GENERALIZED BONDY DEVELOPMENT

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Abstract

Can a single parameter development pattern describe most cases of casualty loss development with reasonable accuracy? The answer is "YES!"

In this paper, the author introduces a generalization of Bondy development and demonstrates its descriptive powers. Generalized Bondy development is defined in terms of a new concept: "the force of development." Generalized Bondy development facilitates estimation of development factors, is easily computed, and affords excellent fits to observed development ratios.

The paper also investigates the characteristics of insurance operations that explain the algorithm. Investigation of this question sheds light on the actuarial ties between the diagnosis of insurance and risk financing situations and the selection of corresponding actuarial models of development.

Introduction

Generalized Bondy development is based on an algorithm developed in the 1960s. The generalized algorithm is easy to compute and affords excellent descriptions of actual development patterns for many types of insurance. This paper will:

1. Afford a logical framework for analysis of development problems.

2. Define the generalized Bondy approach.

3. Apply Bondy development to fractional periods.

4. Illustrate the shape of generalized Bondy development patterns.

5. Introduce an algorithm for estimating Bondy parameters.

6. Present examples of generalized Bondy development.

7.   Analyze the sensitivity of the algorithm.

8.   Review theoretical considerations pertaining to the use of
     generalized Bondy development.

9.   Summarize these results.

1.   The Development Problem

Often the values of observations change as we learn more about the
subject that we are studying. Actuaries call such changes "development." In
insurance and risk financing, actuaries are concerned with various measures of
development for premiums and losses. There are many reasons for this concern.
Perhaps the most common reason is the need to accurately assess the
profitability and financial performance of insurance operations.

Development problems can be described in terms of three component
phenomena:

A.   Point processes

B.   Mixing

C.   Aggregation

A claim or loss occurs when an insurer or other risk financing
institution incurs a legal obligation. When a loss[1] occurs, it occurs at a
well-defined moment in time. As time passes and the loss matures, its
estimated ultimate value and the paid portion will change or develop.
Associated with the loss is a development pattern describing the evolution of
these values. This development pattern illustrates the concept of point
process.

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[1]Depending on coverage (e.g., claims-made v. occurrence) and court
interpretation, losses involving cumulative trauma, gradual pollution, and other
cumulative damages over time may not strictly fit this characterization.

Insurance policies apply to a variety of insureds, types of loss, etc. The development pattern for a book of business is a weighted average of the development patterns for the individual losses. If the mix of business changes, development patterns exhibit corresponding changes. For example, a changing mix of stable development patterns will produce a shifting overall development pattern. And, if different development patterns apply to the constituent elements of the book of business, the resulting development pattern may have mathematical properties distinct from those of the elements' patterns.

In addition to mixing the constituent elements of a book of business, observed development data is aggregated over time. Aggregation can modify development patterns for point processes in determining overall development patterns. For example, if data is grouped on a yearly basis, then an observation at twelve months maturity includes both a loss at zero maturity incurred at the last moment of the period and a loss at twelve months maturity incurred at the first moment of the period. And, to the extent exposures are more heavily concentrated at one of these extremes, observed aggregate development will more closely agree with expected mixed development for the corresponding moment.

These phenomena become more intricate as one moves from primary to reinsurance operations. However, in some cases, aggregate reporting by individual ceding carriers may be characterized as point processes at the reinsurance level.

2.   Generalized Bondy Development

The name "Generalized Bondy Development" is taken from work by Martin Bondy during the 1960s at predecessor organizations of the Insurance Services

Office (ISO). The issue at that time was estimation of the tail factor (i.e.,
the development factor to adjust from an observation at an arbitrary maturity
to the ultimate value).

The following notation is used to define the generalized Bondy approach:

t   —   the duration from the moment a loss is incurred to the
                    observation of the associated variable subject to
                    development.

$r(t)$ —   the expected value of the observation at time t.

h   —   an arbitrary increment in time that defines the periods in
                    which development is measured.

$d(t)$ — $r(t+h) / r(t)$

          —   the development factor to adjust an observed amount at time
                    t to its corresponding value at time $t+h$.

B   —   the Bondy parameter.

Generalized Bondy development is characterized by the following
equation:

$$d(t+h) = d(t)^B \tag{1}$$

Because development ratios approach unity at more mature durations,
values of B are between 0 and 1. Also, equation [1] defines a recursive
relationship among development factors, namely:

$$d(t+2h) = d(t+h)^B = (d(t)^B)^B = d(t)^{B \times B} \tag{2}$$

Because the tail factor is the product of development factors for individual
periods, if $d(t)$ is the last observed development factor, the tail factor for
subsequent periods can be computed by raising $d(t)$ to the power $B/(1-B)$, where
$B/(1-B)$ is the sum of the geometric series:

$$B/(1-B) = B + B^2 + B^3 + B^4 + \ldots \tag{3}$$

In particular, if $B = .5$, the tail factor is identical to the observed
development factor for the most mature observed period.

