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VARIABILITY OF PENSION CONTRIBUTIONS AND FUND LEVELS WITH RANDOM AND AUTOREGRESSIVE RATES OF RETURN

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ABSTRACT

A mathematical model is described which facilitates the comparison of different pension funding methods. Rates of return are assumed firstly to be random and then to be represented by an autoregressive model for the corresponding force of interest. Expressions for the variability of contributions and fund levels can be derived. This leads to a discussion of the "optimal" method of funding. A fuller description of the methodology is given in Dusfresne's doctoral thesis[1] and in recent papers by Haberman and Dufresne[2] and Dufresne[3,4].

1. TYPES OF FUNDING METHOD

Broadly, there are two types of funding methods.

With individual funding methods (e.g. Projected Unit Credit and Entry Age Normal), the normal cost (NC) and the actuarial liability (AL) are calculated separately for each member and then summed to give the totals for the population under consideration. With aggregate funding methods (e.g. Aggregate and Attained Age Normal), there is no hypothecation of normal cost or actuarial liability to individuals; instead the group is considered as an entity, ab initio.

Let C(t) and F(t) be the overall contribution and fund level at time t for a particular pension scheme.

For an individual funding method,

$$C(t) = \sum_x NC(x,t) + ADJ(t) \tag{1}$$

where NC(x,t) is the normal cost for a member aged x at time t, $\sum$ denotes summation over the membership subdivided by attained age and ADJ(t) is an adjustment to the contribution rate at time t represented by the liquidation of the unfunded liability at time t, UL(t). UL(t) is defined by

$$UL(t) = \sum_x AL(x,t) - F(t)$$

where AL(x,t) is the actuarial liability for a member aged x at time t.

For an aggregate method, the overall contribution is directly related to the difference between the present value of future benefits and the fund. Specifically,

$$C(t) = \left[\frac{PVB(t) - F(t)}{PVS(t)}\right] . \; S(t) \qquad (2)$$

where $S(t)$ is the payroll at time t, $PVB(t)$ is the present value of future benefits (of all members including pensioners) at time t and $PVS(t)$ is the present value of future salaries (of active members) at time t.

This paper considers the behaviour of $C(t)$ and $F(t)$ in the presence of random investment returns.

At any time t, a valuation is carried out to estimate $C(t)$ and $F(t)$, based only on the scheme membership at time t. However, as t changes we do allow for new entrants to the membership so that the population remains stationary - see assumptions below.

In the mathematical discussion, we make the following assumptions.

1. All actuarial assumptions are consistently borne out by experience, <u>except</u> for investment returns.

2. The population is stationary from the start.

3. There is no inflation on salaries, and no promotional salary scale. For simplicity, each active member's annual salary is set at 1 unit.

4. The interest rate assumption for valuation purposes is fixed.

5. The real interest rate earned during the period, $(t,t+1)$ is $i(t+1)$. The corresponding force of interest is assumed here to be constant over the interval $(t,t+1)$ and is written as $\delta(t+1)$. Thus $1 + i(t+1) = \exp(\delta(t+1))$.

6. $E[1+i(t)] = E[\exp \delta(t)] = 1+i$, where i is the valuation rate of interest. This means that the valuation rate is correct "on average". This assumption is not essential mathematically but it is in agreement with classical ideas on pension fund valuation. Further, we define $\sigma^2 = \mathrm{Var}(i(t))$.

Assumptions 1., 2., 3., and 4. imply that the following parameters are constant with respect to time, t:

NC the total normal contribution
AL the total actuarial liability
B the overall benefit outgo (per unit of time).

Further, assumptions 1., 2., and 6. imply that

$$AL = (1 + i)(AL + NC - B). \tag{3}$$

The paper adopts a discrete time approach. It is possible to approach this problem using a continuous time formulation; however, the mathematics requires familiarity with stochastic differential equations and the details have been omitted here - the interested reader is referred to Dufresne[1].

2. RANDOM INDEPENDENT, IDENTICALLY DISTRIBUTED RATES OF RETURN: BASIC RESULTS

It is assumed in this section that the earned rates of return $i(t)$ for $t > 1$ are independent, identically distributed random variables with $i(t) > -1$ with probability 1.

2.1 MOMENTS OF C(t) AND F(t): INDIVIDUAL FUNDING METHODS

There are two general ways in which the ADJ(t) term may be computed. Under the "amortization of losses" method, we define

$$ADJ(t) = \sum_{j=0}^{M-1} \frac{L(t-j)}{\ddot{a}_{\overline{M}|}} \tag{4}$$

where $L(t)$ is the loss in year $(t-1,t)$, between two valuation dates. Thus, ADJ(t) is then the total of the intervaluation losses arising during the last M years (ie between t-M and t) divided by the present value of an annuity for a term of M years (ie spread over an M year period). The unfunded liability at time t is then given by

$$UL(t) = \sum_{j=0}^{M-1} L(t-j) \frac{\ddot{a}_{\overline{M-j}|}}{\ddot{a}_{\overline{M}|}} \tag{5}$$

The properties of this method are not pursued here and the interested reader is referred to Dufresne[1,4] for a detailed discussion.

Under the "spread" method, we define $ADJ(t) = \dfrac{UL(t)}{\ddot{a}_{\overline{M}|}}$

ie the adjustment to the normal cost is equal to the overall unfunded liability divuded by the present value of an annuity for a term of M years. Then,

$$C(t) = NC + \frac{(AL - F(t))}{\ddot{a}_{\overline{M|}}} \tag{6}$$

This paper concentrates on the "spread" method.

