Some Remarks on Demography

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There are three textbooks for the Society of Actuaries Course 161 examination, and they do not use the same set of notation. We hope that this note will help alleviate some of the problems.

Actuarial Mathematics (AM) and *Demography Through Problems (DTP)* present stable population theory in terms of continuous models, while *Demographic Techniques (DT)* uses a discrete model. The symbol r in *DTP* denotes the instantaneous growth rate of the population, but the same symbol in *DT* denotes the annual growth rate. (Recall that, in Compound Interest, the symbol δ denotes the force of interest and i the effective annual interest rate; a similar convention does not exist in demography.) In *AM*, the symbol for the instantaneous growth rate is R (p. 526), because the *adjustment coefficient* in Risk Theory is denoted by R; also see Section 5 in Chapter 8 of *An Introduction to Mathematical Risk Theory* by H.U. Gerber.

The Sharpe-Lotka Theorem, as stated on page 101 of *DT*, is that "a closed population which experiences constant age specific fertility and mortality will, *no matter what its initial age distribution*, eventually develop a constant age distribution and will eventually increase in size at a constant rate." (This "most important theorem in the mathematics of population" is also mentioned in section 6.7 on page 94 of *DT*.) A proof of the theorem is not part of the syllabus; however, it can be inferred from the **Remarks** on page 528 of *AM*. It can be shown that, as t tends to $+\infty$, the dominant term of the general solution to equation (18.6.3) is given by (18.6.4). (Cf. formula (12.7.1) on page 366 of *AM*.) Also see the solution to #5.26 on page 106 of *DTP*. The first rigorous mathematical analysis of this result was given by W. Feller in the path-breaking paper "On the Integral Equation of Renewal Theory," *Annals of Mathematical Statistics* 12 (1941).

It follows from the Sharpe-Lotka Theorem that each population with a net maternity function, which depends only on the age of mothers, has an *intrinsic* or *inherent* rate of growth. In *DTP* (p. 56, p. 80) r is the number that satisfies the equation

$$\int_{\alpha}^{\beta} e^{-rx} {}_{x}p_{0}\, m(x)\, dx = 1,$$

while in *DT* (p. 105) r is the number that is the solution of

$$\sum_x (1+r)^{-x}\,{}_xp_0\,f_x = 1.$$

In *AM* (p. 526), R is defined by the equation

$$\int_0^\infty e^{-Rx}\phi(x)\,dx = 1.$$

On page 86 of *DTP* the symbol R_i is defined as the integral

$$\int x^i\,{}_xp_0\,m(x)dx,$$

while on page 105 of *DT* the same symbol denotes the sum

$$\sum_x x^i\,{}_xp_0\,f_x.$$

Note that equation (1) on page 79 of *DTP* is identical to equation (3) on page 105 of *DT* even though the symbols have slightly different meanings.

The *net reproduction rate* is the expected number of female babies that will be born to a female baby who throughout her life will be subject to a set of fixed age-specific mortality and fertility rates. In a continuous model, the net reproduction rate is R_0; in discrete models, there are several slightly different formulas for estimating the net reproduction rate (*DT*, p. 93 and p. 105; *DTP*, p. 106, #27).

The ratio R_1/R_0 is called the *mean length of a generation* in *DT* (p. 105) and is called the *mean age of childbearing in the* stationary *population* in *DTP* (#4.20). The *length of generation* in *DTP* (p. 80) is $(\log_e R_0)/r$, which is not the same as R_1/R_0 (unless $r = 0$). (In terms of the population density function defined in *AM*, the length of generation in a stable population is the number T such that the ratio $l(x, t+T-x)/l(x, t-x)$ equals to the net reproduction rate.) To solve #4.20 in *DTP*, put $h(s) = -\log_e[\psi(s)]$, where $\psi(s) = \int e^{-sx}\,{}_xp_0\,m(x)\,dx$ (*DTP*, #4.7). Then $h'(0)$ and $h'(r)$ are the mean age of childbearing in the stationary population and stable population, respectively. If r is close to zero, $h(r) - h(0) \approx r[h'(0) + h'(r)]/2$ (the trapezoidal rule). Since $h(0) = -\log_e R_0$ and $h(r) = 0$, the length of generation is approximately "the average of the mean age of childbearing in the stationary and in the stable populations."

