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Fluctuations of Pension Contributions and Fund Level

by

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What follows is an account of the talk I gave at the 24th Actuarial Research Conference in Montréal (August 1989). The talk was an outline of a project recently approved by the Actuarial Education and Research Fund. I wish to thank the AERF Board of Directors for funding the project.

1. Goal of project

The goal of the project is to study the variability of pension costs and fund levels under various funding methods. A number of authors have studied the dynamics of pension funding under more or less static conditions (see list of references). A better knowledge of the factors determining the volatility of pension costs (or expense) and fund levels would be of great value, both when choosing a funding method for the valuation of a particular plan, and also when establishing new minimum funding standards which will affect a large number of plans.

In Dufresne (1989), the author used a simplified model of pension funding, including random rates of return and amortization of unfunded liabilities over "n" years. It was shown that increasing n decreases the variance of contributions, but increases the variance of fund levels. It is proposed to use a more comprehensive model and to consider a greater number of funding strategies.

There are many factors causing fluctuations of contributions and fund levels; only variations of rates of return on assets will be considered below. More specifically, suppose

$$R_t = \text{rate of return for period } (t-1, t)$$

$$= r + e_t + \beta_1 e_{t-1} + \ldots + \beta_m e_{t-n} \qquad (1)$$

where $\{e_t\}$ is an i.i.d. sequence with $Ee_t = 0$, $0 < \text{Var } e_t < \infty$ (i.e. $\{e_t\}$ is discrete-time white noise). Eq. (1) says that $\{R_t\}$ is a constant r (the mean rate of return) plus a moving-average process of order* n, i.e. $\{R_t\} \sim MA(m)+r$.

2. The simplest case: the accumulated value of 1 per annum

One of the simplest "pension plan" imaginable is one for which constant contributions are paid in (yearly), while no benefits are paid out. Let $F_o = 0$ and

$$F_t = (1+R_t)F_{t-1}+1.$$

Case m = 0. Here $R_t = r+e_t$ and thus

$$EF_t = E(1+R_t)EF_{t-1} + 1$$

$$= (1+r)EF_{t-1} + 1$$

$$\Rightarrow EF_t = s_{\overline{t}|r}.$$

This is an instance of the *nice property* of some processes $\{R_t\}$: "mean accumulated values (EF_t) grow at the mean rate of return $(ER_t = r)$".

Remark 1. The nice property is equivalent to

$$E(1+R_t \mid F_{t-1}) = r \text{ a.s. } \square$$

All higher moments of F_t can be found recursively:

$$EF_t^k = E[(1+R_t)F_{t-1}+1]^k$$

$$= \sum_{j=0}^{k} \binom{k}{j} E(1+R_t)^j EF_{t-1}^j.$$

Remark 2. This extends the formulas in Boyle (1976). ❑

Case m=1. $F_t = (1+r+e_t+Be_{t-1})F_{t-1}+1$

$$\Rightarrow F_{t-1} \text{ and } R_t \text{ are dependent}$$

$$\Rightarrow \text{above trick does not work.}$$

One way out is to use a *markovian representation*: let $G_t = e_t F_t$ =>

$$G_t = e_t(1+r+e_t+\beta e_{t-1})F_{t-1}+e_t$$

$$= e_t(1+r+e_t)F_{t-1}+\beta e_t G_{t-1}+e_t$$

or

$$\underbrace{\begin{pmatrix} F_t \\ G_t \end{pmatrix}}_{\overline{F}_t} = \underbrace{\begin{pmatrix} 1+r+e_t & \beta \\ e_t(1+r+e_t) & \beta e_t \end{pmatrix}}_{M_t} \underbrace{\begin{pmatrix} F_{t-1} \\ G_{t-1} \end{pmatrix}}_{\overline{F}_{t-1}} + \underbrace{\begin{pmatrix} 1 \\ e_t \end{pmatrix}}_{\overline{e}_t} \tag{2}$$

In Eq. (2) M_t and $\overline{F}_{t-1}$ are independent, so

$$E\overline{F}_t = M\,E\overline{F}_{t-1} + \overline{e}, \qquad M = EM_t, \qquad \overline{e} = E\overline{e}_t.$$

From this vector difference equation we obtain a two-dimensional formula for $E\overline{F}_t$:

$$E\overline{F}_t = (I+M+\ldots+M^{t-1})\overline{e}$$

$$= (M-I)^{-1}(M^t-I)\overline{e}$$

(since $F_o = 0 \Rightarrow \overline{F}_o = 0$).

Example: Effect of dependence of $\{R_t\}$ on $\{EF_t\}$.

