \frac{\lambda_1 u_0 - bu_{-1}}{\lambda_1-\lambda_2}, \quad a_2 = \frac{bu_{-1}-\lambda_2 u_0}{\lambda_1-\lambda_2} . \tag{1.4}$$

Let us now introduce the random variable $B_n(t)$ which is the accumulated value of 1 paid at the beginning of the t^{th} period and accumulated to the end of the n^{th} period, $t \leq n = 1,2,3,\ldots$. Thus

$$B_n(t) = \exp(\sum_{j=t}^{n} \delta_j) = \exp\{(n+1-t)\bar{\delta} + \sum_{j=t}^{n} u_j\}. \tag{1.5}$$

Similarly, the accumulated value of an annuity of 1 paid at the beginning of each period until the end of the n^{th} period is S_n where

$$S_n = \sum_{t=1}^{n} B_n(t). \tag{1.6}$$

In the sequel we develop expressions for the moments of $B_n(t)$ and S_n by exploiting the fact that $\log[B_n(t)]$ is normally distributed (in fact it is a linear combination of $e_1,e_2,\ldots,e_n$), and S_n represents the sum of log-normal random variables.

2. Define $A_n(t)$ as

$$A_n(t) = \sum_{j=t}^{n} u_j \tag{2.1}$$

and its mgf is $M_n(t,\theta)$ for real θ,

$$M_n(t,\theta) = E[\exp(\theta A_n(t))]. \tag{2.2}$$

From (1.3), we see that $A_n(t)$ is a linear combination of n i.i.d. normal random variables. So the existence of $M_n(t,\theta)$ is assured for $-\infty < \theta < \infty$. $A_n(t)$ can be written as

$$A_n(t) = \xi(n,t) + \sum_{k=1}^{n} \gamma_k(n,t)e_k, \quad t = 1,2,\dots,n \tag{2.3}$$

where

$$\gamma_k(n,t) = \sum_{j=0}^{n-t} \beta_{j+t-k} \tag{2.4}$$

$$\beta_r = \begin{cases} b_1\lambda_1^r + b_2\lambda_2^r & r = 0,1,2,\dots \\ 0 & \text{otherwise} \end{cases} \tag{2.5}$$

and

$$\xi(n,t) = \sum_{j=t}^{n} (a_1\lambda_1^j + a_2\lambda_2^j). \tag{2.6}$$

Thus (2.3) immediately gives

$$M_n(t,\theta) = \exp\{\theta\xi(n,t) + \frac{\theta^2\sigma^2}{2} \sum_{k=1}^{n} (\gamma_k(n,t))^2\} \tag{2.7}$$

and hence from (1.5) and (2.7)

$$E[(B_n(t))^k] = \exp\{k(n+1-t)\bar{\delta} + k\xi(n,t) + \frac{k^2\sigma^2}{2} \sum_{r=1}^{n} (\gamma_r(n,t))^2\} \quad (2.8)$$

for $k = 0,1,2,\ldots$.

Let us now write the expressions for both $\gamma_k(n,t)$ and $\xi(n,t)$ in terms of the 'known' constants b_1, b_2, λ_1 and λ_2. Using (2.4) and (2.5) we find

$$\gamma_k(n,t) = \begin{cases} b_1\lambda_1^{t-k}\left(\dfrac{1-\lambda_1^{n-t+1}}{1-\lambda_1}\right) + b_2\lambda_2^{t-k}\left(\dfrac{1-\lambda_2^{n-t+1}}{1-\lambda_2}\right), & k = 1,2,\ldots,t; \\ b_1\left(\dfrac{1-\lambda_1^{n-k+1}}{1-\lambda_1}\right) + b_2\left(\dfrac{1-\lambda_2^{n-k+1}}{1-\lambda_2}\right), & k = t,t+1,\ldots,n. \end{cases} \quad (2.9)$$

Similarly for $t = 1,2,\ldots,n$, we find

$$\xi(n,t) = a_1\lambda_1^{t}\left(\frac{1-\lambda_1^{n-t+1}}{1-\lambda_1}\right) + a_2\lambda_2^{t}\left(\frac{1-\lambda_2^{n-t+1}}{1-\lambda_2}\right) . \quad (2.10)$$