Then

$$\begin{aligned} F(t+1) &= (1 + i(t+1))(F(t) + C(t) - B) \\ &= (1 + i(t+1))[F(t) + NC + (AL - F(t))/\ddot{a}_{\overline{M|}} - B] \\ &= [(1 + i(t+1))/(1 + i)](qF(t) + r) \end{aligned} \tag{7}$$

where $q = (1 + i)(1 - 1/\ddot{a}_{\overline{M|}})$ and $r = (1 + i)\ (NC - B + AL/\ddot{a}_{\overline{M|}})$.

Then, it can be proved that

$$\begin{aligned} E\ F(t+1) &= E\ E\ (F(t+1)\ H_t) \\ &= E\ (q\ F(t) + r) = q\ E\ F(t) + r\ . \end{aligned} \tag{8}$$

This is a recurrence relation which can be solved to give

$E\ F\ (t) = q^t F(0) + r\ (1 - q^t)/(1 - q)$ for $t \geq 0$.

If M 1, then it can be shown that $0 < q < 1$ and so

$$\lim_{t} E\ F(t) = r/(1-q).$$

Using $AL = (1+i)\ (AL + NC - B)$, it can be shown that

$$r/(1-q) = AL. \tag{9}$$

Equation (6) implies that $E\ C(t) = NC + (AL - EF(t))/\ddot{a}_{\overline{M|}}$ and so

$$\lim_{t} E\ C(t) = NC. \qquad (10)$$

A consequence of equations (8) - (10) is that, if $F(0) = AL$, then

$$E\ F(t) = AL \text{ and } E\ C(t) = NC \quad \text{for } t > 0.$$

Concerning second moments, we use the result

$$\text{Var } F(t+1) = E \text{ Var } (F(t+1) | H_t) + \text{Var } E\ (F(t+1) | H_t)$$

from which it can be proved that

$$\text{Var } F(t+1) = a \text{ Var } F(t) + b\ (E\ F\ (t+1))^2 \qquad (11)$$

where $a = q^2(1 + \sigma^2(1+i)^{-2})$ and $b = \sigma^2\ (1+i)^{-2}$.

Equation (11) is also a recurrence relation which may be solved in successive steps to give

$$\text{Var } F(0) = 0$$
$$\text{Var } F(1) = b\ (E\ F(1))^2$$
$$\text{Var } F(2) = a\ b(E\ F\ (1))^2 + b(E\ F(2))^2 \quad \text{and so on.}$$

Generally, $\text{Var } F(t) = b \sum_{k=1}^{t} a^{t-k}\ (E\ F(k))^2$ for $t > 1$. (12)

It can then be shown that

$$\lim \text{Var } F(t) = \left.\begin{array}{ll} bAL^2/(1-a) & \text{if } a < 1 \\ \infty & \text{if } a \geq 1 \end{array}\right\} \qquad (13)$$

Also, $\text{Var } C(t) = \text{Var } F(t)/(\ddot{a}_{\overline{M|}})^2$.

It is also possible to work out covariances. Thus, it can be proved that

$$\mathrm{Cov}(F(t+u), F(t)) = q^u \mathrm{Var}\, F(t), \qquad u > 0.$$

Similarly,

$$\mathrm{Cov}(C(t+u), C(t)) = q^u \mathrm{Var}\, C(t)$$

and

$$\mathrm{Cov}(C(t+u), F(t)) = -q^u \mathrm{Var}(F(t)/\ddot{a}_{\overline{M}|}).$$

Thus, if a < 1, the correlation coefficients satisfy

$$\begin{aligned}\lim_t \mathrm{Cor}(F(t+u), F(t)) &= \lim_t \mathrm{Cor}(C(t+u), C(t)) \\ &= -\lim_t \mathrm{Cor}(F(t+u), C(t)) \\ &= q^{|u|}.\end{aligned}$$

2.2 MOMENTS OF C(t) AND F(t): THE AGGREGATE FUNDING METHOD

As noted in Equation (2), the Aggregate Funding Method is such that

$$C(t) = (PVB - F(t)).S/PVS$$

with

S = Pensionable earnings;
PVB = Present value of future benefits (including pensioners);
PVS = Present value of future earnings.

S, PVB and PVS are aggregate values, relating to the whole population of current members, and here are constants (from assumptions 1., 2., 3. and 4.).

Here

$$
\begin{aligned}
F(t+1) &= (1 + i(t+1))(F(t) + C(t) - B) \\
&= (1 + i(t+1))[F(t)(1 - S/PVS) + S.PVB/PVS - B] \\
&= [(1 + i(t+1))/(1 + i)](q'F(t) + r')
\end{aligned}
$$

where $q' = (1 + i)(1 - S/PVS)$ and $r' = (1 + i)(S.PVB/PVS - B)$.

As before

$$EF(t+1) = q'EF(t) + r'.$$

It can then be shown that $0 < q' < 1$.

Therefore,

$$\lim_{t} EF(t) = r'/(1 - q').$$

Clearly, $EC(t) = (PVB - EF(t)).S/PVS$.

Again,

$$\operatorname{Var}_{t} F(t+1) = a'\operatorname{Var} F(t) + b[EF(t+1)]^2,$$

with $a' = (q')^2(1 + \sigma^2(1 + i)^{-2})$.