Here $R_t = r+e_t+\beta e_{t-1} \Rightarrow ER_t = r$, $\text{Var } R_t = (1+\beta^2)\sigma^2$, $\text{Cov}(R_t, R_{t-1}) = \beta\sigma^2$. One experiment that comes to mind is the calculate EF_t for fixed values of ER_t and $\text{Var } R_t$, but to vary $\text{Cov}(R_t, R_{t-1})$. Let U_t be the accumulated value at time t of one unit invested at time 0. The table below shows EU_{15} and EF_{15} when $ER_t = .10$, $\text{Var } R_t = .01$ and $\rho = \text{Corr}(R_t, R_{t-1}) = \beta/(1+\beta^2)$ varies from -.5 to +.5.

β	ρ	1000 * EU_{15}	1000 * EF_{15}
1.00	.500	4,424	32,805
.75	.480	4,414	32,763
.50	.400	4,374	32,596
.25	.235	4,292	32,255
0	0	4,177	31,772
-.25	-.235	4,065	31,297
-.50	-.400	3,987	30,968
-.75	-.480	3,950	30,809
-1.00	-.500	3,941	30,769
		4,177 (1±6%)	31,772 (1±3%)

Table 1 EU_{15} and EF_{15} when $ER_t = .10$, Var $R_t = .01$

The case $\beta = 0$ corresponds to i.i.d. rates of return. The nice property does not hold when $\beta \neq 0$. In this particular case the rate of growth of mean accumulated values is

$$r' \doteq r + \mathrm{Cov}(R_t, R_{t-1})/(1+r). \tag{3}$$

Remark 3. Approximation (3) is justified as follows. From Eq. (2)

$$EF_t = (1+r)EF_{t-1} + \beta\sigma^2 EF_{t-2} + 1 \tag{4}$$

$$\Rightarrow EF_t = p_t + c_1\lambda_1^t + c_2\lambda_2^t$$

where p_t is a particular solution of (4), $\{\lambda_i\}$ are the solutions of

$$\lambda^2 - (1+r)\lambda - \beta\sigma^2 = 0$$

and $\{c_i\}$ are such that the initial conditions

$$EF_o = 0, \ EF_1 = 1$$

are satisfied. When $r+\beta\sigma^2 \neq 0$ we find

$$p_t \equiv p = -1/(r+\beta\sigma^2).$$

The λ's are

$$\lambda_1 = \frac{1}{2}((1+r)+[(1+r)^2+4\beta\sigma^2]^{1/2})$$

$$\lambda_2 = \frac{1}{2}((1+r)-[(1+r)^2+4\beta\sigma^2]^{1/2}),$$

which are real and distinct when $(1+r)^2+4\beta\sigma^2 > 0$. The constant $\{c_i\}$ are then

$$c_1 = [1+p(\lambda_2-1)]/(\lambda_1-\lambda_2)$$

$$c_2 = [p(1-\lambda_1)-1]/(\lambda_1-\lambda_2).$$

Under most assumptions (e.g. $|\beta| \le 1$, $\sigma^2 < 1/4$, $r \ge 0$) $|\lambda_2| < 1$ and $c_2\lambda_2^t$ quickly dies out as t increases. Thus

$$EF_t \doteq p+c_1\lambda_1^t$$

and the growth rate of EF_t is approximately

$$\lambda_1-1 = \frac{(1+r)}{2}(1+[1+4\beta\sigma^2/(1+r)^2]^{1/2})-1$$

$$\doteq \frac{(1+r)}{2}(1+1+2\beta\sigma^2/(1+r)^2)-1$$

$$= r+\beta\sigma^2/(1+r)$$

$$= r+\mathrm{Cov}(R_t, R_{t-1})/(1+r) = r'.$$

For example, in the case at hand $r = .10$, $\sigma^2 = .01/(1+\beta^2)$, and

$\beta = 1.$ $i = .104027 \Rightarrow s_{\overline{15}|i} = 32.805,$

$\lambda_1 = 1.104527$, $\lambda_2 = -.004527$

$r' = .104545.$

$\beta = -1$ $\quad i = .095945 \Rightarrow s_{\overline{15}|i} = 30.769,$

$\lambda_1 = 1.095436,\ \lambda_2 = .004564$

$r' = .095455.$

The approximation $(1+x)^{1/2} \doteq 1+x/2$ used to derive r' slightly overstates λ_1-1. □

It should be emphasised that approximation (3) only relates to the case $\{R_t\} \sim r+MA(1)$.

Var F_t can also be calculated; from Eq. (2)

$$\overline{F}_t\,\overline{F}_t' = M_t(\overline{F}_{t-1}\overline{F}_{t-1}')M_t'+M_t\,\overline{F}_{t-1}\,\bar{e}_t'$$
$$+\,\bar{e}_t\,\overline{F}_{t-1}'\,M_t'+\bar{e}_t\,\bar{e}_t'$$

(primes denote transposed matrices). A recursive equations is obtained for second-order moments upon taking expectations on both sides and applying the *vec* operation.