The original ISO studies determined that, in some situations, setting the tail factor equal to the most mature development factor (i.e., setting the Bondy parameter equal to 0.5) produced reasonable estimates of ultimate values. This paper presents methods to determine "optimal" values of B that afford a "best fit" to observed development ratios. In this sense it generalizes the original Bondy approach to values other than 0.5.

The "force of development" is a useful concept for analyzing development patterns and greatly facilitates the definition of generalized Bondy development. Its definition parallels the definitions of the force of interest and the force of mortality. The force of development at time t represents the instantaneous change in the variable subject to development at time t and is expressed as a rate spanning the same period as is used to measure development. Thus, if annual data on reported losses is being studied, the force of development is the instantaneous rate of change in reported losses expressed as an annual rate.

Using $f(t)$ to denote the force of development,

$$f(t) = (d/dt \ r(t)) \ / \ r(t) \qquad [4]$$

$$= d/dt \ log_e(r(t)) \qquad [5]$$

Generalized Bondy development can now be defined as describing development patterns for which:

$$f(y) = B^{(y-z)/h} \ f(z) \qquad [6]$$

where B is a single parameter with values between 0 and 1, and y and z are arbitrary maturities. The characteristic Bondy relationships among development factors are logical consequences of this definition. Thus, if $y = t+h$ and $z = t$, equation [6] can be rewritten as:

$$f(t+h) = B \ f(t) \qquad [7]$$

Equation [1] can be derived from equation [6] by integrating over appropriate intervals - (A) from t to $t+h$ and (B) from $t+h$ to $t+2h$ - and comparing results.

(A)
$$\int_t^{t+h} f(x) \, dx \quad = \quad \int_t^{t+h} d/dx(\log_e(r(x)) \, dx$$

$$= \quad \log_e(r(t+h)) - \log_e(r(t))$$

$$= \quad \log_e(r(t+h)/r(t))$$

$$= \quad \log_e(d(t)) \qquad\qquad [8]$$

(B)
$$\log_e(d(t+h)) \quad = \quad \int_{t+h}^{t+2h} f(x) \, dx$$

$$= \quad \int_t^{t+h} B^{((x+h)-x)/h} \, f(x) \, dx$$

$$= \quad B \int_t^{t+h} f(x) \, dx$$

$$= \quad B \, \log_e(d(t)) \qquad\qquad [9]^2$$

Using both sides of equation [9] as exponents of e generates equation [1].

$$d(t+h) = d(t)^B \qquad\qquad [1]$$

3.   Fractional Periods

Analysis of development for fractional periods illustrates the power of generalized Bondy development. Frequently an actuary must determine development factors applicable to periods spanning intervals other than those for which observations are available. Because development intervals can be analyzed as combinations of fractional periods, observed periods, and tail periods, the ability to determine logically consistent development factors for fractional periods significantly enhances the power of any actuarial development model.

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[2] Substitution: $u=x-h$, $du=dx$, $(x=t+h)->(u=t)$, & $(x=t+2h)->(u=t+h)$.

If $d_h(t)$ is a development factor for a Bondy development pattern with parameter B measured using interval h, then $d_h(t)$ can also be written as a product of n development factors measured using interval h/n. Changing the limits of integration in equations [8] and [9] to (A) from t to $t+h/n$ and (B) from $t+h$ to $t+h+h/n$ generates:

$$B = log_e(r(t+h+h/n)/r(t+h)) \; / \; log_e(r(t+h/n)/r(t)) \qquad [10]$$

Using $\hat{B}$ to denote the Bondy parameter for development measured at interval h/n, equation [10] can be rewritten as:

$$B = log_e(d_{h/n}(t+h)) \; / \; log_e(d_{h/n}(t)) = (\hat{B})^n \qquad [11]$$

The development factor $d_{h/n}(t)$ can be computed from:

$$log_e(d_h(t)) = log_e(d_{h/n}(t)) \; (1+\hat{B}+\ldots+\hat{B}^{n-1}) \qquad [12]$$

Development factors for other fractional periods can be derived by raising $d_{h/n}(t)$ to appropriate powers of $\hat{B}$.

4.   The Shape of Bondy Development

Development patterns can be portrayed in terms of the force of development, development ratios, or values of observations (e.g., reported losses).