The mean of a probability distribution is usually denoted by μ. However, in Actuarial Science, the symbol μ denotes the force of mortality. Thus in *DTP* the mean of a probability distribution may be denoted by m or $\bar{x}$. (In the solution to #3.53, the integral

$$\int_x^{x+5} (a-x-2.5)^2\,l(a)\,da$$

is called a variance, even though $\int l(a)\,da = {}_5L_x \neq 1$.)

On page 26 of *DTP* (which is not part of the syllabus), l(x) is defined as the **number** of lives at age x; however, on page 55, l(x) is defined as the **probability** of surviving from birth to age x, i.e., $l(x) = {}_xp_0$. These two definitions are not inconsistent because the *radix* l(0) can be chosen as one; however, for clarity, one may want to replace l(x) by l(x)/l(0) in some of the formulas in *DTP*. It should be noted that in the cases where one is interested in the function l(x) only for nonnegative integers x, l(x) is written as l_x (*DTP*, p. 26). In a continuous model, it is l(x)dx, not l(x), that gives the number of people at exact age x.

In the solution to #3.23 of *DTP* (p. 67) the function c(x) is defined as

$$c(x) = \frac{e^{-rx}l(x)}{\int_0^\infty e^{-ry}\,l(y)\,dy} = \frac{e^{-rx}l(x)}{l(0)/b} = be^{-rx}\,{}_xp_0.$$

In terms of the population density function as defined in *AM*, c(x) can be expressed as

$$c(x) = \frac{l(x, t-x)}{\int_0^\infty l(x, t-x)\,dx};$$

note that the variable t is cancelled out because the population is assumed to be stable.

In the solution to #3.29 of *DTP* (p. 68), the function ${}_nC_x$ is defined as

$${}_nC_x = \int_x^{x+n} c(y)\,dy$$

In this definition, ${}_\infty C_0 = 1$ and ${}_nC_x$ is a number between 0 and 1. In # 3.19, ${}_5C_{10} = 19118/149083$, ${}_5C_{20} = 9393/149083$, ${}_5C_{30} = 8025/149083$, etc.; the number 149083 in the denominators is not really needed since it gets cancelled in the calculations. Note that expression (18.4.5) of *AM* can be written as

$${}_{x_1-x_0}C_{x_0}.$$

It should be noted that, in the solution to #3.20 of *DTP* (p. 66), ${}_nC_x$ is defined as

$${}_nC_x = \int_x^{x+n} e^{-ry}\,l(y)\,dy.$$

Similarly, in #3.13 the symbol ${}_nC_x$ is used to denote the **number**, not the **proportion**, of people. The symbol C_x appears in the solutions to #3.12, #3.16, #3.22 and #3.26; it does not always mean ${}_1C_x$, the number or proportion of people who are aged x last birthday. On the other hand, the symbol

c_x may mean ${}_1C_x$; see the last sentence in the solution to #3.7 (p. 62).

The following theorem on cumulants, which can be found on page 335 of *AM*, is useful for solving many problems in Chapters 3 and 4 of *DTP*. (Also see #4.18 of *DTP*.) Let X be a random variable and

$$\log_e[E(e^{tX})] = \kappa_1 t + \kappa_2 t^2/2! + \kappa_3 t^3/3! + \ldots ;$$

then κ_1, κ_2 and κ_3 are the mean, variance and third moment about the mean, respectively, of X. An immediate application is #4.11, by considering $l(x)m(x)/\psi(0)$ as a probability density function. Another example is the solution to #3.14 and #3.21: Define

$$\theta(s) = \int_0^\infty e^{-sx}\, l(x)\,dx.$$

Then

$$\bar{x} = -\frac{\theta'(r)}{\theta(r)} = -\frac{d}{ds}\log_e[\theta(s)]\Big|_{s=r} = -\frac{d}{ds}\log_e\left[\frac{\theta(s)}{\theta(0)}\right]\Big|_{s=r}.$$

