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Evaluation of Ruin Probabilities

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Abstract

Two series formulas for the probability of eventual ruin are derived by operational calculus method.

This paper is dedicated to Henk Boom, who introduced me to the beautiful subject of Risk Theory.

1. Introduction and notation

Consider the classical collective risk model, in which insurance claims occur according to a Poisson process $N(t)$, $t \geq 0$, and the individual claim amounts $X_1, X_2, X_3, \ldots$ are mutually independent positive random variables, each with probability distribution $\Pr(X \leq x) = P(x)$ and with mean $\mathbf{E}(X) = p_1 < \infty$. Assume that the number of claims process $N(t)$ is independent of the claim amount random variables $\{X_i\}$. Let $\mathbf{E}[N(t)] = \lambda t$. For a given relative security loading θ, $\theta \geq 0$, let c denote the premium rate,

$$c = (1+\theta)p_1\lambda.$$

Put $S_0 = 0$ and, for $k \geq 1$,

$$S_k = X_1 + X_2 + \cdots + X_k. \tag{1.1}$$

The ruin function $\psi(u)$ is defined as the probability that the risk reserve

$$u + ct - S_{N(t)} \tag{1.2}$$

is ever negative. The argument of the ruin function is the amount of risk reserve at time 0.

Let a denote the Lundberg security factor,

$$a = \lambda c^{-1} = [(1+\theta)p_1]^{-1}. \tag{1.3}$$

For $\alpha \geq 0$, define

$$x_+^\alpha = \begin{cases} x^\alpha & x \geq 0 \\ 0 & x < 0 \end{cases}. \tag{1.4}$$

We shall prove that, for $u \geq 0$,

$$\psi(u) = 1 - \frac{\theta}{1+\theta}\sum_{j=0}^{\infty}\frac{(-a)^j}{j!}\mathbf{E}[(u-S_j)_+^j e^{a(u-S_j)}] \tag{1.5}$$

and

$$\psi(u) = \frac{\theta}{1+\theta}\sum_{j=1}^{\infty}\frac{a^j}{j!}\,\mathbf{E}[(S_j - u)_+^j\, e^{-a(S_j - u)}]\,. \qquad (1.6)$$

This paper is motivated and stimulated by Gerber's (1988) fun. Gerber presented his paper in last year's Actuarial Research Conference in Toronto.

2. A convolution series for the probability of eventual ruin

Consider a small time interval (0, s). By the Poisson assumption, the probability that a claim will occur in the interval is $\lambda s + o(s)$. Hence, for $u \geq 0$, the ruin function $\psi(u)$ satisfies the relation [Ross (1983, section 6.6.3)]:

$$\psi(u) = \lambda s\mathbf{E}[\psi(u + cs - X)] + (1 - \lambda s)\psi(u + cs) + o(s). \qquad (2.1)$$

Dividing (2.1) by s, rearranging and letting s tend to 0, we obtain the integro-differential equation

$$c\psi'(u) = \lambda\psi(u) - \lambda\mathbf{E}[\psi(u - X)],$$

or

$$\psi'(u) = a\{\psi(u) - \mathbf{E}[\psi(u - X)]\}. \qquad (2.2)$$

Since $\psi(\text{negative number}) = 1$,

$$\mathbf{E}[\psi(u - X)] = \int \psi(u - x)\, dP(x)$$

$$= \int_0^u \psi(u - x)\, dP(x) + [1 - P(u)]\,. \qquad (2.3)$$

For $u \geq 0$, integrating (2.2) from 0 to u yields

$$\psi(u) - \psi(0) = a\{\psi(u)*[1 - P(u)] - \int_0^u [1 - P(y)]\, dy\}, \qquad (2.4)$$

where

$$f_1(x) * f_2(x) = \int_0^x f_1(x - y)f_2(y)dy = \int_0^x f_1(y)f_2(x - y)dy. \qquad (2.5)$$