The force of development at arbitrary maturities may be computed using the following equation.

$$log_e d(t) = \int_t^{t+h} f(x) \; dx$$

$$= \int_t^{t+h} B^{(x-t)/h} \; f(t) \; dx$$

$$= \int_t^{t+h} e^{(log(B))(x-t)/h} \; f(t) \; dx$$

$$= f(t) \; (h/log(B)) \int_0^1 e^u \; du$$

$$- f(t) \ (h/\log(B)) \ (B-1) \qquad\qquad [13]^3$$

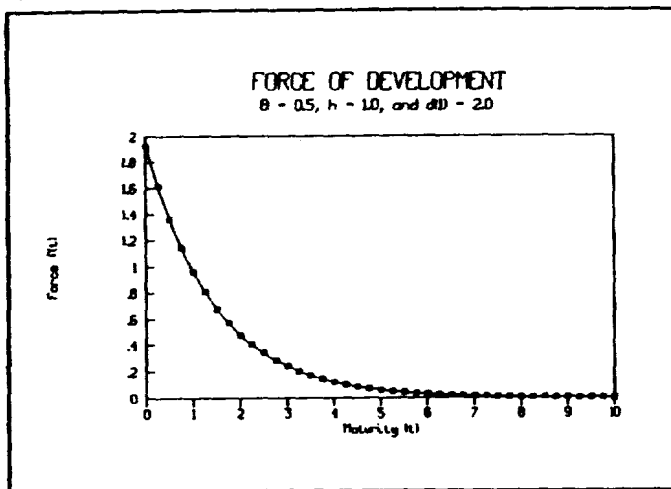Thus, if B is known and $d(\hat{t})$ is also known for some $\hat{t}$,

$$f(\hat{t}) \ - \ \log(B) \ \log(d(\hat{t})) \ / \ (h(B-1)) \qquad [14]$$

and
$$f(t) \ - \ B^{(t-\hat{t})/h} \ f(\hat{t}) \qquad\qquad [15]$$

Chart I shows the force of development at three month intervals for B - 0.5, h - 1, and $d(\hat{t})$ - 2.
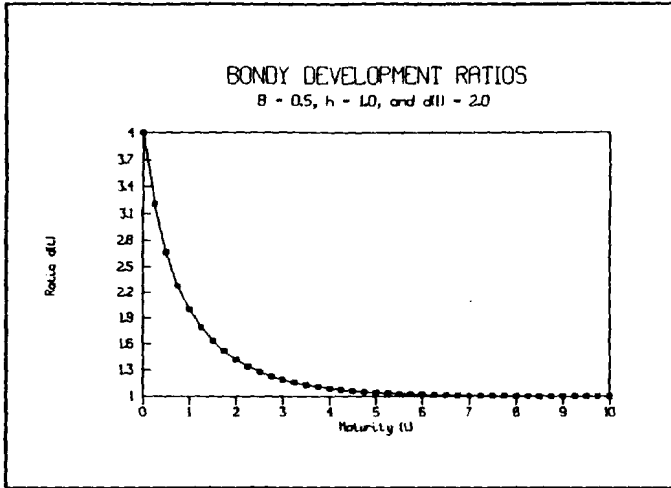
CHART I

FORCE OF DEVELOPMENT
B - 0.5, h - 1.0, and d(1) - 2.0

Using equations [11] and [12] the Bondy parameter for quarterly intervals and the development ratio from maturity 2.00 to maturity 2.25 can be calculated. Equation [1] can then be applied to determine development factors for other three month intervals. Groups of four successive quarterly factors can then be multiplied to determine annual development ratios. Chart II presents the annual development ratios at three month intervals.

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[3] Substitution: $u-(\log(B))(x-t)/h$, $du-(\log(B)/h)dx$, $(x-t)->(u-0)$, & $(x-t+h)->(u-1)$.

CHART II

BONDY DEVELOPMENT RATIOS
B - 0.5, h - 1.0, and d(1) - 2.0

Assuming an observation of $r(t)$ - 100,000 at t-1, it follows that the ultimate
value is 400,000. Combining this information with quarterly development
ratios leads to the following chart of emergence by calendar quarter.

CHART III

BONDY DEVELOPMENT PATTERN
Interval Emergence
B - 0.5, h - 1.0, and d(1) - 2.0
Normalized to Ultimate Value of 400,000
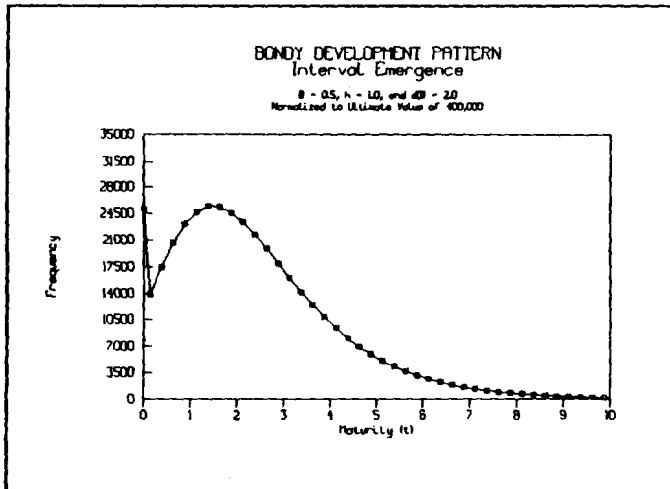
In each of these charts, values have been extended back to a maturity of zero. In applications, the central concern is estimating future development. Analysis of the ability of generalized Bondy development to precisely model immature losses is beyond the scope of this paper.

