A Discrete Equilibrium Model of the Term Structure

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Abstract

This report presents a self-contained development of a single state variable, discrete, equilibrium model of the term structure. The model provides a coherent framework within which to address many of the problems associated with interest-sensitive investment and insurance products, such as valuation of the options implicit in certain contracts, determination of an appropriate portfolio to support a given set of interest-sensitive liabilities, etc., and represents a necessary first step in the resolution of problems of this nature. We emphasize the word "necessary" because although attempts have been made to handle these matters without invoking the entire term structure, we have found that even seemingly plausible assumptions regarding the yields on bonds of various maturities can lead to arbitrary and inconsistent results in the absence of constraints imposed by an equilibrium model.

The development of the model is straightforward. It follows the approach generally associated with the development of continuous term structure models, adapted to a binomial form. Input reflects the user's convictions with respect to spot rate volatility, trend, etc. The model generates a lattice of term structures, each with an associated probability; each path through the lattice represents a particular evolution of the term structure over time and might be regarded as roughly analogous to an interest rate scenario used in reserve testing. There are two important differences, however. First, the model scenarios provide the entire term structure at each point in time, thereby allowing for investigation of portfolio composition; second, because each term structure in the lattice has an associated probability, we can distinguish between reasonably likely, and highly unlikely scenarios.

The only published binomial term structure model of which we are aware is that proposed by Ho and Lee (Journal of Finance, December, 1986), which is well-known within the financial and actuarial communities. Mathematical analysis (given in this report) has made certain serious shortcomings of the Ho-Lee model apparent. In fact, it emerges as a rather unnatural variant of the general model discussed here.

The report contains a section on parameter estimation which should be of interest. One of the difficulties associated with applying term structure models is that while certain parameters can be estimated directly from published data, others cannot. We give an empirical method which uses the actual term structure as a guide.

This report documents the development of a single state variable, discrete, equilibrium model of the term structure. We believe the model will be useful in addressing several problems facing insurers. For example, it can help us better understand and value the options implicit in certain insurance and investment contracts. Duration measures obtained by application of the model will assist in the matching of asset and liability cash flows. Our purpose here is not to treat the applications, however, but to produce a general purpose analytic tool.

The articles by Boyle [5] and Vasicek [10] are useful background reading, but they are optional -- the following development is self-contained. The model we ultimately propose is a discrete analogue of the Cox-Ingersoll-Ross model described by Boyle. However, our basic model is quite general and we show, for example, that it incorporates the model proposed by Ho and Lee [8], as a special case.

<u>Organization of this report</u>

Section 1 endeavors to set the scene for the technical discussion that follows. We will attempt to provide some feel for the meaning of equilibrium and to anticipate and respond to these questions:

Why should we incorporate the equilibrium property in our interest rate model?

Recognizing that several equilibrium models have already been presented in the finance literature, why are we considering and, moreover, promoting, a new model at this time?

Section 2 develops a general equilibrium model of interest rates, goes on to discuss the particular version we are proposing, and concludes with a brief comparative discussion of the model proposed by Ho and Lee [8]. The treatment of each model is complete but largely intuitive. Almost all of the mathematics has been relegated to the appendices in order to present the ideas in the text as clearly as possible. The appendices are comprehensive; every result stated in the text is derived from first principles.

In Section 3 we give an empirical method for estimating values for the model's parameters, using the actual term structure as a guide,

Finally, Section 4 illustrates the proposed model by generating a lattice of term structures and tracing two paths through the lattice (each of which corresponds to a particular evolution of the term structure over time). A sequence of graphs of yield versus maturity is given for each path. The graphs show that under one scenario the yield curve eventually inverts, while under the other, it remains non-inverted throughout.

1. Preliminaries

Background

Interest rate models do not represent an end in themselves; they are developed to assist in the solution of real problems, such as the matching of asset and liability cash flows, or the pricing of financial options. The progression to more sophisticated models proceeds via this logic: if model X gives rise to, say, duration concept x, then a better model Y will give rise to a better duration concept y.

Thus, as the realization was made that the model underlying Macaulay duration allows only parallel displacement of the term structure, the search was on for models built upon more realistic shifts [1,2,3,9]. The expectation was that such models would produce better indices of price sensitivity. These early models bear little resemblance to today's equilibrium models; in fact, they are essentially non-stochastic.[1]

While equilibrium models can generate measures of price sensitivity, this was not the primary motivation behind their development. Rather, development of equilibrium models followed the introduction of trading in options on

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[1] Since the adjective "stochastic" carries a certain panache, some of these models were so labelled by their authors, on the slim justification that it is not known in advance whether the shift will be, say, 5 or 10 basis points.

bonds in much the same way that the (considerably simpler) equilibrium valuation theory of options on equities (exemplified by the Black-Scholes formula) was a response to the introduction of organized trading in stock options.

The equilibrium concept

Section 2 contains a precise mathematical formulation of equilibrium. Necessary and sufficient conditions are established in Appendix A. Our purpose in the following paragraphs is to give an intuitive understanding of this important concept and to suggest why it is a desirable property in an interest rate model.

In the present context equilibrium refers to the relationship among the returns on (default-free) debt instruments. These include not only bonds and mortgages but also their derivative options.

The role of equilibrium in interest rate models is closely related to the more general issue of consistency in any multi-component mathematical model. For example, in pricing a re-entry term insurance product, the actuary should not set the mortality and persistency assumptions independently. Similarly, the actuary should not make assumptions concerning the instantaneous spot rate without considering the implications for yields on other fixed income contracts.

It is a fundamental property of equilibrium interest rate models that there is never an opportunity for gain via riskless arbitrage. The potential user who is dubious about the value or validity of this requirement should ponder the alternatives. Associated with any non-equilibrium model are sure-thing opportunities. Optimal investment strategies developed using these non-equilibrium models make the implicit assumption that the user can systematically outmaneuver the market. Even if such a model is used merely to price an option, there may be problems. Since disequilibriums will occur in a non-systematic fashion, actual maintenance of a replicating portfolio through rebalancing may require a substantial and unplanned infusion of additional funds. According to the model, however, no funds are needed after the initial portfolio is formed, beyond transactions costs and taxes. This problem can of course occur with any model, but it would seem the effects are minimized when the user assumes an essentially neutral position for the portfolio under his management, vis a vis the market in general.

Why a new model?

After surveying existing models we have concluded that none is wholly adequate for our purposes. The two continuous models described in Boyle's survey article have several attractive features, but we feel that the binomial lattice formulation is much more flexible and adaptable. Continuous models are suitable for giving the term structure (i.e., for determining the prices of zero coupon bonds) but are not as convenient as lattice models when the problem is extended to that of pricing an arbitrary stream of interest sensitive cash flows.

We know of only one published binomial lattice model [8]. It is clear that other lattice models exist but the details appear to be proprietary. Publications such as [4] and [7] may have the effect of discouraging the reader from developing a model independently.

In our estimation, the Ho-Lee approach provides little in the way of insight into the underlying dynamics of the term structure. Our analysis of the Ho-Lee model shows that all of the possible yield curves at any time are exactly parallel to each other; consequently the sequence of yield curve shapes is independent of the values taken by the spot rate. We also show that the Ho-Lee model may be expressed as a particular case of the general model we are considering. Expressing it in this form brings out the assumptions regarding spot rate behavior which are implicit in the model, and which, when viewed explicitly appear implausible and somewhat contrived. In short, the Ho-Lee model sacrifices simplicity, understandability, and generality in order to secure one property -- exact reproduction of the initial term structure. This is not necessarily a desirable property, however. It implicitly assumes that the current term structure contains only useful information when in fact it probably contains a certain amount of random "noise". (It should be noted that the finer the lattice, the greater the amount of noise which will be interpreted as information.)

The discussion in Section 2 also points out an inherent limitation of the Ho-Lee model -- expected yields rise steadily (ultimately linearly) over time. This phenomenon, which is more easily isolated in the simpler (but non-equilibrium) Clancy model [6], can be traced to the asymmetry of the stochastic component of the interperiod interest rate change. It is not surprising to find mean yields increasing over time in a model which guarantees that rates never go negative, such as when interest rates follow a logbinomial (lognormal in the limit) distribution. However, a model's usefulness is limited if it does not offer, by suitable choice of parameters, other patterns of long term interest rate behavior, such as the tendency to drift towards a long-term mean value, for example. (At the risk of carping, we note that interest rates can go negative in the Ho-Lee model!)