Letting u tend to $+\infty$ in equation (2.4), we have

$$0 - \psi(0) = a(0 - p_1),$$

or

$$\psi(0) = ap_1 = 1/(1+\theta). \tag{2.6}$$

Hence, with the definitions

$$k(u) = \begin{cases} a[1-P(u)] & \text{if } u \ge 0 \\ 0 & \text{if } u < 0 \end{cases} = a[u_+^0 - P(u)] \tag{2.7}$$

and

$$K(u) = \int_{-\infty}^{u} k(y)\,dy = \int_{0}^{u} k(y)\,dy, \tag{2.8}$$

equation (2.4) becomes

$$\psi(u) - \psi(u) * k(u) = \frac{1}{1+\theta} - K(u), \quad u \ge 0. \tag{2.9}$$

Let the set of all continuous functions defined on the nonnegative real line be denoted by $C[0, \infty)$. With the usual addition and scalar multiplication, $C[0, \infty)$ is a linear space. With the convolution operation (2.5) as multiplication, $C[0, \infty)$ becomes a commutative ring. By the Titchmarsh convolution theorem, the ring has no zero divisors and we can construct the quotient field Q of the ring [Mikusinski (1959), Erdélyi (1962), Yosida (1984)]. The unit element for multiplication in Q is δ, the Dirac delta function. One may rewrite (2.9) as

$$\psi(u) * [\delta(u) - k(u)] = \frac{u_+^0}{1+\theta} - K(u). \tag{2.10}$$

Note that the constant $\frac{1}{1+\theta}$ in (2.9) is replaced by the function $\frac{u_+^0}{1+\theta}$ in (2.10).

To solve for $\psi(u)$ in (2.10), we invert $\delta(u) - k(u)$ as a power series, i.e.,

$$\begin{aligned} \psi(u) &= [\delta(u) - k(u)]^{-1} * [\frac{u_+^0}{1+\theta} - K(u)] \\ &= [\sum_{j=0}^{\infty} k^{*j}(u)] * [\frac{u_+^0}{1+\theta} - K(u)], \end{aligned} \tag{2.11}$$

where

$$k^{*0}(u) = \delta(u)$$

and, for n = 1, 2, 3, ... ,

$$k^{*n}(u) = k^{*(n-1)}(u) * k(u) = \int_0^u k^{*(n-1)}(u-y)\, k(y)\, dy. \tag{2.12}$$

With the definition

$$K^{*n}(u) = \int_0^u k^{*n}(y)\,dy = k^{*n}(u) * u_+^0, \tag{2.13}$$

equation (2.11) becomes

$$\begin{aligned}
\psi(u) &= \frac{1}{1+\theta}\sum_{j=0}^{\infty} K^{*j}(u) - \sum_{j=0}^{\infty} K^{*(j+1)}(u) \\
&= K^{*0}(u) + \left(\frac{1}{1+\theta} - 1\right)\sum_{j=0}^{\infty} K^{*j}(u) \\
&= u_+^0 - \frac{\theta}{1+\theta}\sum_{j=0}^{\infty} K^{*j}(u) .
\end{aligned} \tag{2.14}$$

Note that formula (2.14) is derived under the restriction that u is nonnegative.

Formula (2.11) is formula (26) on page 246 of Dubourdieu (1952). I thank François Dufresne for this reference. J.A. Beekman has consistently promoted the use of (2.14) for evaluating $\psi(u)$. The formula has been rediscovered several times and in different contexts, see Shiu (1988, p. 42).