5.   Parameter Estimation

Bondy development patterns can be quickly estimated from observed development ratios. The algorithm presented in this section can be used to fit a generalized Bondy development pattern to as few as two observations of development ratios. In order to present the algorithm, the following notation is employed:

d_i - the ith observed development ratio, where n development ratios have been observed and the development ratios are arranged in order of maturity with d_n spanning the most mature development period.

l_i - $log_e(d_i)$.

$\hat{B}$ - the estimate of the Bondy parameter.

$\hat{d}$ - the estimated development ratio for the earliest observed period; d will be shown to be a function of B.

$\hat{l}$ - $log_e(\hat{d})$.

The algorithm determines parameter estimates affording a "best fit." A least squares measure of fit is employed. To simplify the mathematics and place greater emphasis on more mature observations, the measure of fit is defined in terms of natural logarithms.

The specific objective of the algorithm is to find values of $\hat{B}$ and $\hat{d}$ that minimize:

$$CRITERION \; - \; \sum_{i=1}^{n} (\; l_i \; - \; \hat{l} \; \hat{B}^{i-1} \;)^2 \qquad [16]$$

The criterion is written in terms of two parameters. For the criterion
to attain a minimum value, the value of its derivative with respect to each
parameter must be zero. The derivative with respect to $\hat{d}$ is:

$$\partial/\partial\hat{d}\ \text{(CRITERION)} = \sum_{i=1}^{n} 2\ (1_i - \hat{1}\ \hat{B}^{i-1})\ (-\hat{B}^{i-1})\ /\ \hat{d} \qquad [17]$$

Because, for relevant cases, development ratios are positive, $\hat{d}$ must be
positive. Setting the derivative equal to zero and solving for $\hat{1}$ leads to:

$$\hat{1} = \sum_{i=1}^{n} 1_i\ \hat{B}^{i-1}\ /\ \sum_{i=1}^{n} \hat{B}^{2i-2} \qquad [18]$$

Equation [17] expresses $\hat{1}$ (and hence $\hat{d}$) as a function of $\hat{B}$. Thus, the problem
of fitting a Bondy development pattern has been reduced to the problem of
computing the single parameter $\hat{B}$.

Writing the criterion as a function $c(\hat{B})$, the value of $\hat{B}$ for which $c(\hat{B})$
takes its minimum value can be determined using a Newton-Raphson iterative
approach to identify the zeroes of the derivative $c'(\hat{B})$ of $c(\hat{B})$. The formula
for $\hat{B}_{x+1}$ (i.e., the estimated value of $\hat{B}$ at the $(x+1)^{st}$ iteration) is:

$$\hat{B}_{x+1} = \hat{B}_x - c'(\hat{B}_x)\ /\ c''(\hat{B}_x) \qquad [19]$$

Because all derivatives are with respect to $\hat{B}$, more simple notation can
be used. For purposes of describing the algorithm for [19], c denotes $c(\hat{B})$,
c' denotes $\partial/\partial\hat{B}(c(\hat{B}))$, and c'' denotes $\partial^2/\partial\hat{B}^2(c(\hat{B}))$. Similarly, $\hat{1}' = d/d\hat{B}(\hat{1})$,
and $\hat{1}'' = d^2/d\hat{B}^2(\hat{1})$. Using this notation, $\hat{B}_{x+1}$ can be evaluated using the
following formulas:

$$c = \sum_{i=1}^{n} (\ 1_i - \hat{1}\ \hat{B}^{i-1}\ )^2 \qquad [16]$$

$$c' = \sum_{i=1}^{n} (-2)\ (1_i - \hat{1}\ \hat{B}^{i-1})\ (\hat{1}'\ \hat{B}^{i-1} + (i-1)\ \hat{1}\ \hat{B}^{i-2})$$

$$- \sum_{i=1}^{n} (-2)\{l_i\hat{l}'\hat{B}^{i-1} + (i-1)l_i\hat{l}\hat{B}^{i-2} - \hat{l}\hat{l}'\hat{B}^{2i-2} - (i-1)\hat{l}^2\hat{B}^{2i-3}\}$$ [20]

$$c'' = \sum_{i=1}^{n} (-2)\{l_i\hat{l}''\hat{B}^{i-1} + 2(i-1)l_i\hat{B}^{i-2} + (i-1)(i-2)l_i\hat{l}\hat{B}^{i-3}$$
$$- (\hat{l}'\hat{l}'+\hat{l}\hat{l}'')\hat{B}^{2i-2} - 2(2i-2)l_i\hat{l}\hat{B}^{2i-3} - (i-1)(2i-3)\hat{l}^2\hat{B}^{2i-4}\}$$ [21]

$$\hat{l} = \sum_{i=1}^{n} l_i \hat{B}^{i-1} / \sum_{i=1}^{n} \hat{B}^{2i-2}$$ [18]