2. Model Description

The model is a single state variable discrete equilibrium model of the term structure, with the following features.

- In common with all single state variable models it assumes that the behavior of the entire term structure is governed by the behavior of a single variable. As with most such models the underlying state variable is taken to be the one period spot rate.

- The model is binomial. It generates a lattice of points, each with an associated probability, and each of which represents a distinct term structure. A given path through the lattice represents a particular evolution of the term structure over time.

- The term structures generated by the model are in equilibrium. The equilibrium condition, which will be discussed in more detail later in the report, ensures that there is no opportunity for risk-free arbitrage between any two distinct bond portfolios.

The procedure for determining the lattice of term structures with a binomial model is as follows.

(i) Choose a form for the stochastic movement of the spot rate over time. Over any single time increment the spot rate may move to one of two values.

(ii) Choose the values of any parameters associated with the form chosen for the spot rate (see Section 3).

(iii) Construct the binomial lattice of spot rates.

(iv) Determine the price of a zero coupon, default-free one-period bond directly from the corresponding spot rate at each point on the lattice.

(v) Determine the price of a zero coupon, default-free two period bond at each point on the lattice in such a way that the prices of the one-period and two-period bonds are in equilibrium at each point.

(vi) Determine the price of a zero coupon, default-free three-period bond at each point such that the prices of the one-period, two-period and three-period bonds are in equilibrium at each point, and so on.

The term structure at any point on the lattice follows directly from the prices of the zero coupon bonds at that point.

Steps (v) and (vi) are achieved by introducing a parameter which reflects both the degree to which yields on longer bonds compensate for lack of liquidity, and the binomial probabilities associated with the movement of the spot rate.

Behavior of the one-period spot rate

We will consider models in which

- the behavior of the one-period spot rate at any time depends upon its current value and upon time, but not upon any earlier values of the spot rate.
- the change in the one-period spot rate at any time can be divided into two components -- a deterministic component reflecting a general trend in spot rate movement, and a stochastic component reflecting random and symmetric fluctuation around the trend.

Mathematically we can express the behavior of the spot rate in such models in the following terms

$$\Delta r(n) = f[n,r(n)]\Delta n + g[n,r(n)]\Delta I \qquad (1)$$

where $r(n)$ is the spot rate at time n

Δn represents a (constant) single time increment

$\Delta r(n) = r(n + \Delta n) - r(n)$

$f[n,r(n)]$ and $g[n,r(n)]$ are functions to be specified

and $P(\Delta I = 1) = 1/2$

$P(\Delta I = -1) = 1/2$

It is a property of models of this kind that

$$E[\Delta r(n)] = f[n,r(n)] \qquad (2)$$

$$Var[\Delta r(n)] = \{g[n,r(n)]\}^2 \qquad (3)$$

In words, the function f (which represents the general trend in spot rate movement) gives the expected value of the change in the spot rate at any time while the function g (which represents the random fluctuation about the trend) gives the standard deviation of that change.

A wide variety of models can be generated from equation (1). We will focus on a subclass of such models defined by

$$\Delta r(n) = \alpha[\gamma - r(n)]\Delta n + \rho\sqrt{r(n)}\,\Delta I \qquad (4)$$

where α, γ and ρ are constants to be specified.

This model has the following properties

(i) The spot rate ultimately approaches the value γ .

This is apparent from the first term on the right hand side of equation (4). The further $r(n)$ is from γ, the stronger the force which tends to draw it back. For given values of $r(n)$ and γ, the parameter α governs the strength of the tendency of $r(n)$ to revert to γ.

(ii) The volatility of the spot rate increases as the spot rate increases. The second term on the right of equation (4) indicates that the variance of the random fluctuation of the spot rate is proportional to the current value of the spot rate.

(iii) The spot rate can never become negative provided that the parameters α, γ and ρ are chosen so that

$$\frac{4\alpha\gamma(1-\alpha)}{\rho^2} \geq 1 \qquad (5)$$

(iv) The model does not automatically create a closed lattice of spot rates because an upstate move followed by a downstate move does not lead to the same spot rate as a downstate move followed by an upstate move.[1]

The issue of closure affects only the interior points of the lattice -- it has no effect upon the upper and lower bounds. We have adopted a mathematical expedient which closes the lattice by choosing a point which always lies between the points generated by equation (4). The method is discussed in detail in the appendix.

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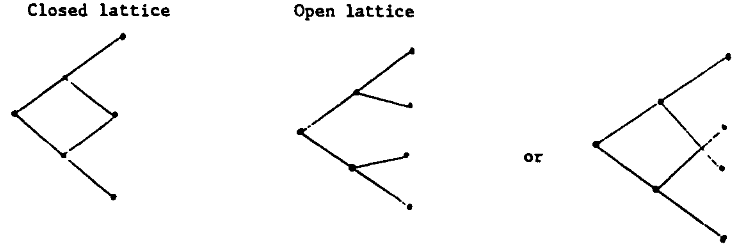
[1] The diagrams below illustrate the distinction between open and closed binomial lattices. It is apparent that if the values taken by the spot rate can be represented by a closed lattice, then the value at any time depends only upon the number of upstate moves (where we arbitrarily designate moves in one direction as "upstate" and those in the other as "downstate"), and not upon the order in which they occur. If the values are represented by an open lattice, however, the order of occurrence is important. While an open lattice model has greater generality, it is achieved at a price; at time n the spot rate may take any of 2^n values, whereas in a closed lattice model it is confined to n+1 values.

Closed lattice   Open lattice



Behavior of the term structure

The yields to maturity available on default-free zero-coupon bonds of all maturities, when taken together form the current term structure. The term structure is often shown as a graph of yield versus maturity.

The relationship between the price, P(T), of a default-free zero-coupon bond which matures for $1 in T periods and the yield to maturity, R(T), on that bond is given by the formula

$$R(T) = \frac{-1}{T} \log_e P(T) \qquad (6)$$

Knowing the price of all default-free zero-coupon bonds is equivalent, therefore, to knowing the yield to maturity on each of these bonds. Knowledge of either enables us to specify the term structure.

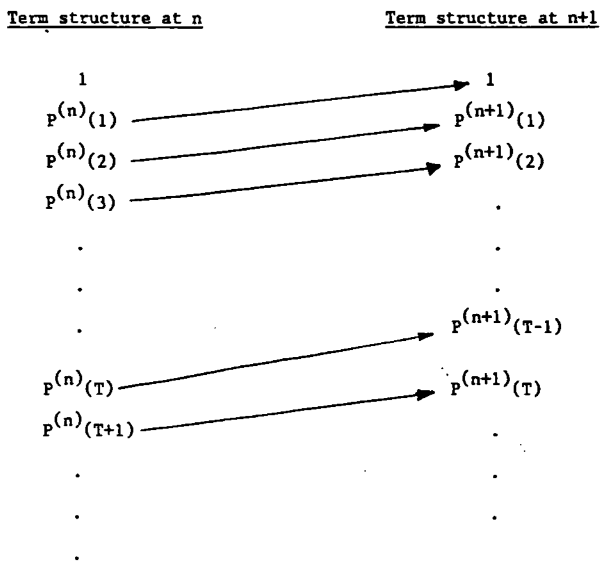
Equilibrium term structures

The term structure is said to be in equilibrium if there is no opportunity for risk-free arbitrage between any two distinct portfolios of default-free zero-coupon bonds. In a single state variable model, the condition extends to any security or portfolio of securities whose value at any time is determined by the value of the state variable at that time.

In the context of a binomial model, an equivalent statement of the equilibrium condition is as follows. Equilibrium requires the current value of any portfolio which is composed of securities governed by the state variable, to be strictly positive whenever each of its two possible values at the end of the next time increment is non-negative and at least one of them is strictly positive.