Since $\log_e[\theta(s)/\theta(0)] \approx -\kappa_1 s + \kappa_2 s^2/2! - \kappa_3 s^3/3!$, the formula in #3.21 follows. (Note that $\theta(0) = {}_\infty L_0 = l(0)\mathring{e}_0$ and $\theta(r) = l(0)/b$; in *DTP* $l(0)$ is usually assumed to be one.) The formulas for #3.29, #3.31 and #3.38 can also be derived in a similar way:

$$\log_e\left(\frac{{}_mC_x}{{}_nC_y}\right) = \log_e\left(\frac{{}_mL_x}{{}_nL_y}\right) + \log_e\int_x^{x+m} e^{-rt}\frac{l(t)}{{}_mL_x}dt - \log_e\int_y^{y+n} e^{-rt}\frac{l(t)}{{}_nL_y}dt.$$

A further example is #3.32: Since

$$m = -\frac{\theta'(r)}{\theta(r)} = -\frac{d}{ds}\log_e[\theta(r+s)]\Big|_{s=0} = -\frac{d}{ds}\log_e\left[\frac{\theta(r+s)}{\theta(r)}\right]\Big|_{s=0},$$

we have

$$\frac{dm}{dr} = -\frac{d^2}{ds^2}\log_e\left[\frac{\theta(r+s)}{\theta(r)}\right]\Big|_{s=0} = -\frac{d^2}{ds^2}[-\kappa_1 s + \kappa_2 s^2/2 - \kappa_3 s^3/6 \pm \ldots]\Big|_{s=0} = \kappa_2,$$

where κ_2 is the variance of the "probability density" function $c(x) = e^{-rx}l(x)/\theta(r)$, $0 \le x < \infty$.

In #4.12 of *DTP* (p. 80), the term "mortality rate" means force of mortality. In #3.39, #4.15 and #4.29 (p. 60, p. 80, p. 81), the term "death rate" means force of mortality. In #4.15, the term "birth rate" means age-specific birth rate.

The variable RSTAR in the program in the solution to #4.4 of *DTP* (p. 83) has to be initialized.

Here is a solution to #4.33 of *DTP*. Consider a population consisting of two stable subpopulations, standard and substandard. Let $l(x)$ denote the survival function of the standard subpopulation and let the survival function of the substandard population be $l^*(x) = e^{-\delta x}l(x)$, where δ is a positive constant. Assume that the two subpopulations are of the same size; then the probability of survival from age x to age y, x < y, for the entire population is

$$\frac{\frac{l^*(y)}{l^*(x)} + \frac{l(y)}{l(x)}}{2} = \left[\frac{e^{-\delta(y-x)} + 1}{2}\right]\frac{l(y)}{l(x)}.$$

Assume that the birth rates of the two subpopulations are such that the subpopulations have the same rate of increase. Now, suppose that the medical improvement will affect the substandard subpopulation only, with its force of mortality uniformly lowered by ε, $\varepsilon < \delta$, i.e., the new survival function for the substandard population is $e^{-(\delta-\varepsilon)x}l(x)$. As the birth rate is assumed to remain unchanged, the rate of increase of the substandard population is increased by ε. At time t after the medical improvement has occurred, the probability of survival from age x to age y, x < y, for the entire population is

$$s(t) = \frac{e^{\varepsilon t}e^{-(\delta-\varepsilon)}\frac{l(y)}{l(x)} + \frac{l(y)}{l(x)}}{e^{\varepsilon t} + 1} = \left[\frac{e^{\varepsilon t}e^{-(\delta-\varepsilon)} + 1}{e^{\varepsilon t} + 1}\right]\frac{l(y)}{l(x)}.$$