Eq. (12) still holds, and the earlier result becomes

$$\lim_{t} \operatorname{Var} F(t) = \begin{cases} b[\lim EF(t)]^2/(1 - a') & \text{if } a' < 1 \\ \infty & \text{if } a' > 1. \end{cases} \tag{14}$$

Clearly, $\operatorname{Var} C(t) = (\operatorname{Var} F(t)).S^2/PVS^2$. Covariances and correlation coefficients are derived in the same fashion, substituting q' for q.

Remarks

(a) Trowbridge[5] has shown that in some cases the Aggregate and Entry Age Normal methods are asymptotically equivalent. The conditions he supposed are assumptions 1. - 6. inclusive plus

7. There is only one entry age into the scheme; and

8. σ^2 = Var i(t) = 0.

Clearly if assumption is maintained but assumption 8. is dropped (i.e. $\sigma^2 > 0$) then Trowbridge's proof still applies, but now to EF(t) and EC(t), yielding

$$\lim_{t} E\,{}^{AGG}F(t) = \lim_{t} E\,{}^{EAN}F(t) = {}^{EAN}AL;$$
$$\lim_{t} E\,{}^{AGG}C(t) = \lim_{t} E\,{}^{EAN}C(t) = {}^{EAN}NC. \qquad (15)$$

(b) It should be noted that in this simple framework the Aggregate method is really a particular case of the Entry Age Normal method (assuming assumption 7. is still in force); equation (15) implies

$$\begin{aligned} {}^{AGG}C(t) &= (PVB - {}^{AGG}F(t)).S/PVS \\ &= (PVB - {}^{EAN}AL).S/PVS + ({}^{EAN}AL - {}^{AGG}F(t))S/PVS \\ &= {}^{EAN}NC + ({}^{EAN}AL - {}^{AGG}F(t))/\ddot{a}_{\overline{N}|} \qquad (16) \end{aligned}$$

where N is defined so that $\ddot{a}_{\overline{N}|}$ = PVS/S. Equation (16) says that the Aggregate and Entry Age Normal methods are identical, if the latter is applied together with an N-year spread of (AL - F(t)). This fact was previously noted by C.J. Nesbitt in his contribution to the discussion of Trowbridge[6].

(c) If $M = 1$, then equation (7) does not apply; instead

$$\begin{aligned} F(t+1) &= (1 + i(t+1))[F(t) + NC + (AL - F(t)) - B] \\ &= [(1 + i(t+1))/(1 + i)]\ (1 + i)(AL + NC - B) \\ &= [(1 + i(t+1))/(1 + i)]\ AL. \end{aligned}$$

Thus, for each $t > 1$,

$$\begin{aligned} EF(t) &= AL, \\ EC(t) &= NC \end{aligned}$$

and

$$\mathrm{Var}\ C(t) = \mathrm{Var}\ F(t) = \sigma^2 (1 + i)^{-2}\ AL^2.$$

3. AUTOREGRESSIVE RATES OF RETURN: BASIC RESULTS

It is apparent from the discussion in Section 2 that the equations for the moments of F(t) and C(t) are of the same type for individual and aggregate funding methods. This section, therefore, considers only one type, viz individual funding methods.

In order to investigate the effects of autoregressive models for the earned real rate of return, the paper follows the suggestion of Panjer and Bellhouse(7) and consider the corresponding force of interest and assume that it is constant over the interval of time (t,t+1): the notation used will be $\delta(t+1)$.

Now it is assumed that the (earned real) force of interest is then given by the following autoregressive process in discrete time of order 1 (AR(1)):

$$\delta(t) = \theta + \varphi\,[\delta(t-1) - \theta\,] + e(t) \qquad (17)$$

where e(t) for t=1,2, ... are independent and identically distributed normal random variables each with mean 0 and variance γ^2. Equation (17) replaces assumption 5. introduced earlier. This model suggests that interest rates earned in any year depend upon interest rates earned in the previous year and some constant level. Box and Jenkins(8) have shown that, under the model represented by equation (17),

$$E[\delta(t)] = \theta$$

$$\mathrm{Var}\,[\delta(t)] = \frac{\gamma^2}{1-\varphi^2} = \nu^2, \text{ say}$$

$$\mathrm{Cov}\,[\delta(t), \delta(s)] = \left(\frac{\gamma^2}{1-\varphi^2}\right)\varphi^{|t-s|} = \gamma(t,s), \text{ say.}$$

The condition for this process to be stationary is that $|\varphi| < 1$.

Boyle[9] investigated the simpler model.

$$\delta(t) = \theta + e(t) \qquad (18)$$

where $\varphi = 0$. Clearly this model bears a close resemblance to that considered in section 2. Appendix I confirms that equation (18) leads to similar results to those presented in section 2.1 for individual funding methods.

In order to apply the autoregressive model (17) to determine moments of F(t) and C(t), it is necessary to abandon the approach of section 2 (and Appendix I) whereby recurrence relations between, for example, E F(t+1) and E F(t) were sought. The presence of a dependence on the past in the autoregressive model would make such an approach problematic.

The approach begins with considering the series generated by the recurrence relation (7), which for convenience is rewritten here as

$$F(t+1) = (1+i(t+1))\,(Q\,F(t) + R) \qquad (19)$$

where $Q = 1 - \dfrac{1}{\ddot{a}_{\overline{M}|}} = vq$, $R = (NC - B + \dfrac{AL}{\ddot{a}_{\overline{M}|}}) = vr$ and $v = (1+i)^{-1}$.

