INTEREST RATE VOLATILITY AND EQUILIBRIUM MODELS
OF THE TERM STRUCTURE: EMPIRICAL EVIDENCE

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ABSTRACT

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This paper examines the justification of using the one-factor general
equilibrium model of Cox, Ingersoll, and Ross (CIR) [1985] to model both the
term structure of interest rates and its associated volatility. A proposed
maximum likelihood approach attempts to match both the yield curve and the
yield volatilities, using a short history of zero coupon bonds (strips) from
the U.S. government securities market. Although the CIR model appears to
match fairly well the yield curve, matching simultaneously both the yield
levels and their volatilities is found to be more difficult. This evidence
raises objections in using a one-factor model to explain yield curve
volatility behavior.

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1. INTRODUCTION

The aim of this paper is to examine the justification of using the one-factor general equilibrium model of Cox, Ingersoll, and Ross (CIR) [1985] to model both the term structure of interest rates and its associated volatility. A maximum likelihood approach that attempts to match both the yield curve and the yield volatilities is proposed, using a short history of zero coupon bonds (strips) from the U.S. government securities market. Although the CIR model appears to match fairly well the yield curve, the simultaneous matching of both the yield levels and their volatilities is found to be more difficult. This evidence raises objections in using a one-factor model to expalin yield curve volatility behavior.

The paper is organized as it follows. Section 2 describes the nature of the CIR model of interest rates, along with the empirical investigations of that model. Section 3 explains the statistical estimation method used to solve the particular identification problem of the CIR model. A maximum likelihood procedure is used and the estimation, and asymptotic properties are presented. Section 4 describes the financial data used for the estimations. Section 5 presents the results and analyzes the likely causes of the model's failure to explain simultaneously both the yield curve and its associated volatility using the same one-factor model. Section 6 concludes.

2. THE COX-INGERSOLL-ROSS (1985) MODEL OF INTEREST RATES

2.1 The One-Factor Model

In Cox, Ingersoll, and Ross [1985], thereafter CIR, the problem of determining the term structure is posed as a problem in general equilibrium theory. A rational asset pricing model to study the term structure of interest rates is used. The current prices and stochastic properties of all contingent claims, including bonds, are derived endogenously. Anticipations of future events, risk aversion, investment alternatives, and preferences about the timing of consumption all play a role in determining the term structure. The model thus includes the main factors traditionally mentioned in a way which is consistent with maximizing behavior and rational expectations.

The dynamics of the simplest form of the CIR [1985] model for the interest rate process are given by

(1) $dr = \kappa(\theta-r)\,dt + \sigma\sqrt{r}\,dz$,

where κ, θ, σ^2 are constants, with $\kappa\theta>0$ and $\sigma^2>0$. The model thus assumes that the term structure is fully specified by the instantaneous riskless rate $r(t)$. For $\theta>0$ and $\kappa>0$, this corresponds to a continuous time first-order autoregressive process where the randomly moving interest rate is elastically pulled toward a central location or long-term value, θ. The parameter κ determines the speed of adjustment.

Let's recall the valuation equation of a default-free discount bond, $P[r,t,T]$, that pays one unit at time T, $\tau=T-t$ periods from now. Using the drift and variance formulae for r as well as the term for the factor risk

premium, λ, the fundamental equation is given by

(2) $(1/2)\sigma^2 r P_{rr} + \kappa(\theta-r)P_r + P_t - \lambda r P_r - rP = 0$.

with the boundary condition $P(r,T,T) = 1$. Ito's Lemma gives the expected price change for the bond as

(3) $(1/2)\sigma^2 r P_{rr} + \kappa(\theta-r)P_r + P_t$

thus $\lambda r P_r + rP$ is the expected return on the bond, and $r + \lambda r P_r/P$ is the rate of that expectation. Obviously, $\pi = \lambda r P_r/P$ is the term premium on the bond, and can be written in two parts as

(4) $\pi = (\lambda r) \ (P_r/P)$

where λr is the covariance of changes in r with the percentage changes in "market portfolio", and P_r/P is the elasticity with respect to the interest rate, or the negative of the modified price duration. Since $P_r < 0$, $\pi > 0$ if $\lambda < 0$ (i.e. negative correlation with optimally invested wealth (the market portfolio).

    In the present single factor model, the returns on all assets are perfectly correlated to the extent they are all exposed to the risk inherent in the interest rate process. The instantaneous interest rate on the zero coupon bond can be written as

(5) $dP/P = [\kappa(\theta-r)P_r/P + P_t/P + (1/2)\sigma^2 r P_{rr}/P] \ dt + \sigma\sqrt{r} \ P_r/P \ dz$

        $= \mu(r,t,T) \ dt + \upsilon(r,t,T) \ dz$.