Applying #4.32 with $\omega(t) = e^{\varepsilon t}$ and $a = e^{-(\delta-\varepsilon)}$, we see that s(t) is a decreasing function. By the way, another solution for #4.32 is:

$$\frac{\omega(t)\,a + 1}{\omega(t) + 1} = a + \frac{1 - a}{\omega(t) + 1}.$$

The symbol L_i in Chapter 5 of *DTP* is not the integral $\int x^i l(x)\,dx$ (Ch. 3, p. 64). The ratio L_{i+1}/L_i on page 102 should read ${}_5L_{5\times i}/{}_5L_{5\times(i-1)}$. The ratios L_{i+1}/L_i on page 103 and L_{i-1}/L_{i-2} on page 108 should be changed accordingly. The factor ${}_5L_0$ in the solution to #5.16 (p. 103) should be replaced by by ${}_5L_0/5$; thus the first row of the matrix **B** is ${}_5L_0\mathbf{F}(\mathbf{I} + \mathbf{S})/2$. Note that, in *DT*, the radix l_0 is not assumed to be one; in paragraph 8.4.(iii) on page 116 of *DT*, the survival ratio ${}_5L_0/(5l_0)$ is prescribed. The corrections in this paragraph will appear in the second printing of *DTP*.

Let S be the survival ratio matrix in #5.15 of *DTP*. Let n be a positive integer. The entry in the i-th column and (i+n)-th row of S^n is the ratio ${}_5L_{5\times(i+n-1)}/{}_5L_{5\times(i-1)}$, $1 \le i \le i + n \le 18$. All other entries in S^n are 0.

Let A denote the n × n matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ b_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & b_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b_{n-1} & 0 \end{bmatrix}.$$

It is easy to check that the characteristic polynomial of A, det(λI − A), is

$\lambda^n - a_1\lambda^{n-1} - b_1a_2\lambda^{n-2} - b_1b_2a_3\lambda^{n-3} - b_1b_2b_3a_4\lambda^{n-4} - \ldots - b_1b_2b_3\ldots b_{n-2}a_{n-1}\lambda - b_1b_2b_3\ldots b_{n-1}a_n$.

Let M be the 18 × 18 projection matrix in #5.16 of *DTP*, with $F = ({}_5F_0\ {}_5F_5\ {}_5F_{10}\ {}_5F_{15} \cdots {}_5F_{85})$. (${}_5F_x$ denotes the age-specific fertility rate for those between ages x and x + 5; see #4.4 and #4.5 for some numerical values.) It follows from the determinant formula that

$$\det(\lambda I - M) = \lambda^{18} - ({}_5L_0 \cdot {}_5F_0 + {}_5L_5 \cdot {}_5F_5)/(2l_0)\lambda^{17} - ({}_5L_5 \cdot {}_5F_5 + {}_5L_{10} \cdot {}_5F_{10})/(2l_0)\lambda^{16}$$
$$- ({}_5L_{10} \cdot {}_5F_{10} + {}_5L_{15} \cdot {}_5F_{15})/(2l_0)\lambda^{15} - \ldots - ({}_5L_{80} \cdot {}_5F_{80} + {}_5L_{85} \cdot {}_5F_{85})/(2l_0)\lambda - ({}_5L_{85} \cdot {}_5F_{85})/(2l_0).$$

(In #5.16, $l_0 = 1$.) This explains the solution to #5.27 (p. 106) that the net reproduction rate is given by the negative of the sum of the coefficients of the second term to the last term (the constant term) of the characteristic polynomial of M.