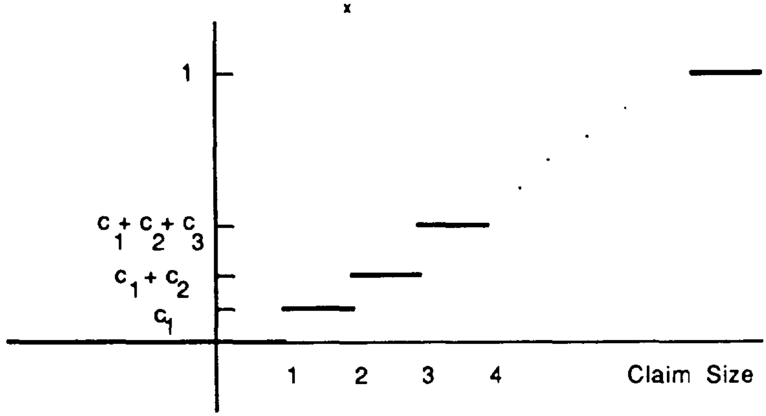
3. First formula

A major difficulty with (2.14) is the calculation of the convolutions $\{K^{*j}\}$. For the moment, let us assume that the random variables $\{X_i\}$ are discrete. Let

$$\Pr(X = x) = c_x , \tag{3.1}$$

$c_x \geq 0$ and $\sum c_x = 1$. Then,

$$P(y) = \sum_x c_x\,(y-x)_+^0. \tag{3.2}$$



Using E to denote the translation operator (forward shift operator), we can write (3.2) as

$$P(y) = \left(\sum_x c_x E^{-x}\right) y_+^0 . \tag{3.3}$$

Let $g(z)$ denote the probability generating function of X,

$$\begin{aligned} g(z) &= E(z^X) \\ &= \sum_x c_x z^x . \end{aligned} \tag{3.4}$$

Thus, formula (3.3) becomes

$$P(y) = g(E^{-1})\, y_+^0 . \tag{3.5}$$

Substituting (3.5) into (2.7) yields

$$k(y) = a[I - g(E^{-1})]\, y_+^0 . \tag{3.6}$$

Formula (3.5) may be considered as an inversion formula for (3.4). In deriving it, we use the assumption that X is a discrete random variable. However, by considering $P(x)$ as a limit of step functions, we see that the discreteness assumption is not necessary [cf. Hirschman and Widder (1955, p. 8), Mikusinski (1959, p. 327) and Erdélyi (1962, p. 57)].

It is easy to check that, for each real number r,

$$E^r (f_1 * f_2) = (E^r f_1) * f_2 = f_1 * (E^r f_2) . \tag{3.7}$$

Hence, for $n = 1, 2, 3, \ldots$, the n-fold convolution of k is

$$k^{*n}(y) = a^n [I - g(E^{-1})]^n (y_+^0 * y_+^0 * \ldots) .$$

Since

$$\frac{y_+^{\alpha-1}}{\Gamma(\alpha)} * \frac{y_+^{\beta-1}}{\Gamma(\beta)} = \frac{y_+^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} , \tag{3.8}$$

we have

$$k^{*n}(y) = a^n [I - g(E^{-1})]^n \frac{y_+^{n-1}}{(n-1)!} . \tag{3.9}$$

It follows from (2.13) and (3.8) that

$$K^{*n}(y) = a^n [I - g(E^{-1})]^n \frac{y_+^n}{n!} . \tag{3.10}$$

Substituting (3.10) into (2.14) yields

$$\psi(u) = 1 - \frac{\theta}{1+\theta} \sum_{j=0}^{\infty} \frac{a^j}{j!} [I - g(E^{-1})]^j u_+^j , \qquad u \geq 0. \tag{3.11}$$

To simplify writing, put $G = g(E^{-1})$. Formula (3.11) can be written symbolically as

$$\psi(u) = 1 - \frac{\theta}{1+\theta} e^{a(I-G)u_+}, \qquad u \geq 0. \tag{3.12}$$

Motivated by the formal identity

$$\exp[a(I-G)u_+] = \exp(-aGu_+)\exp(au_+),$$

we conjecture the formula

$$\psi(u) = 1 - \frac{\theta}{1+\theta} \sum_{j=0}^{\infty} \frac{(-a)^j}{j!} G^j (u_+^j e^{au_+}). \tag{3.13}$$

To prove (3.13), note that

$$(I-G)^n = \sum_{j=0}^{n} (-1)^j \binom{n}{j} G^j = n! \sum_{j=0}^{n} \frac{(-1)^j}{j!\,(n-j)!} G^j. \tag{3.14}$$

Substituting (3.14) into (3.11) and interchanging the order of summation yields (3.13).