Case $m = 2$. $R_t = r+e_t+\beta_1 e_{t-1}+\beta_2 e_{t-2}$. A markovian representaiton is harder to obtain, as the next example shows.

Example. $U_t = (1+r+e_t+\beta_1 e_{t-1}+\beta_2 e_{t-2})U_{t-1}$. Define $U_{1,t} = U_t$, $U_{2,t} = e_t U_t$, $U_{3,t} = e_{t-1}U_t$, $U_{4,t} = e_t^2 U_t$, $U_{5,t} = e_t e_{t-1} U_t$ and

$$\overline{U}_t = (U_{1,t}, \ldots, U_{5,t})'.$$

Then $\overline{U}_t = N_t\,\overline{U}_{t-1}$ where

$$N_t = \begin{bmatrix} 1+r+e_t & \beta_1 & \beta_2 & 0 & 0 \\ e_t(1+r+e_t) & \beta_1 e_t & \beta_2 e_t & 0 & 0 \\ 0 & 1+r+e_t & 0 & \beta_1 & \beta_2 \\ e_t^2(1+r+e_t) & \beta_1 e_t^2 & \beta_2 e_t^2 & 0 & 0 \\ 0 & e_t(1+r+e_t) & 0 & \beta_1 e_t & \beta_2 e_t \end{bmatrix}$$

With this representation, calculating EU_t is a problem in dimension 5, while calculating EU_t^2 is a problem in dimension 25 (or 15 using the *vech* operation instead of *vec*). (The representation above may not be minimal, however.)

3. Aggregate funding method

Suppose a pension model with no inflation, stationary population and a fixed valuation rate of interest, and let F, B and C stand for fund level, benefit payments and overall contributions, respectively. F is the value of the fund at the beginning of the year, just before B is paid out and C paid in. Under the Aggregate method

$$C_t = (PVFB - F_t) / (PVFS/S)$$

where PVFB is the present value of future benefits, PVFS the present value of future salaries and S the annual payroll. We obtain

$$\begin{aligned} F_t &= (1+R_t)[F_{t-1}+C_{t-1}-B] \\ &= a(1+R_t)(F_{t-1}+b), \quad a, b \text{ constants,} \end{aligned}$$

which says that $\{F_t\}$ has nearly the same structure it had in the previous section. The mean and variance of F_t (and C_t) can be calculated when $\{R_t\} \sim MA(m)$ + constant, $m = 0$, 1 or 2. The case $m = 0$ (that is to say i.i.d. rates of return) has been dealt with in Dufresne (1986) and (1988b).

Further points to be studied in connection with this model include:

- non-stationary populaton
- variable valuation rate of interest
- random inflation on salaries
- minimum funding requirements.

4. Amortizing annual gains/losses

Consider an individual funding method (e.g. unit credit, entry age normal) and suppose:

- no inflation
- stationary population
- valuation rate of interest (denoted "i_V") is fixed
- L_t = actuarial loss in $(t-1, t)$ (positive or negative)
- NC = normal cost
- AL = actuarial liability.

What is meant by "amortizing losses" over n years is that overall contributions are

$$C_t = NC + \frac{L_t}{\ddot{a}_{\overline{n}|}} + \frac{L_{t-1}}{\ddot{a}_{\overline{n}|}} + \ldots + \frac{L_{t-n+1}}{\ddot{a}_{\overline{n}|}}.$$

It can be shown (Dufresne, 1989) that

- $L_t = (R_t - i_v)\Big[\sum_{k=1}^{n-1} \beta_k L_{t-k} - AL/(1+i_v)\Big]$ where $\{\beta_k\}$ are constants
- if $\{R_t\}$ is i.i.d. and $ER_t = i_v$, then $\{L_t\}$ is an uncorrelated sequence, and the mean and variance of (F_t, C_t) can be calculated from those of $\{L_t\}$
- when the amortization period n is lengthened Var C decreases while Var F increases.

Other questions of interest include:

- $\{R_t\}$ ~ constant + MA(m), m = 0, 1, 2
- constant non-zero inflation
- different treatment of gains ans losses
- making i_v variable
- random inflation.

5. Final remark

All the processes described above are members of the class of bilinear autoregressive processes with general form

$$X_t = \sum_{i=0}^{P} \sum_{j=1}^{Q} (a_i + b_i e_{t-i}) X_{t-j} + \sum_{i=0}^{R} c_i e_{t-i}$$

where $\{e_t\}$ is i.i.d. . See Granger and Andersen (1978), Subba Rao and Gabr (1984).

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