Evolution of the term structure over time

The relationship between the prices of default-free zero coupon bonds of various maturities does not remain constant over time -- as the one-period spot rate changes, the graph of yield versus maturity takes different shapes. Over the single time interval [n,n+1], the price of a zero coupon bond which matures at time n+T goes from $P^{(n)}(T)$ at time n, say, to $P^{(n+1)}(T-1)$ at time n+1. Over the same interval the term structure at time n (defined by the values $[P^{(n)}(T)]_{T=0}^{\infty}$) changes into the term structure at time n+1 (defined by the values $[P^{(n+1)}(T-1)]_{T=1}^{\infty}$) as shown below.



We wish to investigate possible evolutions of the term structure, subject to the constraint that the term structure remains in equilibrium at all times.

Within a binomial model, over the time interval [n,n+1], the price of a bond which matures at n+T will change from its current value of $P^{(n)}(T)$ to one of two possible values, which we will call $P_1^{(n+1)}(T-1)$ and $P_0^{(n+1)}(T-1)$.

Since at time n it is not known which of the values $P_1^{(n+1)}(T-1)$ and $P_0^{(n+1)}(T-1)$ will obtain at time (n+1), it is natural that the current price $P^{(n)}(T)$ should reflect both of them. We might expect therefore that $P^{(n)}(T)$ will take the form

$$P^{(n)}(T) = hP_1^{(n+1)}(T-1) + kP_0^{(n+1)}(T-1) \tag{7}$$

where h and k reflect the relative likelihood of the values $P_1^{(n+1)}(T-1)$ and $P_0^{(n+1)}(T-1)$

We can show mathematically that the evolving term structure will always be in equilibrium provided that the values h and k are the same for zero coupon bonds of <u>all</u> maturities at a given time. In other words all term structures generated by the model will be in equilibrium provided that h and k do not vary with T. This result is proved in the appendix.

The specific formula for $P^{(n)}(T)$ in terms of $P_0^{(n+1)}(T-1)$ and $P_1^{(n+1)}(T-1)$ is

$$P^{(n)}(T) = P^{(n)}(1)[\pi P_1^{(n+1)}(T-1) + (1-\pi) P_0^{(n+1)}(T-1)] \tag{8}$$

where in general π may vary with time and with the current value of the spot rate but may not vary with T.

Interpretation of π

If π were the probability of an upstate move (and $(1-\pi)$ were the probability of a downstate move), equation (8) would indicate that the price $P^{(n)}(T)$ at time n of a zero coupon bond is equal to the present value of the expected price of the bond at time n+1. In fact π is related to the binomial probability of an upstate move by the formula

$$\pi = q - Q\sqrt{q(1-q)} \tag{9}$$

where q is the probability of an upstate move and
Q is an elusive concept often referred to as the liquidity premium

Q generally takes values between 0 and 1 and measures the extent to which bond issuers have to increase rates on longer bonds in order to compensate for lack of liquidity. When $Q = 0, \pi = q$ and we have the case discussed above. In what follows Q is taken to be constant with regard to time, and q is taken to be $\frac{1}{2}$, so that

$$\pi = \frac{1}{2}(1 - Q) \tag{10}$$

Summary

The operation of the model can be summarized as follows

Input $r(0), \alpha, \gamma, \rho$

Process (calculation of spot rate lattice)

$$\Delta r(n) = \alpha[\gamma - r(n)]\Delta n + \rho\sqrt{r(n)}\Delta I$$

Output (spot rate lattice)

$\{r_i(n)\}$ $\quad n = 0,1,\ldots$
$\quad i = 0,1,\ldots n$

Input $\{r_i(n)\}, \pi$

Process (calculation of lattice of term structures)

$$P_i^{(n)}(0) = 1$$

$$P_i^{(n)}(1) = \exp\{-r_i(n)\}$$

$$P_i^{(n)}(T) = P_i^{(n)}(1)\,\{\pi P_{i+1}^{(n+1)}(T-1) + (1-\pi)P_i^{(n+1)}(T-1)\}$$

$$R_i^{(n)}(T) = \frac{-1}{T}\log_e P_i^{(n)}(T)$$

Output (lattice of term structures)

$\{P_i^{(n)}(T)\}, \quad \{R_i^{(n)}(T)\}$

The Ho-Lee binomial model

Ho and Lee [8] have proposed a single state variable binomial equilibrium model of the term structure. They have taken a different approach from that described above in that rather than define the behavior of the spot rate and derive the behavior of the term structure (subject to the equilibrium condition), they start from a given term structure. They then derive permissible movements in the prices of zero coupon bonds subject to the equilibrium condition and certain other assumptions designed to make the mathematics manageable. In this way they are able to obtain explicit expressions for the possible term structures at any time, and in particular to derive a formula for the corresponding values of the one-period spot rate.

Mathematical analysis reveals some peculiarities of the Ho-Lee model, however. It is shown in the appendix that

- If the initial term structure is such that yields approach a limiting value as maturities lengthen (and this is usually the case), then the one-period spot rate will ultimately increase without limit.

- All of the possible yield curves at a particular time are exactly parallel to each other; consequently the sequence of yield curve shapes is the same regardless of which path through the lattice is chosen. This is in sharp contrast to the model proposed in this paper in which the shape of the yield curve is highly dependent upon the value of the spot rate. The example in Section 4 illustrates how different paths through the lattice lead to different sequences of yield curve shapes.

- It is possible to obtain the Ho-Lee model by choosing an appropriate stochastic process of the form given in equation (1) to describe the behavior of the one-period spot rate and deriving the behavior of the term structure exactly as described above for the proposed model. Expressing the Ho-Lee model in this form enables us to examine the implicit assumptions regarding the behavior of the one-period spot rate. It is shown in the appendix that at any time the expected value of the change in the spot rate reflects the initial term structure but does not vary according to the current value of the spot rate (hence the spot rate cannot be drawn towards a long term value) and that the volatility of the spot rate is constant -- it does not change over time and does not depend on the current value of the spot rate.

3. Parameter estimation

Input to the model consists of an initial spot rate, $r(0)$, and the parameters ρ, γ, α and $\pi\ (= \frac{1}{2}(1 - Q))$

Of the parameters, ρ (which governs volatility) and γ (the ultimate value of the spot rate) correspond in a direct way to the user's convictions regarding spot rate behavior; α, which reflects the rate at which the spot rate can be expected to approach γ is less directly accessible, but can be obtained from the following relationship (proved in the appendix):

$$\alpha = 1 - \left(\frac{\varepsilon}{|r(0) - \gamma|} \right)^{1/n_0} \tag{11}$$

where n_0 is the number of periods required for the expected value of the spot rate to move from its current value $r(0)$ to within $\pm\ \varepsilon$ of the ultimate value γ, and,

$$|r(0) - \gamma| > \varepsilon > 0$$

Thus if the current of the spot rate is 0.05, the ultimate value is thought to be 0.08, and we expect the spot rate to reach 0.0795 in 10 periods, then $\alpha = 0.164$ is consistent with our expectations.

The parameter Q (implicit in the value chosen for π) does not have an obvious intuitive interpretation and is therefore not directly available from published data.

In order to assign a value to it, we suggest an empirical approach which makes use of the information in the current term structure.

Ideally, the user would proceed as follows.

(i) Establish values of ρ and γ which reflect his outlook regarding spot rate behavior.

(ii) Use (10) to derive a value of α consistent with his expectations.

(iii) Plot the current term structure (a graph of yield versus time to maturity for default free zero coupon bonds).

(iv) Run the model with the established values of ρ, γ and α to ascertain the value of Q for which the model's initial term structure most closely approaches the actual current term structure.

It should be noted that in step (iv) we do not attempt to reproduce the actual term structure exactly. In addition to information, the actual term structure also contains a certain amount of random "noise" which should not be allowed to influence the choice of model parameter values. By choosing a smooth curve with the same general configuration as that of the current term structure, we hope to include information from that term structure while eliminating much of the noise.

It may turn out to be impossible to obtain a good fit if the values of ρ, and γ are fundamentally inconsistent with the actual term structure. If this is the case, the user must decide whether his own views or those of the market (implicit in the actual term structure) should prevail. In practice we expect the procedure to be applied interactively. The user will input an initial set of parameter values; then through a series of iterations he will modify them until he obtains a parameter set which adequately balances his preconceptions against the information contained in the current term structure.