$$\hat{l}' = (\sum_{i=1}^{n} (i-1)l_i\hat{B}^{i-2} \sum_{i=1}^{n} \hat{B}^{2i-2}$$
$$- \sum_{i=1}^{n} l_i\hat{B}^{i-1} \sum_{i=1}^{n} (2i-2)\hat{B}^{2i-3}) / (\sum_{i=1}^{n} \hat{B}^{2i-2})^2$$ [22]

$$\hat{l}'' = ((\sum_{i=1}^{n}(i-1)(i-2)l_i\hat{B}^{i-3} \sum_{i=1}^{n} \hat{B}^{2i-2} - \sum_{i=1}^{n} l_i\hat{B}^{i-1} \sum_{i=1}^{n}(2i-2)(2i-3)\hat{B}^{2i-4})(\sum_{i=1}^{n}\hat{B}^{2i-2})^2$$
$$- (\sum_{i=1}^{n} (i-1)l_i\hat{B}^{i-2} \sum_{i=1}^{n} \hat{B}^{2i-2} - \sum_{i=1}^{n} l_i\hat{B}^{i-1} \sum_{i=1}^{n} (2i-2)\hat{B}^{2i-3})$$
$$(2 \sum_{i=1}^{n} \hat{B}^{2i-2} \sum_{i=1}^{n} (2i-2)\hat{B}^{2i-3})) / (\sum_{i=1}^{n} \hat{B}^{2i-2})^4$$ [23]

Although equations [16] through [23] appear detailed and complex, values are readily computed using generally available software for personal computers. Sample calculations for medical malpractice are illustrated in Appendices I through III.

Appendix I presents the computation for annual paid loss development factors for medical malpractice insurance based on data published by A. M. Best & Company in Casualty Loss Reserve Development. Three year weighted averages are used.

Appendix II summarizes the results of the generalized Bondy estimation procedure. There are many ways to evaluate the fit of the estimates to the data. Appendix II presents the percentage of total variation in the

logarithms of the observed development ratios that is explained. Despite the

mixing of claims-made and occurrence policies, aggregation, and other

complications, the generalized Bondy approach explains 99.62% of the variation

in observed development ratios (i.e., the logarithms of these ratios). The

arrangement of the exhibit also facilitates direct comparison of pairs of

observed and estimated ratios. Finally, the factor to ultimate 1.6939 can be

compared to the corresponding value in Appendix I.

Appendix III presents one iteration in spreadsheet format. It should be

noted that columns (T) through (V) use the iterated value of B, not the value

at the start of the iteration.

6.   Examples

A. M. Best & Co. publishes casualty loss development data annually for

both primary insurers and reinsurance companies. Primary lines include

automobile liability, general liability, medical malpractice, multi-peril

coverages, and workers' compensation. Computations analogous to the medical

malpractice example in the Appendices generate the following results:

Line	Bondy Parameter	Explained Variation
Automobile Liability	.370	99.48%
General Liability	.629	99.53%
Medical Malpractice	.627	99.62%
Multi-Peril	.321	96.97%
Workers' Compensation	.398	99.28%

In each case, the generalized Bondy development affords an excellent fit to

the observed development ratios.

In his paper "Extrapolating, Smoothing, and Interpolating Development

Factors" Richard Sherman illustrates the ability of inverse power curves to

fit development patterns. Sherman discusses several forms of inverse power
curve. The most common form is:

$$d(t) - 1.0 + a\ t^{-b} \qquad \{24\}$$

where a and b are parameters to be estimated.

The following table compares generalized Bondy and inverse power fits
for data in Sherman's paper and the author's reply to discussion. Unlike the
preceding table, the percentage of variation explained is in terms of actual
development ratios (not their logarithms).

	Type of Business	Inverse Power Explained Variation	Bondy Parameter	Bondy Explained Variation
1.	Workers' Compensation Paid Loss Development - 1969 Accident Year	99.73%	.354	99.52%
2.	Automobile Bodily Injury Liability - 5 Carriers	99.29%	.190	99.97%
3.	General Liability - 5 Carriers	99.14%	.488	98.83%
4.	Workers' Compensation - 5 Carriers	99.87%	.457	98.81%
5.	RAA Automobile Liability	96.25%	.403	99.67%
6.	RAA General Liability	99.94%	.582	99.21%
7.	RAA Medical Malpractice	92.20%	.474	98.91%
8.	RAA Workers' Compensation	99.24%	.633	95.93%
9.	Automobile Bodily Injury Claim Count	99.96%	.091	99.96%
10.	Other Bodily Injury Claim Count	99.26%	.279	99.58%
11.	Medical Malpractice Claim Count	99.41%	.262	99.57%

Of eleven comparisons, each technique explained more variance in five cases
and there was one tie. These results might change somewhat with different
rounding conventions. The important point is that both approaches explain
significant portions of the variation in observed development ratios.