Then $F(t) = F(0).\,Q^t\,e^{\Delta(t)} + Q^{t-1}Re^{\Delta(t)} + Q^{t-2}R\,e^{\Delta(t)-\Delta(1)}$

$$+ \ldots\ldots\ldots + R\,e^{\Delta(t)-\Delta(t-1)} \qquad (20)$$

where $\Delta(t) = \sum_{u=1}^{t} \delta(u)$.

In order to obtain an expression for E F(t), it is necessary to consider terms of the form $E(e^{\Delta(t)-\Delta(s)})$ for $s=0,1,\ldots,t-1$.

Given the distributional assumption for e(t), and that

$$\left.\begin{aligned} E(\Delta(t)) &= E\left(\sum_{u=1}^{t}\delta(u)\right) = t\,\theta \\ Var(\Delta(t)) &= Var\left(\sum_{u=1}^{t}\delta(u)\right) = \sum_{u=1}^{t}\sum_{w=1}^{t}\gamma(u,w) \\ &= \sum_{u=1}^{t}\sum_{w=1}^{t}\nu^2\varphi^{|u-w|} \end{aligned}\right\} \qquad (21)$$

then $E(e^{\Delta(t)-\Delta(s)}) = \exp[(t-s)\theta + \frac{1}{2}\sum_{s+1}^{t}\sum_{s+1}^{t}\gamma(u,w)]$

$$= \exp[(t-s)\theta + \nu^2 G(t,s)] ,$$

where $G(t,s) = \frac{1}{2}\sum_{s+1}^{t}\sum_{s+1}^{t}\varphi^{|u-w|} = \frac{1}{2}(t-s) + \sum_{u=s+1}^{t}\sum_{w=u+1}^{t}\varphi^{w-u}$

$= \frac{1}{2}(t-s) + \sum_{u=s+1}^{t}\sum_{x=1}^{t-u}\varphi^{x}$ where x=w-u

$= \frac{1}{2}(t-s) + \sum_{x=1}^{t-s-1}(t-s-x)\varphi^{x}$ on changing the order of summation and simplifying.

$= \frac{1}{2}(t-s) + J$

An expression for $J = \sum_{x=1}^{t-s-1}(t-s-x)\varphi^{x}$, a decreasing geometric progression, can be obtained by standard techniques. Hence

$$G(t,s) = \frac{1}{2}\left(\frac{1+\varphi}{1-\varphi}\right)(t-s) - \frac{\varphi\,(1-\varphi^{t-s})}{(1-\varphi)^2} \qquad (22)$$

Thus $E[e^{\Delta(t)-\Delta(s)}] = \exp\left[(t-s)\left(\theta + \frac{1}{2}\left(\frac{1+\varphi}{1-\varphi}\right)\nu^2\right) - \varphi\nu^2\frac{(1-\varphi^{t-s})}{(1-\varphi)^2}\right]$ (23)

If the subsidiary parameters $c = \exp\left(\theta + \frac{1}{2}\left(\frac{1+\varphi}{1-\varphi}\right)\nu^2\right)$, and

$$d = \nu^2\varphi(1-\varphi)^{-2} ,$$

are introduced then $E[e^{\Delta(t) - \Delta(s)}] = c^{t-s}\,e^{-d(1-\varphi^{t-s})}$. (24)

Equation (20) implies that

$$EF(t) = F(0)\, Q^t\, E\, e^{\Delta(t)} + R \sum_{s=0}^{t-1} Q^{t-s-1}\, E(e^{\Delta(t)-\Delta(s)})$$

$$= (F(0) Q^t c^t e^{d\varphi^t} + \frac{R}{Q} \sum_{s=0}^{t-1} Q^{t-s}\, c^{t-s}\, e^{d\varphi^{t-s}})e^{-d}. \qquad (25)$$

The second term is of the form of the present value in conventional life contingencies of a temporary annuity based on Gompertz's or Makeham's law of mortality.

- In section 2.1, it was noted that $0 < q < 1$.

So $$cQ = \exp(-\theta - \tfrac{1}{2}\nu^2)\exp(\theta + \tfrac{1}{2}\left(\frac{1+\varphi}{1-\varphi}\right)\nu^2)\cdot q$$

$$= q\exp\left(\frac{\varphi\nu^2}{1-\varphi}\right) = q \text{ if } \varphi = 0.$$

For convergence as $t\to\infty$, we require $cQ < 1$. And we note that $|\varphi| < 1$.

It can then be shown that

$$\lim_{t} E(F(t)) = \frac{R}{Q}\,\frac{Qc}{1-Qc}\, e^{-d}$$

$$= \frac{vrc}{1-vqc}\, e^{-d}. \qquad (26)$$

If $\varphi = 0$ then $c = \exp(\theta + \tfrac{1}{2}\nu^2) = 1+i$ and $d=0$ and

hence $\lim_{t} E\,F(t) = \dfrac{r}{1-q}$ as in equation (11).

The above result simplifies since

$$vc = \exp\left[\theta + \tfrac{1}{2}\left(\frac{1+\phi}{1-\phi}\right)\nu^2 - \theta - \tfrac{1}{2}\nu^2\right] = \exp\left[\frac{\phi\nu^2}{(1-\phi)}\right]$$

to give $\displaystyle \lim_{t} E\,F(t) = \frac{r \exp\left[-\frac{\phi^2\nu^2}{(1-\phi)^2}\right]}{1-q\exp\left[\frac{\phi\nu^2}{(1-\phi)}\right]} \neq AL.$

Then $E\,C(t) = NC + \dfrac{AL - E\,(F(t))}{\ddot{a}_{\overline{M|}}}$ from equation (6).