The graphs below illustrate the process. The yields on Treasury notes, published in the Wall Street Journal of February 22, 1988, were used as a proxy for an actual yield curve. (We recognize that Treasury notes are not zero coupon bonds, but for purposes of illustration, we ignore the distortions introduced by coupon payments.)

Graph 1 shows the model's initial term structure where

ρ = 0.070
γ = 0.088
α = 0.5
Q = 0.4 ,

and each binomial transition represents one year.

Graph 2 shows the model's initial term structure with

ρ = 0.070
γ = 0.088
α = 0.5
Q = 0

The change in parameter values has shifted the major part of the model's curve from above the actual yield curve to below it.

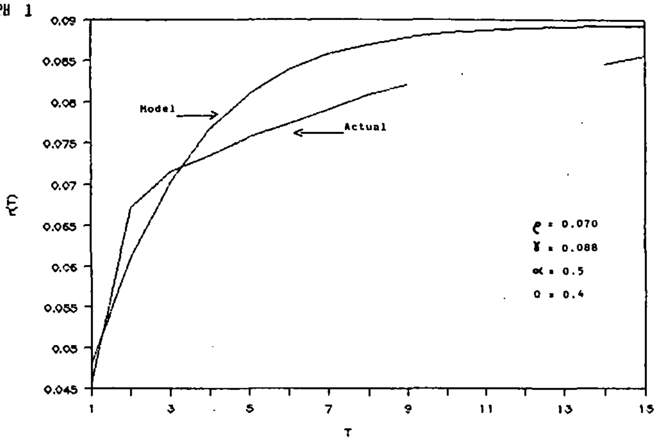
A better fit is shown in Graph 3 with

ρ = 0.070
γ = 0.088
α = 0.730
Q = 0

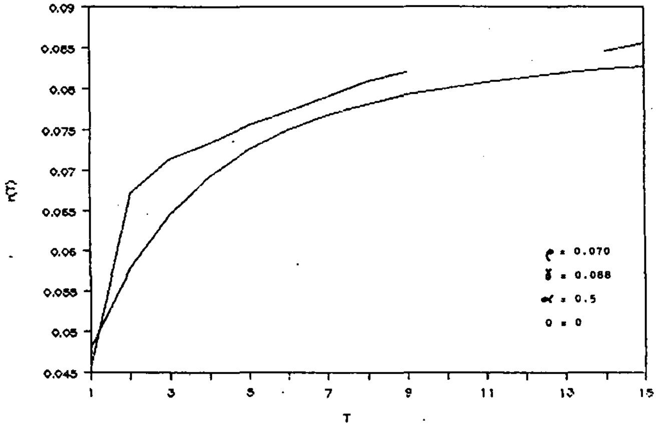
With practice the user will rapidly acquire a feel for the sensitivity of the model's initial yield curve to the values of the various parameters.

Since no formal curve fitting procedures were applied, the model parameter values for graph 3 do not necessarily provide the best possible fit to the actual term structure, but it is clear that the fit is reasonably close.

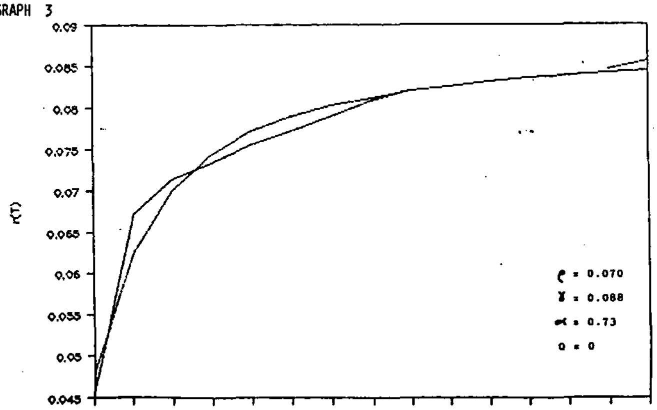
GRAPH 1



GRAPH 2



GRAPH 3



4. Examples

Suppose that the current value of the one-period spot rate is 5%, that the ultimate value is thought to be 8% and that the spot rate is expected to reach 7.7% in 8 periods. The following values are consistent with this situation.

$$r(0) = 0.05$$

$$\gamma = 0.08$$

$$\alpha = 0.25$$

$$\rho = 0.014$$

$$\pi = \frac{1}{2}$$

Exhibit 1 develops the one-period spot lattice generated by these values through five periods. Each point on the lattice corresponds to an entire term structure. A given path through the lattice therefore represents a particular evolution of the term structure over time. Two such paths have been marked on Exhibit 1 -- an upper path (the points in the rectangles) which shows the spot rate taking the larger of the two possible values at each transition, and a lower path (marked by ovals) in which the spot rate sometimes takes the larger value and sometimes takes the smaller. Both paths reflect the tendency of the spot rate to increase from 5% to 8%.

Exhibit 2 shows the evolution of the term structure which corresponds to the upper path. As the spot rate increases from 5% to 8% and beyond, the shape of the yield curve changes from non-inverted to humped to fully inverted. (The labels on the axes are somewhat illegible; the important thing is the sequence of shapes, however.)

Exhibit 3 shows the evolution of the term structure which corresponds to the lower path. In this case the yield curve remains non-inverted throughout.

Exhibit 4 shows the yield curves of the upper path all placed on the same set of axes. Exhibit 5 shows those of the lower path all on the same set of axes. These exhibits convey an idea of the relative levels and degrees of curvature of the various term structures in each path.

It should be noted that while the probability of each path is equal to $(1/2)^5$, the probabilities of the final points of each path are not equal. The probability of the final point of the upper path is $(1/2)^5$, because the only route through the lattice which reaches it is the one shown, whereas the probability of the final point of the lower path is $10 \times (1/2)^5$ because there are ten routes through the lattice which terminate at that point.

Exhibit 1 - Lattice of one period spot rates

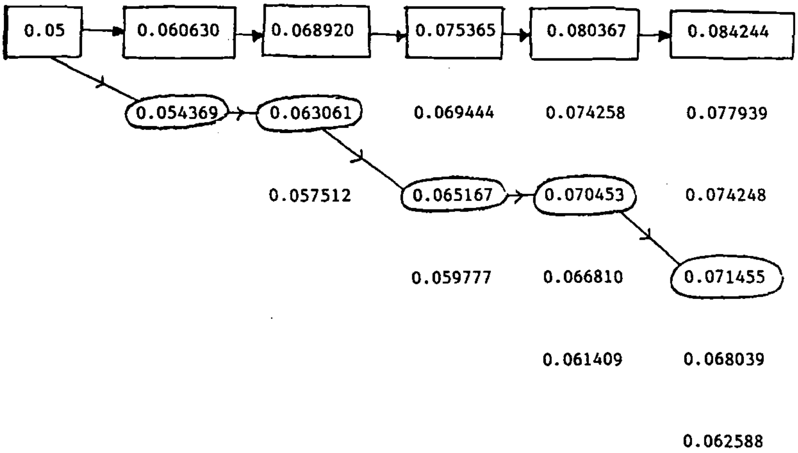
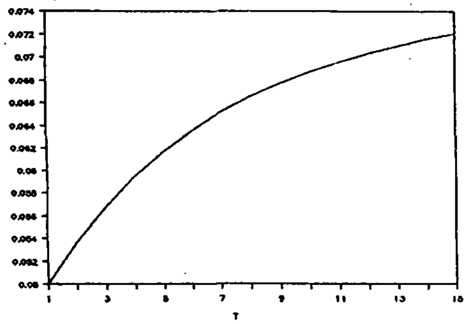
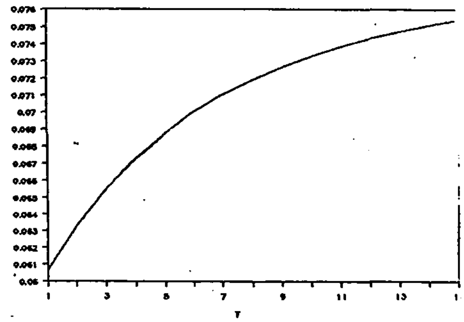


EXHIBIT 2 - EVOLUTION OF TERM STRUCTURE -- UPPER PATH



1.