Since $g(z) = \mathbf{E}(z^X)$ and the random variables $\{X_i\}$ are identically and independently distributed,

$$[g(z)]^j = \mathbf{E}(z^{X_1+X_2+\dots+X_j}) = \mathbf{E}(z^{S_j}). \tag{3.15}$$

Hence,

$$G^j f(y) = [g(E^{-1})]^j f(y) = \mathbf{E}[f(y - S_j)] \tag{3.16}$$

and, for $u \geq 0$,

$$\psi(u) = 1 - \frac{\theta}{1+\theta} \sum_{j=0}^{\infty} \frac{(-a)^j}{j!} \mathbf{E}[(u - S_j)_+^j e^{a(u-S_j)_+}], \tag{3.17}$$

which is (1.5).

Using (3.17), Willmot (1988) has obtained formulas for $\psi(u)$ when X is gamma (with arbitrary non-scale parameter) or continuous uniform. As in Shiu (1988), let us now consider the case that the individual claim amount random variable X takes on positive integer values only:

$$\Pr(X = n) = c_n, \qquad n = 1, 2, 3, \dots. \tag{3.1'}$$

Let

$$c_k^{*j} = \Pr\left(\sum_{i=1}^{j} X_i = k\right) = \Pr(S_j = k).$$

Since $c_0 = 0$,

$$(g(z))^j = \sum_{k=0}^{\infty} c_k^{*j} z^k$$

$$= \sum_{k=j}^{\infty} c_k^{*j} z^k. \tag{3.18}$$

The coefficients $\{c_k^{*j}\}$ can be evaluated recursively by the formula

$$c_k^{*(m+n)} = \sum_{i+j=k} c_i^{*m} c_j^{*n}. \tag{3.19}$$

An *APL* program for evaluating (3.19) can be found in Grenander (1982, p. 402).

It follows from (3.17) and (3.18) that

$$\psi(u) = 1 - \frac{\theta}{1+\theta} \sum_{j=0}^{\infty} \frac{(-a)^j}{j!} \sum_{k=j}^{\infty} c_k^{*j} (u-k)_+^j \, e^{a(u-k)}$$

$$= 1 - \frac{\theta}{1+\theta} \sum_{k=0}^{\infty} e^{a(u-k)} \sum_{j=0}^{k} \frac{(-a)^j c_k^{*j} (u-k)_+^j}{j!}$$

$$= 1 - \frac{\theta}{1+\theta} \sum_{k=0}^{\lfloor u \rfloor} e^{a(u-k)} \sum_{j=0}^{k} \frac{c_k^{*j} [a(k-u)]^j}{j!}, \tag{3.20}$$

where $\lfloor u \rfloor$ denotes the greatest integer less than or equal to u.

Note that

$$c_k^{*0} = \begin{cases} 1 & k = 0 \\ 0 & \text{otherwise} \end{cases}.$$

For $u \geq 1$, formula (3.20) can be written as

$$\psi(u) = 1 - \frac{\theta}{1+\theta} \{e^{au} + \sum_{k=1}^{\lfloor u \rfloor} e^{a(u-k)} \sum_{j=1}^{k} \frac{c_k^{*j} [a(k-u)]^j}{j!}\}$$

$$= 1 - \frac{\theta e^{au}}{1+\theta} \{1 + \sum_{k=1}^{\lfloor u \rfloor} e^{-ak} \sum_{j=1}^{k} \frac{c_k^{*j} [a(k-u)]^j}{j!}\}. \tag{3.21}$$