To obtain an explicit expression for Var $(F(t))$, it will be necessary to consider $E(F(t)^2)$, which itself will depend on terms of the form

$$E(e^{\Delta(t) - \Delta(s) + \Delta(t) - \Delta(r)})$$

for $r,s = 0, 1, \ldots, t-1$.

For such cross-product moment terms, we begin with consideration of

$$\text{Var }(\Delta(t) - \Delta(s) + \Delta(t) - \Delta(r)).$$

Without loss of generality, we take $r > s$ and rewrite the argument as

$$\Delta(r) - \Delta(s) + 2\,(\Delta(t) - \Delta(r)).$$

Then $\text{Var }[\Delta(t) - \Delta(s) + \Delta(t) - \Delta(r)]$

$$= \text{Var }[\Delta(r) - \Delta(s)] + 4\text{ Var }[\Delta(t) - \Delta(r)] + 4\text{ Cov }[\Delta(r) - \Delta(s), \Delta(t) - \Delta(r)]$$

$$= \sum_{s+1}^{r}\sum_{s+1}^{r}\gamma(u,w) + 4\sum_{r+1}^{t}\sum_{r+1}^{t}\gamma(u,w) + 4\sum_{s+1}^{r}\sum_{r+1}^{t}\gamma(u,w)$$

$$= \sum_{s+1}^{r}\sum_{s+1}^{r}\gamma(u,w) + 4\sum_{s+1}^{t}\sum_{r+1}^{t}\gamma(u,w)\ .$$

Given the distributional assumptions for e(t), we thus have

$$E(e^{(t)-\ (s)+\ (t)-\ (r)}) = \exp\left((t-s)\ +(t-r)\ +\qquad (u,w)\right.$$

$$\left. +\ 2 \qquad (u,w)\right)$$

where, in this case of an AR(1) model,

$$(u,w) = {}^{2} \qquad {}^{u-w} .$$

For convenience, we can write

$$E(e^{\ (t)-\ (s)+\ (t)-\ (r)}) = \exp\left[(t-s)\ \ +(t-r)\ \ +\ \ {}^{2}H(t,r,s)\right]$$

and consider the simplification of H(t,r,s) in Appendix II.

Rewriting equation (20) gives

$$F(t) = (F(0)Q^{t} + Q^{t-1}R)e^{\ (t)} + Q^{t-2}\,R\,e^{\ (t)\ -\ \ (1)}$$
$$+\ Q^{t-3}\,R\,e^{\ (t)\ -\ \ (2)} + \ldots\ldots + R\,e^{\ \ (t)\ -\ \ (t-1)}$$

For convenience, we will take F(0) = 0.

Then, $E(F(t))^{2} = E\,[\qquad e^{\ (t)-\ (s)}\ e^{\ \ (t)-\ (r)}\ Q^{t-1-s}\ Q^{t-1-r}\ R^{2}]$

$$= \frac{2R^{2}}{Q^{2}} \qquad Q^{t-s}\ Q^{t-r}\ \ E(e^{\ (t)-\ (s)+\ (t)-\ (r)})$$

$$+\ \frac{R^{2}}{Q^{2}} \qquad Q^{2(t-s)}\ E(e^{\ 2(\ (t)-\ (s))})$$

Given the distributional assumptions for e(t), we thus have

$$E(e^{\Delta(t)-\Delta(s)+\Delta(t)-\Delta(r)}) = \exp\left((t-s)\theta+(t-r)\theta+\tfrac{1}{2}\sum_{s=1}^{r}\sum_{s=1}^{r}\gamma(u,w) + 2\sum_{s=1}^{t}\sum_{r=1}^{t}\gamma(u,w)\right)$$

where, in this case of an AR(1) model,

$$\gamma(u,w) = \nu^2 \; \phi^{|u-w|}.$$

For convenience, we can write

$$E(e^{\Delta(t)-\Delta(s)+\Delta(t)-\Delta(r)}) = \exp\left[(t-s)\theta+(t-r)\theta + \nu^2 H(t,r,s)\right]$$

and consider the simplification of H(t,r,s) in Appendix II.

Rewriting equation (20) gives

$$F(t) = (F(0)Q^t + Q^{t-1}R)e^{\Delta(t)} + Q^{t-2}\,R\,e^{\Delta(t)-\Delta(1)} + Q^{t-3}\,R\,e^{\Delta(t)-\Delta(2)} + \ldots\ldots + R\,e^{\Delta(t)-\Delta(t-1)}$$

For convenience, we will take F(0) = 0.