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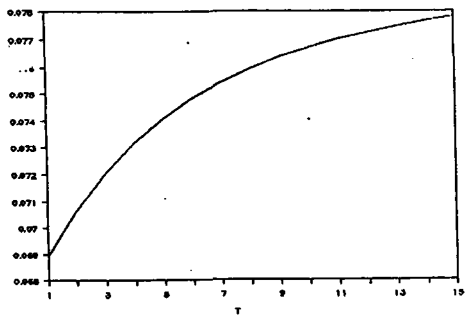
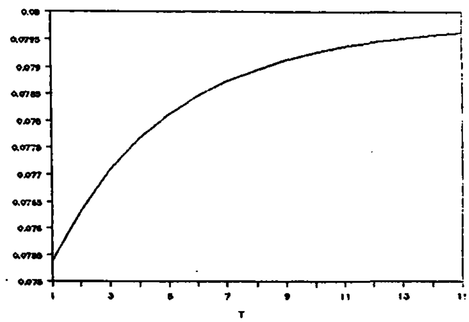


3.



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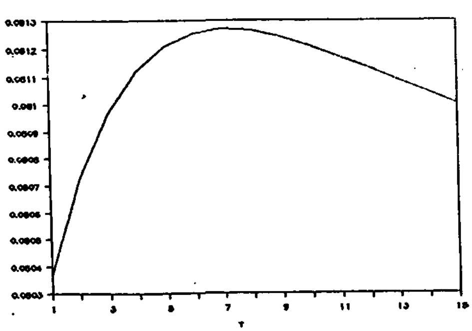

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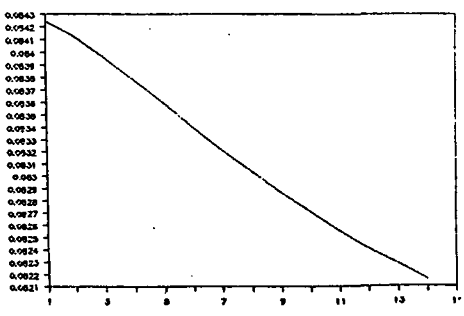

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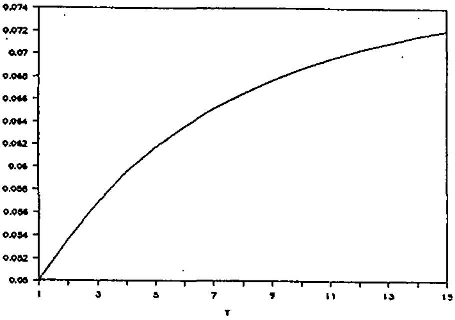
EXHIBIT 3 -- EVOLUTION OF TERM STRUCTURE -- LOWER PATH



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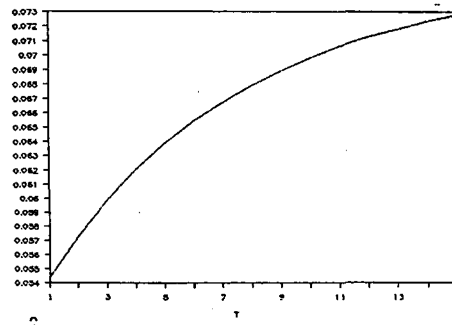

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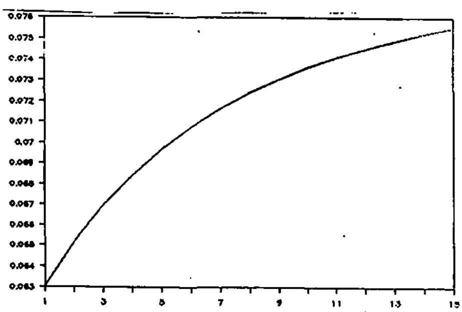

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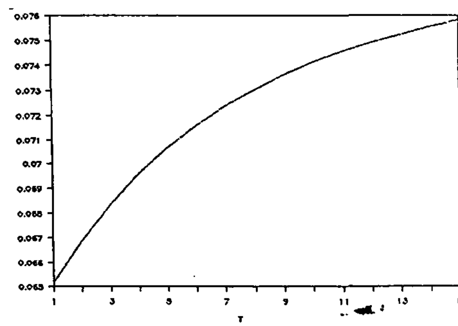

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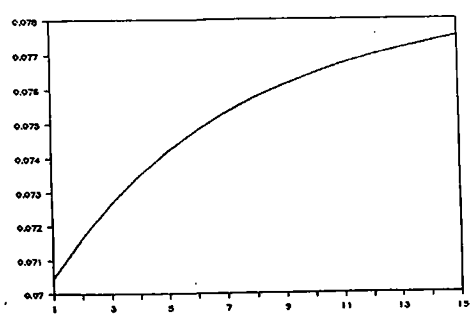

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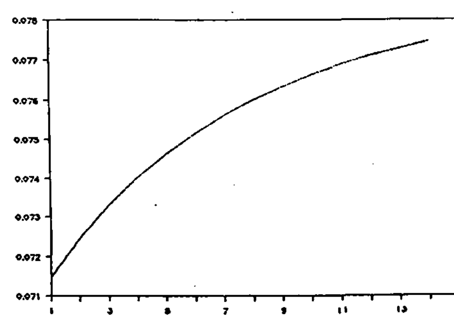


EXHIBIT 4 - EVOLUTIOH OF THE TERM STRUCTURE - UPPER PATH

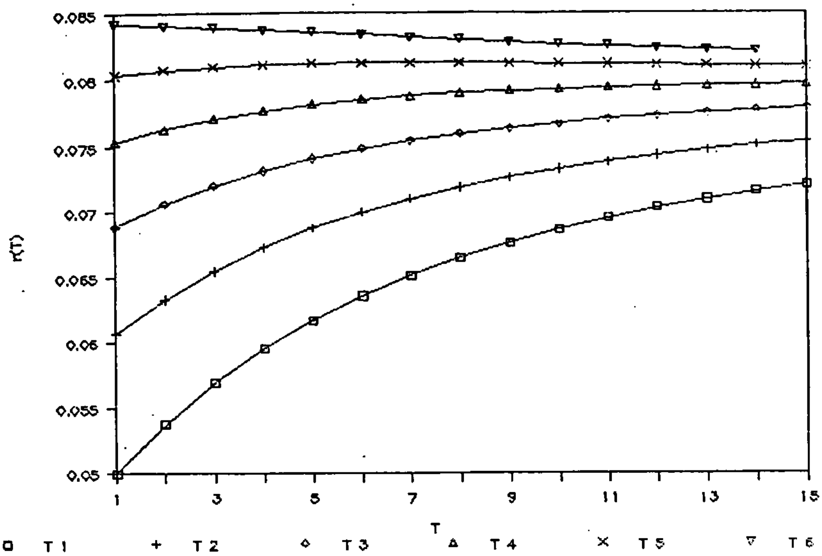
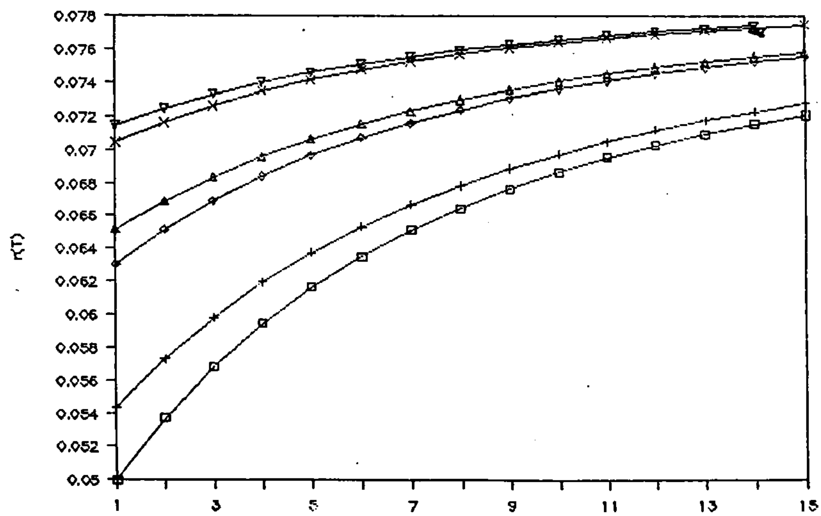