These examples suggest that the Bondy parameter affords a quantitative measure of the tail of a casualty line. Because generalized Bondy development is determined by a single parameter, the parameter value affords a means of classifying insurance and risk financing tails. Indeed, Bondy development describes premium development as well as loss development. Bondy parameters close to unity correspond to long-tailed lines. Bondy parameters close to zero correspond to short-tailed lines.

7.   Sensitivity

In order to analyze the sensitivity of the algorithm, an underlying Bondy development pattern with parameter 0.5 is used to generate samples of from 4 to 10 development ratios. A 5% distortion is introduced in one ratio with an offsetting distortion in an adjacent development ratio. In other words, a +5% movement in observed losses at one valuation increases the development ratio for which it serves as the numerator and decreases the development ratio for which it serves as the denominator. Thus, a 5% increase in the first ratio in a sample is offset by an adjustment of 1/1.05 in the second ratio, and a 5% increase in the last ratio is offset by an adjustment of 1/1.05 in the preceding ratio. Results are summarized in the following table. Other types of distortion are possible, but are beyond the scope of the present analysis.

Disturbance	Sample Size	Bondy Parameter	Explained Variation
+5.0% at 1st Pt	4	.4759	99.45%
	6	.4791	99.61%
	10	.4795	99.71%
+5.0% at Lst Pt	4	.4937	97.79%
	6	.4949	98.58%
	10	.4992	98.95%

In reviewing this table it should be noted that fit is measured against the logarithms of the modified development ratios. The parameter values are consistent with the direction of the distortion.

The table suggests that the development ratios for less mature periods exert greater influence on generalized Bondy fits than ratios for more mature periods. In a sense the algorithm assigns greater weights to the ratios generated by larger differences in observed values.

8. Theoretical Considerations

This paper began with a discussion of general components of development - namely, point processes, mixing, and aggregation. Generalized Bondy development has been presented as a point process. This interpretation is consistent with mixing and aggregation considerations.

Mixed Bondy development patterns are not strictly Bondy. However, to the extent insurance classification schemes identify exposures with like loss characteristics, the corresponding Bondy parameters for the individual patterns should be within a fixed range. Accordingly, a fitted generalized Bondy pattern should afford reasonable estimates of actual development.

More direct results are available for aggregation. For instance, Appendix IV demonstrates that, if the underlying distribution of losses is uniform across the aggregation period, then pointwise Bondy development implies that the aggregate data will exhibit Bondy development with the same parameter value.

There is one restriction to this result. At immature valuations (i.e., valuation dates at which not all data from the aggregation period are included in the observed sample), observed development ratios are not Bondy. Fitting

Bondy development to immature data can create misleading estimates of tail development. Although the tail in generalized Bondy development always extends to infinity, the generalized Bondy algorithm will underestimate tail development if valuation dates truncate observation periods so that they are not fully earned. Appendix IV affords a mathematical analysis of this phenomenon.

For example, in the medical malpractice example of Appendices I-III, the twelve month observed value of paid losses probably does not fully recognize losses incurred during the twelfth month of the accident year. If the procedure is applied to the two most mature observed ratios, the estimated factor to ultimate is 2.186 - compared to 1.6939 in Appendix II and the industry average of 2.173 in Appendix I.

The algorithm presented in this paper is not unique. There are other ways to calculate Bondy parameters. For example, a least squares criterion could be applied to the logarithms of the term in equation [16]. Insofar as the logarithm of a logarithm is a further abstraction, this alternative approach does not generally produce as good a fit to observed development ratios. Also, because logarithms of negative numbers are not defined, the alternative cannot always be computed. Nonetheless, it does describe generalized Bondy development with an alternative measure of goodness of fit.

Two important theoretical questions are beyond the scope of this paper. The first question concerns the determination of ultimate values and reserves. A good fit to observed development ratios is not the same as an accurate determination of adequate reserves. The concern in this paper is predicting observed development ratios. This paper has not addressed the issue of estimating ultimate values and corresponding reserves.

For example, when development factors are employed in the estimation of ultimate values and corresponding reserves, they are generally applied to observed values. However, it is possible that adjustments to observed values could produce better estimates of reserves. Whether the observed values, to which Bondy development ratios are applied, should be normalized to be consistent with Bondy development is an issue in the estimation of ultimate values, not the fitting of development ratios? The issues are intimately related, but questions concerning estimates of ultimate values are beyond the scope of this paper.

The second theoretical question beyond the scope of this paper is why generalized Bondy point processes should describe insurance and risk financing operations. For example, if annuities are the subject, the point process is described by the development factors of the form $(n+1)/n$. Likewise, if claims are reported in accordance with a waiting line model and an exponential reporting pattern, the force of development at time t is $\hat{\delta}e^{-\hat{\delta}t}/(1-e^{-\hat{\delta}t})$. In both of these examples, characteristics of the insurance operations can be used to infer the mathematical form of the development ratio or force of development, respectively. This paper has not identified characteristics of insurance operations that imply description by generalized Bondy development. Further research is appropriate.