Then, $E(F(t))^2 = E\left[\sum_{s=0}^{t-1}\sum_{r=0}^{t-1} e^{\Delta(t)-\Delta(s)}\, e^{\Delta(t)-\Delta(r)}\, Q^{t-1-s}\, Q^{t-1-r}\, R^2\right]$

$$= \frac{2R^2}{Q^2}\sum_{r=1}^{t-1}\sum_{s=0}^{r-1} Q^{t-s}\, Q^{t-r}\; E(e^{\Delta(t)-\Delta(s)+\Delta(t)-\Delta(r)})$$

$$+ \frac{R^2}{Q^2}\sum_{s=0}^{t-1} Q^{2(t-s)}\; E(e^{2(\Delta(t)-\Delta(s))})$$

$$= \frac{2R^2}{Q^2} \sum_{r=1}^{t-1}\sum_{s=0}^{r-1} Q^{t-s}Q^{t-r}\exp[\theta(t-s)+(t-s)\frac{(1+\varphi)}{(1-\varphi)}\nu^2]\exp[\theta(t-r)+(t-r)\frac{(1+\varphi)}{(1-\varphi)}\frac{\nu^2}{2}]$$

$$\cdot \exp\left[\frac{-3\varphi\nu^2}{(1-\varphi)^2}\right] \quad .\exp(\nu^2[2\frac{(r-s)\varphi^{r-s}}{(1-\varphi)} + \frac{2\varphi^{t-r+1} + 2\varphi^{t-s+1} - \varphi^{r-s+1}}{(1-\varphi)^2}])$$

$$+\frac{R^2}{Q^2} \exp\left[\frac{-4\varphi\nu^2}{(1-\varphi)^2}\right] \sum_{s=0}^{t-1} Q^{2(t-s)}\exp[2(t-s)\theta+2(t-s)\frac{(1+\varphi)}{(1-\varphi)}\nu^2]\exp[\frac{4\varphi^{t-s+1}\nu^2}{(1-\varphi)^2}]$$

using the simplified versions of H(t,r,s) from Appendix II.

These double summations are in a complicated form but are again related to annuity values.

It can then be shown that (see Appendix III for details)

$$\lim_{t} E(F(t)^2) = e^{-3d} \frac{2R^2Qc^2w}{(1-Qc)(1-Q^2cw)} + e^{-4d} \frac{R^2cw}{(1-Q^2cw)}$$

where $d = \frac{\nu^2\varphi}{(1-\varphi)^2}$, $c = \exp(\theta + \frac{(1+\varphi)\nu^2}{2(1-\varphi)})$, Q=vq, R=vr as before and

$$w = \exp(\theta + \frac{3}{2}\frac{(1+\varphi)}{(1-\varphi)}\nu^2).$$

We note that $|\varphi|<1$, by assumption, and that

$$Q^2 c\, w = q^2 v^2\, cw = q^2\, \exp\left(\frac{(1+3\varphi)\nu^2}{(1-\varphi)}\right) .$$

For convergence, as $t \to \infty$, we require $Qc < 1$ and $Q^2 cw < 1$.

$$\text{Then, } \lim_{t} \text{Var } F(t) = e^{-3d}\, \frac{2R^2 Qc^2 w}{(1-Qc)(1-Q^2 cw)} + e^{-4d}\, \frac{R^2 cw}{(1-Q^2 cw)} - \frac{R^2 c^2}{(1-Qc)^2}\, e^{-2d} .$$

Then, formulae for $\lim_{t} \text{Var } C(t) = \dfrac{\text{Var } F(t)}{(\ddot{a}_{\overline{M|}})^2}$ may be obtained.

The next stage in this study is to investigate further the properties of these moment equations and relationships between them.

4. CONCLUSIONS

The variability of contributions ($C(t)$) and fund levels ($F(t)$) resulting from random (real) rates of return has been studied mathematically. The funding methods considered are Aggregate Method and those methods that prescribe the normal cost to be adjusted by the difference between the actuarial liability and the current fund, divided by the present value of an annuity for a term of "M" years. A simple demographic/financial model permits the derivation of formulae for the first two moments of F and C, when the earned rates of return form an independent identically distributed sequence of random variables.

The approach has been extended to include the case of a first order autoregressive model for the real rates of return. Expressions for the first two moments of $C(t)$ and $F(t)$ have been obtained and their detailed properties are currently under investigation.

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SH/MVP/VPCFLRRR

APPENDIX I

Consider $\delta(t) = \theta + e(t)$ (18)

where $e(t)$ for $t=1,2,$ ..... are independent and identically distributed normal random variables with mean 0 and variance γ^2.

Using the notation of section 2.1, equation (7) becomes

$$F(t+1) = e^{(\delta(t+1) - \theta - \frac{1}{2}\gamma^2)} (q\, F(t) + r) \tag{7a}$$

$$\begin{aligned} \text{Then } E(F(t+1)) &= E\, E\, (F(t+1|H_t) \\ &= E(e^{\delta(t+1) - \theta - \frac{1}{2}\gamma^2})\, E(q\, F(t) + r) \\ &= (q\, E\, F\, (t) + r) \end{aligned} \tag{8}$$

Since, using moment generating functions, $E(e^{\delta(t+1)}) = e^{\theta+\frac{1}{2}\gamma^2} = 1+i$.

And so, equations (9) and (10) and the associated results would follow.

Concerning second moments, we again use the result that

$$\text{Var}\ (F(t+1)) = E\ \text{Var}\ (F(t+1)|H_t) + \text{Var}\ E\ (F(t+1)|H_t).$$

$$\text{Var}\ (F(t+1)|H_t) = \text{Var}\ (e^{\delta(t+1)-\theta-\frac{1}{2}\gamma^2})\ (q\, F(t) + r)^2$$

$\text{Var}\ (e^{\delta(t+1)}) = e^{2\theta + \gamma^2}(e^{\gamma^2} - 1)$ using moment generating functions.

$$\begin{aligned} \text{So Var}\ (F(t+1)|H_t) &= e^{-2\theta-\gamma^2} e^{2\theta+\gamma^2} (e^{\gamma^2} - 1)\ (q\, F(t) + r)^2 \\ &= (e^{\gamma^2} - 1)\ (q(F(t) - E\, F(t)) + q\, E\, F(t) + r)^2 \end{aligned}$$

$$\text{Hence } E\ \text{Var}(F(t+1)|H_t) = (e^{\gamma^2} - 1)\ [q^2\ \text{Var}\ F(t) + (E\, F(t+1))^2]$$

From (8), $E(F(t+1)|H_t) = (q\, F(t) + r)$

Hence $\text{Var}\ (E(F(t+1)|H_t)) = q^2\ \text{Var}\ F(t)$.