EXHIBIT 5 - EVOLUTION OF THE TERM STRUCTURE - LOWER PATH



Appendix A

1) If the behavior of the one-period spot rate is given by

$$\Delta r(n) = f[n,r(n)]\Delta n + g[n,r(n)]\Delta I \qquad (A.1.1)$$

where $P[\Delta I = 1] = 1/2$

$P[\Delta I = -1] = 1/2$

Then

$$E[\Delta r(n)] = f[n,r(n)]$$

$$Var[\Delta r(n)] = \{g[n,r(n)]\}^2$$

Proof

The distribution of Δr is given by

$$P[\Delta r = f + g] = 1/2$$

$$P[\Delta r = f - g] = 1/2$$

so,

$$E(\Delta r) = 1/2(f + g) + 1/2(f - g)$$

i.e. $$E(\Delta r) = f \qquad (A.1.2)$$

$$E(\Delta r^2) = 1/2[f^2 + 2fg + g^2) + 1/2[f^2 - 2fg + g^2)$$

$$= f^2 + g^2$$

and $$Var(\Delta r) = E(\Delta r^2) - [E(\Delta r)]^2$$

i.e. $$Var(\Delta r) = g^2 \qquad (A.1.3)$$

2) If the behavior of the spot rate is defined by

$$\Delta r(n) = \alpha[\gamma - r(n)]\Delta n + \rho\sqrt{r(n)}\Delta I \qquad (A.2.1)$$

Then,

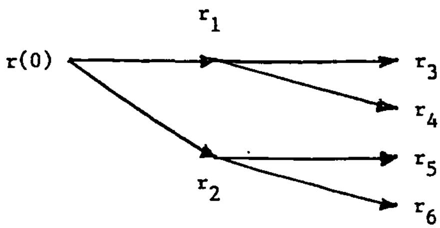
$$E[r(n)] = \gamma + (1 - \alpha)^n[r(0) - \gamma] \qquad (A.2.2)$$

and if $0 < \alpha < 2$

$$\lim_{n \to \infty} E[r(n)] = \gamma \qquad (A.2.3)$$

Proof

Consider two transitions



From (A.2.1)

$$r_1 = r(0) + \alpha[\gamma - r(0)] - \rho\sqrt{r(0)} \qquad (A.2.4)$$

$$r_2 = r(0) + \alpha[\gamma - r(0)] + \rho\sqrt{r(0)} \qquad (A.2.5)$$

$$r_3 = r_1 + \alpha(\gamma - r_1) - \rho\sqrt{r_1} \qquad (A.2.6)$$

$$r_4 = r_1 + \alpha(\gamma - r_1) + \rho\sqrt{r_1} \qquad (A.2.7)$$

$$r_5 = r_2 + \alpha(\gamma - r_2) - \rho\sqrt{r_2} \qquad (A.2.8)$$

$$r_6 = r_2 + \alpha(\gamma - r_2) + \rho\sqrt{r_2} \qquad (A.2.9)$$

Then,

$$E[r(1)] = \frac{1}{2}[r_1 + r_2]$$

i.e.

$$E[r(1)] = r(0)(1 - \alpha) + \alpha\gamma \qquad (A.2.10)$$

from (A.2.4) and (A.2.5)

$$E[r(2)] = \frac{1}{4}[r_3 + r_4 + r_5 + r_6]$$

$$= \frac{1}{4}[2r_1(1 - \alpha) + 2r_2(1 - \alpha) + 4\alpha\gamma]$$

from (A.2.6) - (A.2.9)

i.e. $$E[r(2)] = (1 - \alpha)E[r(1)] + \alpha\gamma \qquad (A.2.11)$$

Proceeding in this way, it is clear that in general

$$E[r(n)] = (1 - \alpha)E[r(n - 1)] + \alpha\gamma \qquad (A.2.12)$$

From (A.2.12),

$$E[r(n)] = (1 - \alpha)^n E[r(0)] + \alpha\gamma[1 + (1 - \alpha) + \ldots + (1 - \alpha)^{n-1}]$$

$$= (1 - \alpha)^n r(0) + \alpha\gamma\left[\frac{1 - (1 - \alpha)^n}{\alpha}\right]$$

i.e. $$E[r(n)] = \gamma + (1 - \alpha)^n[r(0) - \gamma] \qquad (A.2.13)$$

(A.2.3) follows directly from (A.2.13).

3) If the behavior of the spot rate is given by (A.2.1), and the number of periods required for the expected value of the spot rate to move from its current value r(0) to within $\pm\varepsilon$ of the ultimate value γ, is n_0, then

$$\alpha = 1 - \left(\frac{\varepsilon}{|r(0) - \gamma|}\right)^{1/n_0} \qquad (A.3.1)$$

where $|r(0) - \gamma| > \varepsilon > 0$

and $0 < \alpha < 1$

Proof

(i) Assume that $r(0) > \gamma$

From (A.2.2), if

$$E[r(n_0)] = \gamma + \varepsilon ,$$

$$\gamma + (1 - \alpha)^{n_0}[r(0) - \gamma] = \gamma + \varepsilon \qquad (A.3.2)$$

Which gives

$$\alpha = 1 - \left(\frac{\varepsilon}{r(0) - \gamma}\right)^{1/n_0} \qquad (A.3.3)$$

(ii) A similar result holds when $r(0) < \gamma$

4) The stochastic process (A.2.1) will not generate negative values of the spot rate provided that

$$\frac{4\alpha\gamma(1-\alpha)}{\rho^2} \geq 1 \qquad \text{(A.4.1)}$$

Proof

Suppose that the current value of the spot rate is r(0). From (A.2.1), the next value will be either

$$r(1) = r(0) + \alpha[\gamma - r(0)] + \rho\sqrt{r(0)} \qquad \text{(A.4.2)}$$

or

$$r(1) = r(0) + \alpha[\gamma - r(0)] - \rho\sqrt{r(0)} \qquad \text{(A.4.3)}$$

If neither of these values is to be negative, then

$$r(0)(1-\alpha) + \alpha\gamma - \rho\sqrt{r(0)} \geq 0 \qquad \text{(A.4.4)}$$

Consider the function

$$f(x) = x(1-\alpha) - \rho\sqrt{x} + \alpha\gamma \qquad \text{(A.4.5)}$$

We wish to find the condition that

$$f(x) \geq 0 \text{ for all } x \geq 0 \qquad \text{(A.4.6)}$$

(A.4.6) will be satisfied if f(x) has a minimum value at $x = x_0$, say, and $f(x_0) \geq 0$

From (A.4.5),

$$f'(x) = (1 - \alpha) - 1/2\ \rho/\sqrt{x} \tag{A.4.7}$$

and hence f(x) has a minimum point at

$$x_0 = \frac{\rho^2}{4(1 - \alpha)^2} \tag{A.4.8}$$

$$f\left(\frac{\rho^2}{4(1 - \alpha)^2}\right) = \alpha\gamma - \frac{\rho^2}{4(1 - \alpha)} \tag{A.4.9}$$

and therefore (A.4.6) is satisfied provided that

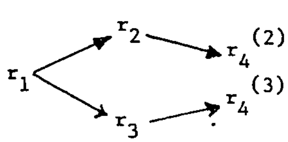
$$\alpha\gamma \geq \frac{\rho^2}{4(1 - \alpha)} \tag{A.4.10}$$

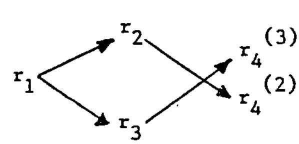
5) The stochastic process defined by

$$\Delta r = \alpha(\gamma - r)\Delta n + \rho\sqrt{r}\,\Delta I \tag{A.5.1}$$

does not generate a closed lattice of spot rates. The value of the spot rate after an upstate and a downstate move depends upon the order in which those moves occurred. We have eliminated this difficulty by modifying the form taken by the stochastic process for the one-period spot rate. The following example will illustrate the method.

Consider two transitions, one of which is upstate and the other downstate. Suppose that r_1 is the current one-period spot rate, that at time 1 the spot rate is either r_2 or r_3 and that at time 2 the spot rate is $r_4^{(2)}$ if at time 1 it was r_2, and $r_4^{(3)}$ if at time 1 it was r_3. The diagram should clarify the definitions.