By taking the relevant expectations, and using the Brown and Dybvig [1986] notation, the solution of the fundamental equation is of the form

(6) $P[r,t,T] = A[t,T] * \ e^{-B[t,T]r}$

where for $\tau = T-t$,

(7a) $A[t,T] = \{ \phi_1 \exp(\phi_2 \tau) / (\phi_2[\exp(\phi_1 \tau)-1]+\phi_1) \}^{\phi_3}$

(7b) $B[t,T] = [\exp(\phi_1 \tau)-1] / [(\phi_2[\exp(\phi_1 \tau)-1]+\phi_1)]$

where

(7c) $\phi_1 = \{ (\kappa+\lambda)^2 + 2\sigma^2 \}^{1/2}$

(7d) $\phi_2 = (\kappa+\lambda + \phi_1) / 2$

(7e) $\phi_3 = 2\kappa\theta / \sigma^2$.

2.2 Previous Empirical Tests of the CIR Model

In a recent paper, Brown and Dybvig [1986] tested the nominal form of the one-factor CIR model of the term structure on 373 cross-sections of monthly US Treasury, default free coupon bond prices data from 1952 through 1983. Each cross-section was used to produce maximum likelihood estimates of the identifiable parameters. Using their cross-sectional methodology, they pointed out that the CIR [1985] model was not fully identified. Nevertheless, they came out with estimates of the short-rate of interest (which was treated as an additional parameter), the long-term rate or mean reversion rate, and the variance parameter.

Stambaugh [1986] offered another test of the Cox-Ingersoll-Ross [1985] model by focusing on the nature of the expected excess returns. In such a model, the expected excess returns are linear functions of forward premiums, where the number of latent variables captured by the forward premiums equals the number of state variables in the pricing relation. Stambaugh pointed out that when the maturities of the forward premiums match those of the excess returns, the number of state variables required to describe expected returns can appear to be large due to measurement errors. An alternative set of forward premiums with non-matching maturities is postulated to reduce the

problems arising from measurement error. Tests of the number of latent variables in expected returns on U.S. Treasury bills were performed in a Generalized Method of Moments (GMM) framework and rejected a single-variable specification of the term structure. But Stambaugh provided some evidence that two or three latent variables appear to describe expected returns on bills of all maturities. Expected returns estimated using two latent variables vary with business cycles in a manner similar to what Fama [1986] observed for forward rates for the full range of maturities. Nevertheless, Stambaugh concluded by affirming that expected returns on U.S Treasury bills appear to change over time in a manner that is consistent with parsimonious models of the term structure, such as models developed by Cox, Ingersoll, and Ross [1985].

3. THE ECONOMETRIC METHODS

In the nominal form of the CIR [1985] model, the successful identification and estimation of the parameters are sufficient to price nominal bonds of any maturity. Moreover, the yields on these bonds are perfectly positively correlated. This follows from

(8a) $P_t(r,\tau) = A(\tau) \exp\{-B(\tau)r(t)\}$,

(8b) $y_t(r,\tau) = - \ln[P_t(r,\tau)]/\tau = a(\tau) + b(\tau)r(t)$,

(8c) $a(\tau) = -\ln[A(\tau)]/\tau$, $b(\tau) = B(\tau)/\tau$,

(8d) $COV[y_t(r,\tau_1),y_t(r,\tau_2)] = b(\tau_1)b(\tau_2)VAR[r(t)]$,

(8e) $VAR[y_t(r,\tau_1)] = b(\tau_i)^2 VAR[r(t)]$,

where COV and VAR are unconditional covariances and variance operators, and where $VAR[r(t)] = \sigma^2 r(t)$.

In this section, we will lay out the econometric method used to estimate the parameters of the interest rate model. Both the generalized method of moments and the maximum likelihood estimation procedures will be explained, along with their statistical properties and inference. A statistical testing methodology for contingent claims models will be developed, using the principle of invariance. But before, let's address the identification problem of the CIR [1985] model, as suggested by Brown and Dybvig [1986].