We therefore obtain

$$\text{Var } F(t+1) = (e^{\gamma^2}-1)\, q^2 \text{ Var } F(t) + q^2 \text{ Var } F(t) + (e^{\gamma^2}-1)(E\ F(t+1))^2$$

$$= e^{\gamma^2} q^2 \text{ Var } F(t) + (e^{\gamma^2} - 1)\ (E\ F(t+1))^2$$

$$= a \text{ Var } F(t) + b\ (E\ F(t+1))^2$$

where a and b correspond exactly to the definitions given earlier.

And so, equations (12) and (13) and the associated results would follow.

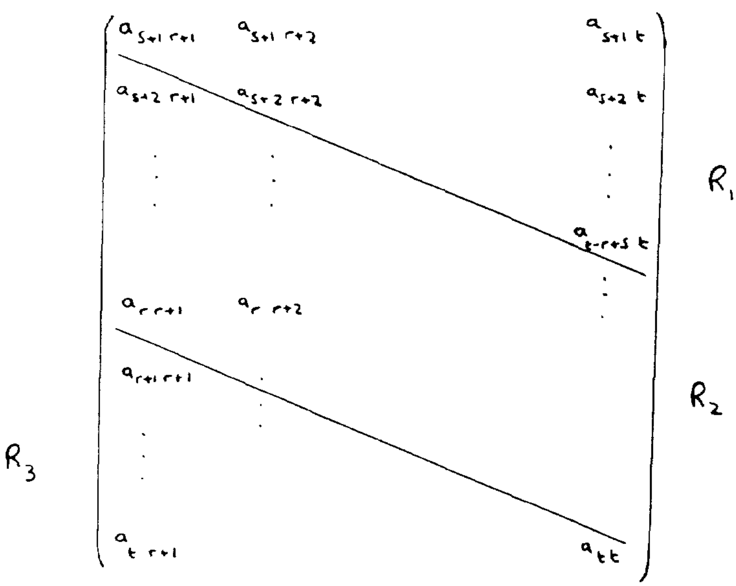
Appendix II

$$H(t,r,s) = \frac{1}{2} \sum_{w=S+1}^{r} \sum_{u=S+1}^{r} \varphi^{|u-w|} + 2 \sum_{w=S+1}^{t} \sum_{u=r+1}^{t} \varphi^{|u-w|} = A + B, \text{ say.}$$

$$\text{Then } A = \frac{1}{2} (r-s) + \sum_{u=S+1}^{r} \sum_{w=u+1}^{r} \varphi^{w-u} = \frac{1}{2} (r-s) + \sum_{1}^{r-s-1} (r-s-x)\ \varphi^{x}$$

as proved in section 3, in the discussion leading up to equation (22).

To obtain, a simplified expression for the second term, B, we can think of the summands $2\varphi^{u-w}$ as being entries a_{wu} in an array with (t-s) rows and (t-r) columns and r > s without loss of generality. It is convenient to divide up the array into three regions (following Appendix I of Panjer and Bellhouse[7]).

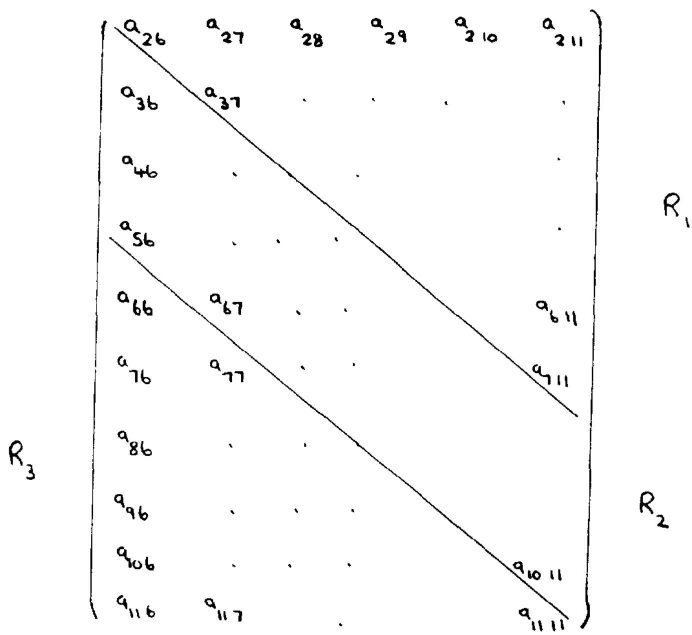


R_1 is the set of values in the array of a_{wu} bounded by $w=s+1$, $u=t$ and $u-w=r-s$.

R_2 is the set of values in the array bounded by $u=r+1$, $u=t$, $u-w=r-s-1$ and $u-w=1$.

R_3 is the set of values in the array bounded by $u=r+1$, $w=t$ and $u-w=0$.