Results

(i) If we replace the function $\rho\sqrt{r}$ in equation (A.5.1) with the function $(1-\alpha)(\sqrt{r_2}-\sqrt{r_3})\sqrt{r}$ for the second transition, we obtain a unique value of r_4, regardless of path.

(ii) With the modification described above, the unique value r_4 will always lie between $r_4^{(3)}$ and $r_4^{(3)}$, regardless of which of these is larger.

Proof

(i) Consider the second transition, assuming that the value of the spot rate at time 1 is r_2, using the modification described above.

$$\Delta r_2 = \alpha(\gamma - r_2) - (1-\alpha)(\sqrt{r_2} - \sqrt{r_3})\sqrt{r_2}$$

$$r_4 = r_2 + \Delta r_2$$

$$= r_2 + \alpha(\gamma - r_2) - (1-\alpha)(\sqrt{r_2} - \sqrt{r_3})\sqrt{r_2}$$

$$r_4 = \alpha\gamma + (1-\alpha)\sqrt{r_3}\sqrt{r_2} \qquad (A.5.2)$$

Consider the second transition, assuming that the value of the spot rate at time 1 is r_3, and use the modification

$$\Delta r_3 = \alpha(\gamma - r_3) + (1 - \alpha)(\sqrt{r_2} - \sqrt{r_3})\sqrt{r_3}$$

$$r_4 = r_3 + \Delta r_3$$

$$= r_3 + \alpha(\gamma - r_3) + (1 - \alpha)(\sqrt{r_2} - \sqrt{r_3})\sqrt{r_3}$$

i.e. $$r_4 = \alpha\gamma + (1 - \alpha)\sqrt{r_2}\sqrt{r_3} \qquad (A.5.3)$$

Since (A.5.2) and (A.5.3) give the same expression, the modification achieves the required result.

(ii) Using (A.5.1) unmodified for the second transition, assuming that the value of the spot rate at time 1 is r_2,

$$r_4^{(2)} = r_2 + \alpha(\gamma - r_2) - \rho\sqrt{r_2}.$$

i.e. $$r_4^{(2)} = r_2(1 - \alpha) - \rho\sqrt{r_2} + \alpha\gamma \qquad (A.5.4)$$

Similarly,

$$r_4^{(3)} = r_3(1 - \alpha) + \rho\sqrt{r_3} + \alpha\gamma \qquad (A.5.5)$$

From (A.5.3) and (A.5.4)

$$r_4 - r_4^{(2)} = \sqrt{r_2}[\rho - (1 - \alpha)(\sqrt{r_2} - \sqrt{r_3})] \qquad (A.5.6)$$

From (A.5.3) and (A.5.5)

$$r_4 - r_4^{(3)} = \sqrt{r_3}[(1 - \alpha)(\sqrt{r_2} - \sqrt{r_3}) - \rho] \qquad (A.5.7)$$

From (A.5.6) and (A.5.7), if

$\rho < (1 - \alpha)(\sqrt{r_2} - \sqrt{r_3})$, then

$$r_4^{(2)} > r_4 > r_4^{(3)} \quad \text{(A.5.8)}$$

and if

$\rho > (1 - \alpha)(\sqrt{r_2} - \sqrt{r_3})$, then

$$r_4^{(3)} > r_4 > r_4^{(2)} \quad \text{(A.5.9)}$$

and if

$\rho = (1 - \alpha)(\sqrt{r_2} - \sqrt{r_3})$

$$r_4^{(3)} = r_4 = r_4^{(2)} \quad \text{(A.5.10)}$$

and therefore r_4 always lies between $r_4^{(2)}$ and $r_4^{(3)}$

6. (i) The term structures defined by $\{P_i^{(n)}(T)\}$ where $P_i^{(n)}(T)$ is the price at time n and state i of a zero coupon default free bond which matures for 1 at time (n+T) and

$$n = 0,1,...$$
$$i = 0,1,..., n$$
$$T = 0,1,...$$

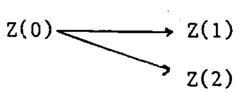
are in equilibrium if and only if there exists π, $(0<\pi<1)$, possibly dependent on i and n, but <u>not</u> dependent on T such that

$$P_i^{(n)}(T) = P_i^{(n)}(1)\ [\pi\ P_{i+1}^{(n+1)}(T-1) + (1-\pi)P_i^{(n+1)}(T-1)] \qquad (A.6.1)$$

(ii) The principle of equilibrium (that risk free gain is prohibited) can be extended to all securities whose price is governed by the value of the state variable. (A.6.1) then applies to all securities with the definition of T generalized to represent an index whose values distinguish one security from another. As in (i), π is the same for all securities at a particular state and time.

<u>Proof</u>

Let $\mathcal{B}$ denote the set of all default free zero coupon bonds. Let Z be any member of $\mathcal{B}$. Consider a single binomial transition and let Z(0) be the current price of Z; at the end of the transition the price of Z will be either Z(1) or Z(2).



Assume that bonds can be held in any fractional denominations and define a <u>portfolio</u> to be a linear combination of zero coupon bonds. That is, for any portfolio W we can write

$$W = \sum_{i \in I_W} a_i\ Z_i \qquad (A.6.2)$$

where $\{Z_i : i \in I_W\}$ is a subset of $\mathcal{B}$

and the $\{a_i\}$ are real numbers.

We will show that

1) If the prices of all bonds in $\mathcal{B}$ satisfy the relation

$$Z(0) = U(0)\,[\pi\, Z(1) + (1 - \pi)\, Z(2)\,] \qquad \text{(A.6.3)}$$

where U(0) is the current price of a bond which matures for 1 at the end of one transition and

π takes the same value for all bonds $(0<\pi<1)$

then all portfolios W are such that

$$W(1) > 0 \text{ and } W(2) \geq 0 \quad \Rightarrow \quad W(0) > 0 \qquad \text{(A.6.4)}$$

That is, risk free gain is excluded.

2) If risk free gain is excluded, (i.e. all portfolios W satisfy (A.6.4)), then the prices of all bonds in $\mathcal{B}$ satisfy (A.6.3).

1) Let W be an arbitrary portfolio given by (A.6.2)

If the prices of all bonds in $\mathcal{B}$ satisfy (A.6.3), then

$$W(0) = \sum_{i\varepsilon I_W} a_i\, Z_i(0)$$

$$= \sum_{i\varepsilon I_W} a_i U(0)\{\pi\, Z_i(1) + (1 - \pi)\, Z_i(2)\}$$

$$= \pi\, U(0) \sum_{i\varepsilon I_W} a_i Z_i(1) \;+\; (1 - \pi)\, U(0) \sum_{i\varepsilon I_W} a_i Z_i(2)$$

i.e., $$W(0) = U(0)\,\{\pi W(1) + (1 - \pi)\, W(2)\} \qquad \text{(A.6.5)}$$

From (A.6.5), if $W(1) > 0$ and $W(2) \geq 0$, then $W(0) > 0$.

2) Given any bond T and a bond, U, which matures for 1 at the end of one transition, we can form a portfolio which takes the same values at the end of one transition as any specified bond S. That is, for any bond S, there exist real numbers a and b such that

$$a + bT(1) = S(1) \qquad (A.6.6.)$$

$$a + bT(2) = S(2) \qquad (A.6.7.)$$

If risk free gain is to be excluded from the system, we require that

$$a\,U(0) + bT(0) = S(0) \qquad (A.6.8.)$$

For any T, there exists a real number π such that

$$T(0) = U(0)\ \{\pi\ T(1) + (1 - \pi)\ T(2)\} \qquad (A.6.9)$$

From (A.6.6) and (A.6.7),

$$a = \frac{S(2)T(1) - S(1)T(2)}{T(1) - T(2)} \qquad (A.6.10)$$

$$b = \frac{S(1) - S(2)}{T(1) - T(2)} \qquad (A.6.11)$$

Substituting from (A.6.9), (A.6.10) and (A.6.11) into (A.6.8), we see that if risk free gain is excluded, then,

$$S(0) = U(0)\{\pi S(1) + (1 - \pi)S(2)\} \qquad (A.6.12)$$

In other words, if risk free gain is excluded, there exists a real number π such that every bond Z in $\mathcal{B}$ satisfies

$$Z(0) = U(0)\{\pi Z(1) + (1 - \pi)Z(2)\} \qquad (A.6.13)$$

With a change of notation, (A.6.1.) follows.