9.   Conclusion

Generalized Bondy development has been defined in terms of an unifying underlying concept - the force of development. The interaction of various theoretical considerations with procedures for estimating generalized Bondy development has been discussed. The general definition agrees with historic definitions of Bondy development. More importantly, generalized Bondy

development can be characterized in terms of a single parameter and this parameter can be used to compute development factors spanning arbitrary intervals. Estimation of the parameter is simple and straightforward. And, generalized Bondy development affords excellent fits to observed development ratios for many cases of insurance development.

Medical Malpractice
Development Factors based on
Paid Losses & Loss Adjustment Expenses

I. Observations*

Accident Year	12 mos	24 mos	Value** at 36 mos	48 mos	60 mos	Est. Ult. Value**
1981	$35,173	$136,175	$308,436	$539,488	$799,488	
1982	$74,296	$196,917	$391,307	$680,142	$940,320	$2,085,013
1983	$93,189	$223,976	$504,246	$804,961	$1,101,762	$2,346,222
1984	$66,461	$296,442	$598,322	$952,219		$2,503,708
1985	$90,771	$290,598	$642,083			$2,957,598
1986	$95,891	$309,099				$3,207,604
1987	$89,408					$3,404,750

II. Development Ratios

Accident Year	24-12	36-24	Span of Maturity 48-36	60-48	Ult-60
1981	3.872	2.265	1.749	1.482	
1982	2.650	1.987	1.738	1.383	2.217
1983	2.403	2.251	1.596	1.369	2.130
1984	4.460	2.018	1.591		
1985	3.201	2.210			
1986	3.223				
3 Yr Avg	3.628	2.160	1.642	1.411	2.173
Cumulative	39.462	10.876	5.036	3.067	2.173
Percent Pd	2.53%	9.19%	19.86%	32.61%	46.01%

*   Data from Best's Casualty Loss Reserve Development (1988 edition)
**   (000's omitted)

GENERALIZED BONDY DEVELOPMENT

SAMPLE ITERATION

		Input		Newton Raphson Results	
Index	Maturity	Development Ratio	Fitted Ratio	Factor to Ultimate	Percent Developed
1	12	3.628	3.5814	30.4839	3.28%
2	24	2.160	2.2244	8.5118	11.75%
3	36	1.642	1.6504	3.8266	26.13%
4	48	1.411	1.3688	2.3186	43.13%
5		.000	1.2174	1.6939	59.04%
				1.3913	71.87%

Initial

Bondy Parameter — B —   .627000
Logarithm   — l — 1.275377
Criterion   — c —   .001976
Explained variation —   99.62%

Iterated

Bondy Parameter — B —   .626668
Logarithm   — l — 1.275748
Criterion   — c —   .001976
Explained variation —   99.62%

Calculations for Appendix II

(A) Index	(B) Sample Count	(C) Obs Log l_i	(D) Squared Deviation	(E) l_iB^{i-1}	(F) B^{2i-2}	(G) $(i-1)l_iB^{i-2}$
1	1	1.288682	.318018	1.288682	1.000000	.000000
2	1	.770108	.002057	.482858	.393129	.770108
3	1	.495915	.052366	.194959	.154550	.621877
4	1	.344299	.144744	.084867	.060758	.406061
5	0	.000000	.000000	.000000	.000000	.000000
Total	4	2.899003	.517185	2.051365	1.608438	1.798047
Average	xx	.724751	.129296	1 =	1.275377	xx

(A) Index	(H) $(2i-2)B^{2i-3}$	(I) $(i-1)(i-2)$ $l_i B^{i-3}$	(J) $(2i-2)(2i-3)$ B^{2i-4}	(K) $l_i - lB^{i-1}$	(L) Squared Error	(M) $l_i lB^{i-1}$
1	.000000	.000000	.000000	.013304	.000177	-1.442390
2	1.254000	.000000	2.000000	-.029553	.000873	-.540451
3	.985968	.991830	4.717548	-.005473	.000030	-.218212
4	.581419	1.295252	4.636512	.029929	.000896	-.094989
5	.000000	.000000	.000000	.000000	.000000	.000000
Total	2.821386	2.287082	11.354060	.008207	.001976	-2.296042
1' = -1.119275		1" =	-3.654361			

(A) Index	(N) $(i-1)l_i lB^{i-2}$	(O) $ll'B^{2i-2}$	(P) $(i-1)l^2B^{2i-3}$	(Q) $l_i l"B^{i-1}$	(R) $2(2i-2)l'$ $l B^{2i-3}$	(S) $(i-1)(2i-3)$ l^2B^{2i-4}
1	.000000	-1.427498	.000000	-4.709308	.000000	.000000
2	.982179	-.561191	1.019870	-1.764537	-3.580166	1.626587
3	.793128	-.220620	.801881	-.712449	-2.814934	3.836751
4	.517881	-.086732	.472864	-.310134	-1.659948	3.770846
5	.000000	.000000	.000000	.000000	.000000	.000000
Total	2.293188	-2.296042	2.294615	-7.496428	-8.055048	9.234184
		c' = .002854			c" =	8.604556