3.1 The Identification Problem of the CIR Model

In practice, what we observe is the nominal price of a discount bond at date t for a dollar deliverable at $t+\tau$. In a single model of the term structure of interest rates, if the state variable is observable, and if there are several bonds of different maturities to permit identification of the problem, then simply equating the functional form with the observed discount bond prices or yields, at a point in time, would represent a way to estimate the parameters of the model from a cross-section of securities, using non-linear regression techniques. To that effect, Gibbons and Ramaswamy [1986], in their two-factor model of the real term structure, add

> Even without observing the state variable, non-linear cross-sectional regressions may allow identification and estimation of the underlying parameters as long as the state variables are treated as additional parameters. However, it is not clear how to link the time series properties of the state variables to the estimated parameters [p. 7].

Brown and Dybvig [1986] mentioned an identification problem, associated with the CIR model. Originally cast using four parameters and a state variable, or five parameters, $(\kappa, \theta, \lambda, \sigma, r)$, the model cannot be separately identified, using cross-sectional data. The reason lies in the appearance of κ and λ as $\kappa+\lambda$ almost everywhere in the model, so that, when considering a single cross-section, the four parameters $(\kappa, \theta, \lambda, \sigma)$ can be expressed as three (ϕ_1, ϕ_2, ϕ_3). One way to

alleviate the problem is to impose the no-arbitrage condition, the so-called Local Expectations Hypothesis, i.e set $\lambda = 0$.

Another way to solve the identification problem is to take advantage of the availability of the dynamic specification of the single forcing variable, the instantaneous riskless rate, $r(t)$. Specifically, one can observe the term structure at different points in time, the so-called time series approach, and make use of the availability of a functional form within the CIR model for the relevant densities of the unobservable forcing variable, r. In doing so, our system becomes "overidentified", since a single forcing exogenous variable is used to explain two sets of endogenous variables. One is then able to exploit this availability in testing the overidentifying restrictions and arriving at parameter estimates.

In estimating a nonlinear system related to the proposed one, Gibbons and Ramaswamy [1986] used the Generalized Method of Moments (GMM) first developed by Hansen [1982] and employed in Hansen and Singleton [1983], Brown and Gibbons [1985] and elsewhere. Gibbons and Ramaswamy used the restrictions on the population first and second moments of the CIR model to identify the parameters. The first moment restrictions are given by the functional form of the CIR model for the discount bonds, taken separately, at the same given point in time. The second moment restrictions, or cross and autocovariances, are given by the functional form of the variance and autocovariance functions of the model, for two discount bonds, identical or different, taken at two different points in time.

Our approach somewhat differs form the one taken by Gibbons and Ramaswamy [1986], but nevertheless, carries the essential of the moment matching idea. In a first step, the term structure of interest rate, i.e. the yield curve for discount securities of different maturities, is calculated at a given point in time, assuming starting values that would made mean reversion possible $(\kappa, \theta > 0)$. In a second step, we use

these estimates of the yield curve equation as initial estimates, ϕ_{10}, ϕ_{20}, and σ_0, for the yield curve volatility or second moment restrictions. Alternatively, we can use these initial estimates to estimate jointly both the yield and yield volatilities simultaneously, by using also a non-linear estimation technique. In that case, the minimand uses both the errors in the yields and in their volatilities. The One-Step Theorem ensures us that, for the parameters estimated in the first procedure, i.e. ϕ_1, ϕ_2, and σ, the final estimates for those estimates have the same statistical properties as the estimates of the first step.

The yield curve volatilities are measured using the daily yield observations of the recent, past trading days. Usually, a three-month period, consisting of approximately 60 trading days is chosen; but periods such as one month (last 20 trading days), or one year (252 trading days) could be chosen. Different methods of estimations could be chosen to obtain a standard deviation of the yield estimates, the proxy for the volatility. Usually, the simple maximum likelihood estimate of the standard deviation will be chosen. If the data is thought to be unreliable, a robust estimate, one that minimizes the absolute mean deviation, as opposed to the square of the deviations, could be used.

Working with daily bond price data also uncovers the problems associated with nonsynchronous trading, which is thought to cause the observed autocorrelations in daily returns on asset prices. Fama [1970] found slightly positive average autocorrelations in examining daily security returns with a lag of one day and no empirical evidence of significant autocorrelations for higher lags. Also, daily market- index returns exhibit a pronounced positive first- order autocorrelation. This index phenomenon has been called the Fisher effect since Lawrence Fisher [1966] hypothesized its probable cause. Hence a correction for auto-correlation of daily errors in yields would be used, in calculating the second moments of the yield relationships.

Since the volatility restrictions do not depend on the mean-reversion parameter, θ (CIR notation), or equivalently, ϕ_3 (Brown-Dybvig notation, thereafter BD), values for the parameters ϕ_1, ϕ_2, and σ can be found by a non-linear estimation procedure.