The appeal of formula (3.21) is that it is a finite sum, not an infinite series. Since the number of terms depends on the size of u, the formula is inefficient for large u. However, for large values of u, Lundberg's asymptotic formula [Cramér (1955, p. 68), Seal (1969, formula (4.64)), Feller (1971, p. 378), Beekman (1974, p. 52), Gerber (1979, formula (5.27)), Asmussen (1987, p. 284)]

$$C e^{-Ru} \tag{3.22}$$

is an effective method for evaluating $\psi(u)$. In (3.22), R denotes the adjustment coefficient, which is the positive number satisfying the equation

$$\int_0^\infty e^{Rx} k(x)\, dx = 1 . \tag{3.23}$$

Another way to express (3.23) is the equation

$$E(e^{RX}) = 1 + (1+\theta)p_1 R.$$

The coefficient C in (3.22) is [Seal (1969, p. 130), Gerber (1979, (5.28))]

$$\frac{\int_0^\infty e^{Rx} \int_x^\infty [1 - P(y)]\, dy\, dx}{\int_0^\infty x\, e^{Rx} [1 - P(x)]\, dx} = \frac{\theta\, p_1}{R \int_0^\infty x\, e^{Rx} [1 - P(x)]\, dx} . \tag{3.24}$$

By an integration by parts, the denominator in the right-hand side of (3.24) can be written as

$$E(Xe^{RX}) - (1+\theta)p_1 .$$

4. Second formula

Formula (3.17) is an alternating series. From a numerical point of view, it might be better to have a formula whose terms have only one sign. Such a formula has been given by Prabhu (1965, formula (5.55)) and Gerber (1988, formula (27)). In this section, we derive this formula by means of operational calculus.

Following Gel'fand and Shilov (1964, p. 49), we define, for each nonnegative number α,

$$x_-^\alpha = \begin{cases} |x|^\alpha & \text{if } x < 0 \\ 0 & \text{if } x \ge 0 \end{cases} . \tag{4.1}$$

Note that, for each nonnegative integer n and for each real number x,

$$x_+^n + (-1)^n x_-^n = x^n .$$

In particular,

$$x_+^0 + x_-^0 = 1 .$$

If we let 1(x) denote the constant function that takes on the value 1 for all x, then it follows from (3.5) that

$$\begin{aligned} P(x) &= g(E^{-1})[1(x) - x_-^0] \\ &= 1(x) - g(E^{-1})\, x_-^0 . \end{aligned} \tag{4.2}$$

Hence,

$$k(x) = a[x_+^0 - P(x)] = a[g(E^{-1}) - I]\,x_-^0 = a(G - I)\,x_-^0. \tag{4.3}$$

Similar to (3.8), we have

$$\frac{x_-^{\alpha-1}}{\Gamma(\alpha)} * \frac{x_-^{\beta-1}}{\Gamma(\beta)} = \frac{x_-^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}. \tag{4.4}$$

Here, we should remark that the usual definition for the convolution $f_1 * f_2$ is

$$f_1(x) * f_2(x) = \int_{-\infty}^{\infty} f_1(x-y)\, f_2(y)\, dy. \tag{4.5}$$

If the functions f_1 and f_2 are assumed to take the value 0 on the negative axis, then (4.5) is reduced to (2.5). It follows from (3.7) and (4.4) that

$$k^{*n}(x) = a^n [G - I]^n \frac{x_-^{n-1}}{(n-1)!}, \tag{4.6}$$

which is similar to (3.9).