Appendix B   Ho-Lee Model

1) Equation (25) of the Ho-Lee paper gives

$$E[r(n)] = \log_e\left(\frac{P(n)}{P(n+1)}\right) + \log_e [\pi\delta^{-n} + (1-\pi)] + nq\log_e\delta \qquad (B.1.1)$$

where

E[r(n)] is the expected value of the one-period spot rate at n,

P(n) is the price at time zero of a zero-coupon, default-free bond which matures for 1 at n,

δ is a constant such that $0 < \delta < 1$, and

q is the probability of an upstate move

(B.1.1) can be written,

$$E[r(n)] = \log_e\left(\frac{P(n)}{P(n+1)}\right) + \log_e[\pi + (1-\pi)\delta^n] - n(1-q)\log_e\delta \qquad (B.1.2)$$

We will consider the behavior of each of the terms of (B.1.2) as n gets large.

1st Term

Since

$$R(n) = -\frac{1}{n}\log_e P(n) \qquad (B.1.3)$$

where R(n) is the yield to maturity at time zero of a bond which matures for 1 at n,

$$\log_e\left(\frac{P(n)}{P(n+1)}\right) = (n+1)R(n+1) - nR(n) \qquad (B.1.4)$$

If the initial term structure is such that yields approach a limiting value as maturities lengthen, i.e.

$$\lim_{n \to \infty} R(n) = R \quad \text{then,} \tag{B.1.5}$$

$$\lim_{n \to \infty} \log_e\left(\frac{P(n)}{P(n+1)}\right) = R \tag{B.1.6}$$

2nd Term

$$\log_e[\pi + (1 - \pi)\delta^n] \longrightarrow \log_e \pi \quad \text{as} \quad n \longrightarrow \infty$$

3rd Term

Since $0 < \delta < 1$ and q and δ are fixed with regard to time,

$-n(1-q)\log_e\delta$ is a positive linear term in n.

It is apparent, then, that according to the Ho-Lee model, if $\lim_{n \to \infty} R(n) = R$,

$$E[r(n)] \to R + \log_e \pi - n(1 - q)\log_e \delta \,, \tag{B.1.7}$$

as n gets large.

(B.1.7) implies that according to the Ho-Lee model, the one-period spot rate can be expected ultimately to increase as a straight line over time, with slope equal to $-(1 - q)\log_e\delta$

2) Parallel yield curves in the Ho-Lee model

Equation (22) of the Ho-Lee paper gives

$$P_i^{(n)}(T) = \frac{P(T+n)h(T+n-1)h(T+n-2)...h(T)}{P(n) \quad h(n-1)h(n-2)...h(1)} \delta^{T(n-i)} \tag{B.2.1}$$

Since

$$R_i^{(n)}(T) = \frac{-1}{T} \log_e P_i^{(n)}(T) \tag{B.2.2}$$

where $R_i^{(n)}(T)$ is the yield to maturity at time n and state i of a bond which matures for 1 at time n + T

$$R_i^{(n)}(T) = \frac{-1}{T} \log_e\left(\frac{P(T+n)h(T+n-1)h(T+n-2)...h(T)}{P(n) \quad h(n-1)h(n-2)...h(1)}\right) \frac{-1}{T} \log_e \delta^{T(n-i)} \tag{B.2.3}$$

Let

$$G(T,n) = \frac{-1}{T} \log_e\left(\frac{P(T+n)h(T+n-1)...h(T)}{P(n) \quad h(n-1)...h(1)}\right)$$

Then (B.2.3) can be written

$$R_i^{(n)}(T) = G(T,n) - (n-i)\log_e \delta \tag{B.2.4}$$

From (B.2.4)

$$R_i^{(n)}(T) - R_{i-1}^{(n)}(T) = \log_e \delta \tag{B.2.5}$$

It is apparent from (B.2.5) that at time n, the yield curve at any state i is parallel to the yield curve at state (i-1). It follows that all yield curves at time n are parallel.

3) Ho-Lee model as a special case of equation (A.1.1)

Equation (24) of the Ho-Lee paper gives

$$r_i(n) = \log_e\left(\frac{P(n)}{P(n+1)}\right) + \log_e\,[\pi\delta^{-n} + (1-\pi)] + i\,\log_e\delta \qquad \text{(B.3.1)}$$

where $r_i(n)$ is the one-period spot rate at time n if i upstate moves have occurred.

Result

The Ho-Lee model with q = 1/2 is a special case of the class of models given by

$$\Delta\, r_i(n) = f_i(n)\Delta n \;+\; g_i(n)\,\Delta I \qquad \text{(B.3.2)}$$

with

$$f_i(n) = \log_e\left(\frac{P(n+1)P(n+1)}{P(n)P(n+2)}\right) + \log_e\left(\frac{\pi + (1-\pi)\,\delta^{n+1}}{\pi + (1-\pi)\,\delta^{n}}\right) - \frac{1}{2}\log_e\delta \qquad \text{(B.3.3)}$$

and

$$g_i(n) = \frac{-1}{2}\,\log_e\delta \qquad \text{(B.3.4)}$$

Proof

To prove the result, it is sufficient to show that the spot rate process (B.3.2) generates the spot rate lattice defined by (B.3.1)

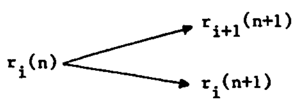
Let

$$H(n) = \log_e\left(\frac{P(n)}{P(n+1)}\right) + \log_e[\pi\delta^{-n} + (1-\pi)]$$

Then (B.3.1) can be written

$$r_i(n) = H(n) + i\,\log_e\delta \qquad \text{(B.3.5)}$$

Consider a one-period transition



To generate the lattice given by (B.3.5) using (B.3.2) we require that

$$r_{i+1}(n+1) \quad = \quad r_i(n) + f_i(n) - g_i(n) \qquad \text{(B.3.6)}$$

and

$$r_i(n+1) \quad = \quad r_i(n) + f_i(n) + g_i(n) \qquad \text{(B.3.7)}$$

(B.3.6) and (B.3.7) can be written

$$H(n+1) + (i+1)\log_e \delta = H(n) + i\log_e \delta + f_i(n) - g_i(n) \qquad \text{(B.3.8)}$$

$$H(n+1) + i\log_e \delta \quad = H(n) + i\log_e \delta + f_i(n) + g_i(n) \qquad \text{(B.3.9)}$$

which give

$$f_i(n) \quad = H(n+1) - H(n) + \frac{1}{2} \log_e \delta \qquad \text{(B.3.10)}$$

$$g_i(n) \quad = \frac{-1}{2} \log_e \delta \qquad \text{(B.3.11)}$$

(B.3.3) follows directly from (B.3.10)

Implications

(i) It follows from this result that in the Ho-Lee model, the expected change in the one-period spot rate varies with time, but cannot vary with the current value of the spot rate (i.e. it varies with n, but not with i). The variation with time is governed to a great extent by the initial term structure. Properties such as the tendency to drift toward an ultimate value are therefore excluded from this model.

(ii) The volatility of the spot rate (represented by the function g) is fixed -- it does not vary with time, and it does not vary with the current value of the spot rate.

(iii) (B.3.3) can be written,

$$f_i(n) = (n+2)R(n+2) - 2(n+1)R(n+1) + nR(n)$$
$$+ \log_e[\pi + (1-\pi)\delta^{n+1}] - \log_e[\pi + (1-\pi)\delta^{n}]$$
$$\frac{-1}{2} \log_e \delta \qquad \text{(B.3.12)}$$

If the initial term structure is such that

$$\lim_{n \to \infty} R(n) = R, \text{ we have}$$

$$\lim_{n \to \infty} f_i(n) = \frac{-1}{2} \log_e \delta \qquad \text{(B.3.13)}$$

The ultimate value of the expected change in the one-period spot rate is therefore constant [and equal to $g_i(n)$] which leads to the ultimate linear increase of $E[r(n)]$.

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