(A) Index	(T) $l_i B^{i-1}$	(U) B^{2i-2}	(V) Sqrd. Error
1	1.288682	1.000000	.000167
2	.482602	.392713	.000862
3	.194752	.154224	.000026
4	.084732	.060566	.000920
5	.000000	.000000	.000000
Total	2.050769	1.607503	.001976
iter. 1 =	1.275748		

Aggregation for Uniform Distribution

A generalized Bondy development pattern (with parameter B) describes the point process. Observations for accident dates x during an accident year from a through b are observed at valuation date v. The density function $p(x)$ describes the relative exposures during the accident year. For purposes of the example, the density function is uniform, i.e., $p(x) = 1/(b-a)$. The observed value for accident year losses $R(v)$ can now be written as:

$$R(v) = \int_a^{min(b,v)} p(x) \; r(v-x) \; dx$$

$$= (1/(b-a)) \int_a^{min(b,v)} r(v-x) \; dx \qquad [A1]$$

To evaluate $R(v)$ the point process $r(v-x)$ is first rewritten in terms of $r(1)$ and $f(1)$. Using equation [8] it follows that:

$$log_e(r(v-x)/r(1)) = \int_1^{v-x} f(1) \; B^{t-1} \; dt \qquad [A2]$$

Evaluating the integral:

$$\int_1^{v-x} B^{t-1} \; dt = \int_0^{log(b)(v-x-1)} e^u/log(B) \; du \qquad [A3]$$

$$= (B^{v-x-1}-1) \; / \; log(B) \qquad [A3]^4$$

Thus,

$$r(v-x) = r(1) \; EXP(f(1)(B^{v-x-1}-1)/log(B)) \qquad [A4]$$

and

$$R(V) = (1/(b-a))\int_a^{min(b,v)} r(1) EXP(f(1)(B^{v-x-1}-1)/log(B))dx \qquad [A5]$$

Substituting:

$$u \quad = \quad f(1) \; (B^{v-x-1}-1) \; / \; log(B)$$

$$du \quad = \quad -f(1) \; B^{v-x-1} \; dx$$

$$(x=a) \quad -> \quad (u= f(1) \; (B^{v-a-1}-1) \; / \; log(B)$$

$$(x=min(b,v))-> \quad (u= f(1) \; (B^{v-min(v,b)-1}-1) \; /log(B)$$

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[4] Substitution: $u=log(B)(t-1)$, $du=log(B)dt$, $(t=1)=>(u=0)$, & $(t=v-x)=>(u=log(B)(v-x-1))$.

generates:

$$R(v) = (r(1)/(b-a)) \int_{f(1)\ B^{v-a-1}-1\ /\ log(B)}^{f(1)\ B^{v-min(v,b)-1}\ /\ log(B)} e^{u}\ du$$

$$= (r(1)/(b-a))\ e^{(-f(1)/log(B))(B^{v-min(v,b)-1}-B^{v-a-1})} \qquad [A6]$$

From equation [A6], it follows that the development factor $D(v)$ for aggregated losses is:

$$D(v) = e^{(-f(1)/log(B))\ (B^{v-min(v,b)}-B^{v-min(v,b)-1})} \qquad [A7]$$

and

$$log(D(v)) = (-f(1)/log(B))\ (B^{v-min(v+1,b)}-B^{v-min(v,b)-1}) \qquad [A8]$$

Taking the ratio of the logarithms for $D(v)$ and $D(v+1)$ shows that:

$$\frac{log(D(v+1))}{log(D(v))} = \frac{(-f(1)/log(B))\ (B^{v-min(v+2,b)+1}-B^{v-min(v+1,b)})}{(-f(1)/log(B))\ (B^{v-min(v+1,b)}-B^{v-min(v,b)-1})}$$

$$= B \qquad [A9]$$

Equation [1] follows immediately from equation [A9]. In other words, aggregated generalized Bondy development over a uniform distribution generates Bondy development with the same parameter B at the aggregate level.

The example corresponds to accident year development. Treaty years, policy years, and other forms of aggregation would generate somewhat different sets of equations.

It should be noted that the limits of integration include terms of the form $min(v,b)$ and that the derivation of [A9] from [A8] presumes that the minimum is b for each occurrence of these term in equation [A9]. At immature durations (i.e, situations in which a valuation date is less than b) the aggregated data will not follow the identified generalized Bondy development pattern.

Lastly, the proof demonstrates that pointwise Bondy development can generate aggregate Bondy development. It does not demonstrate that aggregate Bondy development necessarily implies pointwise Bondy development.

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