We are now in a position to develop an expression for the moments of an annuity of 1 per period payable at the start of each period. Call this annuity S_n, i.e.

$$S_n = \sum_{t=1}^{n} B_n(t).$$

To find the k^{th} moment of S_n, $k = 0,1,2,\ldots$ we proceed as follows:

$$(S_n)^k = \sum_{C(k)} \binom{k}{\mathbf{r}} (B_n(1))^{r_1} (B_n(2))^{r_2} \cdots (B_n(n))^{r_n} \tag{2.11}$$

where the summation is over the elements $\mathbf{r}$ of the set $C(k)$ and

$C(k) = \{\mathbf{r} = (r_1, r_2, \ldots, r_n) \in R^n:\ r_i = 0,1,2,\ldots,k$ and $\sum_{i=1}^{n} r_i = k\}$, and

$$\binom{k}{\mathbf{r}} = \binom{k}{r_1\ r_2\ \cdots\ r_n} \tag{2.12}$$

is the multinomial coefficient necessary in the expansion of $(S_n)^k$. From (1.5), (2.1) and (2.3) we see that

$$(S_n)^k = \sum_{C(k)} \binom{k}{\mathbf{r}} \exp\{\sum_{i=1}^{n} r_i[(n+1-i)\bar{\delta} + \xi(n,i) + \sum_{j=1}^{n} \gamma_j(n,i)e_j]\} \tag{2.13}$$

A slight rearrangement of (2.13) yields

$$(S_n)^k = \sum_{C(k)} \binom{k}{\mathbf{r}} \exp\left\{\sum_{j=1}^{n} [r_j(n+1-j)\bar{\delta} + r_j\xi(n,j) + \phi(n,j:\mathbf{r})e_j]\right\} \tag{2.14}$$

where

$$\phi(n,j:\mathbf{r}) = \sum_{i=1}^{n} r_i \gamma_j(n,i). \tag{2.15}$$

Thus (2.14) demonstrates that $(S_n)^k$ is the sum of log-normal variates. It also enables us to immediately write down an expression for $E[(S_n)^k]$, $k = 0,1,2,\ldots$ as

$$E[(S_n)^k] = \sum_{C(k)} \binom{k}{\mathbf{r}} \exp\{\sum_{j=1}^{n} [r_j(n+1-j)\bar{\delta} + r_j\xi(n,j) + \tfrac{1}{2}\sigma^2(\phi(n,j:\mathbf{r}))^2]. \tag{2.16}$$

$$E[(S_n)^k] = \sum_{C(k)} \binom{k}{\mathbf{r}} \exp\{\mathbf{r}(\boldsymbol{\delta}+\boldsymbol{\xi})^T + \frac{1}{2}\sigma^2(\mathbf{r}\boldsymbol{\Gamma})(\mathbf{r}\boldsymbol{\Gamma})^T\} \tag{2.17}$$

where $\boldsymbol{\delta} = (n\bar{\delta},(n-1)\bar{\delta},\ldots,2\bar{\delta},\bar{\delta})$, 1xn vector

$\boldsymbol{\xi} = (\xi(n,1),\xi(n,2),\ldots,\xi(n,n-1),\xi(n,n))$, 1xn vector

and $\boldsymbol{\Gamma} = \{\gamma_{ij}(n)\}$ with $\gamma_{ij}(n) = \gamma_j(n,i)$ for $i,j = 1,2,\ldots,n$, nxn matrix

Note that for fixed n the vectors $\boldsymbol{\delta}$ and $\boldsymbol{\xi}$, and the matrix $\boldsymbol{\Gamma}$ are 'known'. The number of elements $\mathbf{r}$ in $C(k)$ is $\binom{n+k-1}{k}$ (see for example Feller 1968, Chapter II.5). These vectors $\mathbf{r}$ can easily be found with the aid of a computer while the matrix and vector multiplications can quickly be accomplished as well. Thus the moments of S_n can be found exactly using either equations (2.16) or (2.17) and a computer. It should be pointed out that the summation over $C(k)$ can be written in the form of successive summations in series as $\Sigma\Sigma\Sigma\ldots$.

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