The characteristic equation of M has exactly one positive real root, which is also the eigenvalue of M with the largest absolute value. (This is a consequence of the Perron-Frobenius Theorem; that the characteristic equation of M has exactly one positive root can also be inferred by using the Descartes rule of sign.) Call it λ_+. According to the second clause in the solution to #5.26 of *DPT*, $\lambda_+ = e^{5r}$. To see this, we return to #4.4 and #4.5, where the equation $l(0) = \int_0^\infty e^{-rx}l(x)m(x)dx$ is discretized as

$$l(0) \approx \sum_{j=0}^{17} e^{-r(2.5+5j)}\ {}_5L_{5j}\ {}_5F_{5j}. \qquad (\bigstar)$$

Put $e^{5r} = \lambda$, replace the term $e^{-r(2.5+5j)}$ in the right-hand side of ($\bigstar$) by its "average"

$$[e^{-r5j} + e^{-r5(j+1)}]/2 = [\lambda^{-j} + \lambda^{-(j+1)}]/2,$$

and multiply both sides of the equation by λ^{18}. Since ${}_5F_0 = 0$, the characteristic equation of M is "equivalent" to ($\bigstar$).

The third clause in the solution to #5.26 (*DTP*, p. 106) indicates that the eigenvectors of M corresponding to λ_+ are approximately proportional to the vector $({}_5C_0\ {}_5C_5\ {}_5C_{10}\ {}_5C_{15} \cdots {}_5C_{85})^T$. That the entries of an eigenvector corresponding to the eigenvalue λ_+ are all of the same sign is guaranteed by the Perron-Frobenius Theorem.

In the last sentence on page 516 of *AM*, the word "die" should be replaced by the words "are alive".

For detailed exposition on stable population theory, we refer the reader to the two books by Nathan Keyfitz: *Introduction to the Mathematics of Population* (1968) and *Applied Mathematical Demography* (1977). Although the first book is out of print, a second edition of the second book was published by Springer-Verlag in 1985. A recent book on stable population theory is *Deterministic*

Aspects of Mathematical Demography by John Impagliazzo. This excellent book was published in 1985 as Volume 13 of the Springer-Verlag Biomathematics Series. A paper which we would recommend highly is "Resolving a Historical Confusion in Population Analysis" by the Nobel laureate Paul A. Samuelson. It is Chapter 15 of the book *Mathematical Demography: Selected Papers*, which was edited by D. Smith and N. Keyfitz and published in 1977 as Volume 6 of the Springer-Verlag Biomathematics Series. The paper by W. Feller which we mentioned earlier and many other important papers on mathematical demography are reprinted in this book.

Below is a list of equivalent symbols:

Actuarial Mathematics (Ch. 18)		*Demography Through Problems* (Ch. 3–5)	
$s(x)$, ${}_{x}p_{0}$		$l(x)$	See * below
μ_x		$\mu(x)$	
R		r	
$N(t)$	(p. 519, p. 529)	P_t	(Chapter 5)
$b_f(t)$	(p. 526)	$B(t)$	(#4.1)
b	(§ 18.4)	B	(#3.41, #3.45, #3.49)
$i(t)$	(#18.7.b)	b	(p. 55) Also see ** below
$f(t)$	(#18.16)	b	(p. 73, #3.38), ${}_{\infty}C_{65} / {}_{45}C_{20}$
$\beta(x, \cdot)$	(§ 18.6)	$m(x)$	(p. 55)
$\phi(x)$	(§ 18.6)	$l(x)m(x)$	
$H(r)$	(§ 18.6)	$\psi(r)$	(#4.7)
$H(0)$, β		$\psi(0)$, R_0	
$H^{(i)}(0)$		$\psi^{(i)}(0)$, R_i	

* In *DTP* the radix $l(0)$ is usually set to be one; thus $l(x)/l(0) = l(x)$.

** The symbol $b(t)$ is also introduced in the solution to #5.13 of *DTP* (p. 102). Note that integrating the differential equation $dP_t/dt = [b(t) - d(t)]P_t$ yields

$$P_{t_1} = P_{t_0} \exp\{\int_{t_0}^{t_1} [b(t) - d(t)]\, dt\}.$$