This sub-division is best illustrated by an example. Let $t=11$, $r=5$, $s=1$ then the array is as follows



Then $2 \sum\sum_{R_1} \varphi^{|u-w|} = 2 \sum_{x=r-s}^{t-s-1} (t-s-x)\ \varphi^x$

$$2 \sum\sum_{R_2} \varphi^{|u-w|} = 2\ (t-r) \sum_{x=1}^{r-s-1} \varphi^x$$

$$2 \sum\sum_{R_3} \varphi^{|u-w|} = 2(t-r) + 2 \sum_{x=1}^{t-r-1} (t-r-x)\ \varphi^x$$

Hence $H(t,r,s) = A + B$

$$= \frac{1}{2}\ (r-s) + \sum_{x=1}^{r-s-1} (r-s-x)\ \varphi^x + 2 \sum_{x=r-s}^{t-s-1} (t-s-x)\ \varphi^x$$

$$+ 2\ (t-r) \sum_{x=1}^{r-s-1} \varphi^x + 2(t-r) + 2 \sum_{x=1}^{t-r-1} (t-r-x)\ \varphi^x$$

$$= \frac{1}{2}\ (r-s) + \frac{(r-s)\varphi}{1-\varphi} - \frac{\varphi(1-\varphi^{r-s})}{(1-\varphi)^2} + \frac{2(t-s)\ \varphi^{r-s}}{(1-\varphi\ )}$$

$$- 2\varphi\ \frac{(\varphi^{r-s} - \varphi^{t-s})}{(1-\varphi)^2}$$

$$+ 2(t-r)\ \frac{\varphi\ (1-\varphi^{r-s-1})}{(1-\varphi)} + 2(t-r) + \frac{2(t-r)\ \varphi}{(1-\varphi)}$$

$$- 2\varphi\ \frac{(1-\varphi^{t-r})}{(1-\varphi)^2}$$

which simplifies further to

$$H(t,r,s) = \frac{1}{2}\ (t-s)\ \frac{(1+\varphi)}{(1-\varphi)} - \frac{\varphi^{r-s+1}}{(1-\varphi)^2} + \frac{2(r-s)\ \varphi^{r-s}}{(1-\varphi)}$$

$$+ \frac{3}{2}\ (t-r)\ \frac{(1+\varphi)}{(1-\varphi)} + 2 \left(\frac{\varphi^{t-r+1} + \varphi^{t-s-1}}{(1-\varphi)^2} \right) - \frac{3\varphi}{(1-\varphi)^2} .$$

If r=s, this simplifies to

$$H(t,s,s) = 2(t-s) \left(\frac{1+\varphi}{1-\varphi} \right) + 4 \frac{\varphi^{t-s+1}}{(1-\varphi)^2} - \frac{4\varphi}{(1-\varphi)^2} .$$

Appendix III

Consider the double summation

$$I_1(t) = \frac{2R^2}{Q^2} \sum_{r=1}^{t-1} \sum_{s=0}^{r-1} Q^{t-s} Q^{t-r} c^{t-s} w^{t-r}$$

where $c = \exp\left[\theta + \frac{1}{2} \nu^2 \frac{1+\varphi}{1-\varphi}\right]$ and $w = \exp\left[\theta + \frac{3}{2} \nu^2 \frac{1+\varphi}{1-\varphi}\right]$

$$I_1(t) = \frac{2R^2}{Q^2} \sum_{r=1}^{t-1} Q^{t-r} w^{t-r} \sum_{s=0}^{r-1} Q^{t-s} c^{t-s}$$

$$= \frac{2R^2}{Q^2} \sum_{r=1}^{t-1} Q^{t-r} w^{t-r} \frac{Qc}{1-Qc} \left((Qc)^{t-r} - (Qc)^t\right)$$

$$= \frac{2R^2}{Q^2} \left(\frac{Qc}{1-Qc}\right) \left[\sum_{r=1}^{t-1} (Q^2cw)^{t-r} - (Qc)^t \sum_{r=1}^{t-1} (Qw)^{t-r}\right]$$

$$= \frac{2R^2}{Q^2} \left(\frac{Qc}{1-Qc}\right) \left[\frac{Q^2c\,w - (Q^2c\,w)^t}{1-Q^2\,c\,w} - \frac{(Qc)^t[Qw - (Qw)^t]}{1-Qw}\right]$$

As $t \to \infty$, $\lim I_1(t) = \frac{2R^2}{Q^2} \left(\frac{Qc}{1-Qc}\right) \left(\frac{Q^2cw}{1-Q^2cw}\right)$.

Consider the double summation

$$I_2(t) = \frac{R^2}{Q^2} \sum_{s=0}^{t-1} Q^{2(t-s)} (c\ w)^{(t-s)}$$

$$= \frac{R^2}{Q^2} \quad (Q^2\ cw) \qquad \left[\frac{1-(Q^2cw)^t}{1-Q^2cw}\right] .$$

As $t \to \infty$, $\quad \mathrm{Lim}\ I_2(t) = \dfrac{R^2}{Q^2} \quad \dfrac{Q^2\ cw}{1-Q^2cw}$.

Hence $\lim\limits_{t\to\infty} (e^{-3d} I_1(t) + e^{-4d}\ I_2(t))$

$$= e^{-3d} \frac{2R^2Qc^2w}{(1-Qc)(1-Q^2cw)} \quad + \quad e^{-4d} \frac{R^2cw}{1-Q^2cw} \quad .$$

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