By (2.13),

$$K^{*n}(x) = x_+^0 * k^{*n}(x) = [1(x) - x_-^0] * k^{*n}(x) = \int_{-\infty}^{\infty} k^{*n}(y)\, dy - a^n[G - I]^n \frac{x_-^n}{n!}. \tag{4.7}$$

Since $(1+\theta)^n k^{*n}(y)$ is the n-fold convolution of the probability density function

$$\frac{y_+^0 - P(y)}{p_1},$$

the value of the integral in (4.7) is $(1+\theta)^{-n}$. Substituting (4.7) into (2.14) and simplifying yields, for $u \ge 0$,

$$\psi(u) = \frac{\theta}{1+\theta} \sum_{j=0}^{\infty} \frac{[a(G - I)]^j u_-^j}{j!} \tag{4.8}$$

$$= \frac{\theta}{1+\theta} \sum_{j=1}^{\infty} \frac{[a(G - I)]^j u_-^j}{j!},$$

from which (1.6) follows.

Comparing (3.11) with (4.8), one sees that

$$1 = \frac{\theta}{1+\theta} \sum_{j=0}^{\infty} \frac{[a(I - G)]^j}{j!} [u_+^j + (-1)^j u_-^j] = \frac{\theta}{1+\theta} \sum_{j=0}^{\infty} \frac{[a(I - G)]^j}{j!} u^j. \tag{4.9}$$

Symbolically, (4.9) may be written as $1 + \theta^{-1} = e^{a(I-G)u} = e^{-aGu} e^{au}$. One may conjecture that

$$1+\frac{1}{\theta} = \sum_{j=0}^{\infty} \frac{(-aG)^j}{j!}[u^j e^{au}] = \sum_{j=0}^{\infty} \frac{(-a)^j}{j!} \mathbf{E}[(u-S_j)^j e^{a(u-S_j)}], \tag{4.10}$$

which turns out to be Theorem 1.(b) of Gerber (1988).

The formula that corresponds to (3.20) is:

$$\psi(u) = \frac{\theta}{1+\theta} \sum_{k=\lfloor u \rfloor+1}^{\infty} e^{-a(k-u)} \sum_{j=1}^{k} \frac{c_k^{*j}[a(k-u)]^j}{j!}. \tag{4.11}$$

Although each term in (4.11) is nonnegative, my student Aftab Ali has demonstrated that (4.11) is not a practical formula for computer implementation.

5. Inversion formula

In the discussion above, we rely on the relation that, if

$$g(z) = \mathbf{E}(z^X) = \int z^x \, dP(x), \tag{3.4}$$

then

$$P(y) = g(E^{-1})\, y_+^0. \tag{3.5}$$

A symbolic proof is as follows. By (3.4), the right-hand side of (3.5) is

$$\left[\int_{-\infty}^{\infty} E^{-x} \, dP(x)\right] y_+^0 = \int_{-\infty}^{\infty} (y-x)_+^0 \, dP(x) = (y-x)_+^0 P(x)\Big|_{x=-\infty}^{x=\infty} - \int_{-\infty}^{\infty} P(x)\, d(y-x)_+^0.$$

Since the derivative of the Heaviside unit function is the Dirac delta function

$$\frac{d}{dx} x_+^0 = \delta(x),$$

we have

$$g(E^{-1}) y_+^0 = \int P(x)\,\delta(y-x)\,dx = P(y),$$

as required.

For a discrete random variable X, that (3.4) implies (3.5) is straightforward. It is interesting to see how the implication works in a continuous case. Let X be an exponential random variable with mean 1. Then $g(z) = (1-\log_e z)^{-1}$. Hence,

$$g(E^{-1}) = [I - \log_e(E^{-1})]^{-1} = [I + \log_e(E)]^{-1} = (I+D)^{-1},$$

where D denotes the differentiation operator. Now,

$$(I+D)^{-1} = D^{-1}(I+D^{-1})^{-1} = D^{-1}(I - D^{-1} + D^{-2} - D^{-3} + \cdots).$$

Interpreting D^{-1} as integration yields

$$g(E^{-1})\, y_+^0 = D^{-1}\left(y_+^0 - \frac{y_+}{1!} + \frac{y_+^2}{2!} - \ldots\right) = D^{-1}(y_+^0 e^{-y}) = y_+^0 - y_+^0 e^{-y} = P(y).$$

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