3.2 The Maximum Likelihood Estimation Procedure

The maximum likelihood estimation procedure proposed here, which differs from the ones used in estimating the CIR model by Brown and Dybvig [1986] and Gibbons and Ramaswamy [1986], nevertheless carries the essential of the moment matching idea. It exploits the dynamic specification of the CIR model in testing the overidentifying restrictions and arriving at parameter estimates. Specifically, we still want to minimize the quantity (minimand)

(9)   Minimize $M(\beta) = g_T(\beta)'\Omega^{-1}g_T(\beta)$

$\qquad \beta \epsilon B$

where $\beta = \beta(r,\kappa,\theta,\sigma,\lambda)$ are the model parameters, Ω is a weighting matrix, the asymptotic covariance matrix of the vector of sample moment conditions, $g_T(\beta)$; the vector $g_T(\beta)$ is now defined by stacking the vectors from the relations on the first moment (the yield curve) and the second moment (the yield curve volatility):

(10a) $h_{1t}(r_i;\beta) = y_t(r_i) - a(r_i) - b(r_i)r(t)$

(10b) $h_{2t}(r_i;\beta) = SD[y_t(r_i)] - b(r_i)SD[r(t)]$, i, j = 1,..., n

where SD is the unconditional (relative) standard deviation operator, and where n is the number of available maturities for the pure discount instruments. In practice, the unconditional standard deviation is estimated using recent daily observations, correcting for the Fisher effect (first autocorrelation, due to nonsynchronous trading). Since

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h_{1t} and h_{2t} are assumed to be normally distributed, our procedure involves choosing β from a feasible region B to minimize these deviations from the model. Note that the state variable, $r(t)$, can be either left outside of the estimation, if it is observable with precision at the valuation date, or can be incorporated in the maximization procedure, as an additional parameter.

The estimation problem we face is the joint estimation of the system (β, Ω). Usual applications of the MLE require a two-step procedure to find β, where Ω is first set equal to the identity matrix, and the resulting set of estimates for β are used to construct a second Ω matrix, different from the identity matrix. In the procedure described above, the special structure of the orthogonality conditions, $h_t(\beta)$, is used to avoid this first step. Looking at the special nature of the estimation problem, the assumption of serially uncorrelated $h_t(\beta)$, due to a rationality assumption that agents use all past information, would be reasonable. There is no reason, a priori, to expect the errors in the yield curve matching, the h_{1t} condition, could be correlated with the ones in the yield curve volatility matching, the h_{2t} condition, nor there is any reason to expected correlations in the two h_{1t}s or h_{2t}s of different maturities. For these reasons, the diagonal structure will be adopted for the weighting matrix. The asymptotic justification for MLE only requires that the weighting matrix be a consistent estimator of the asymptotic variance- covariance matrix for $h(\beta)$.

The covariance matrix of the asymptotic distribution of the MLE estimator for β can then be consistently estimated by:

(11a) $Var(\beta^*) = [D'(\beta^*)\Omega^{-1}D(\beta^*)]^{-1}$,

(11b) $D(\beta^*) = T^{-1} \Sigma_t \ \partial h_t / \partial \beta \ |_{\beta = \beta^*}$.

Asymptotic tests and inference can be conducted on the MLE estimates, using the well known three asymptotic tests mentioned in Engle [1984]; these are the likelihood ratio test (LRT), the Wald's test, and the

Lagrange Multiplier (LM) or Rao test. All three tests can be shown to have the same limit Chisquare distribution, $\chi^2(q)$, under the null hypothesis, where q is the number of parameters to estimate. In our case, Wald's test would be the easiest to use, since it requires only to estimate the score and Hessian functions under the unrestricted form. This contrasts with the Lagrange Multiplier test, which requires a new estimation for each restricted values of the parameter, and the LRT, which requires both restricted and unrestricted estimations.

4. DATA SOURCES AND DESCRIPTION

The empirical results reported in this research are based on a cross-section of data on the Government securities market. These data can be obtained from several sources, such as the CRSP tape and the Wall Street Journal. Using the methodology described previously, we estimate the parameters of the term structure of interest rates on the basis of data on the prices of U.S. Treasury issues trading at a given point in time. Treasury issues, other than Treasury bills, are coupon bonds and can be represented in the model if we regard each as the sum of a series of discount issues corresponding to each coupon payment, plus a discount issue corresponding to the terminal payment on the bond. In fact, this decomposition of coupon bonds into coupons and principal repayment of different maturities make possible the estimation of the highly non-linear bond price function with only a handful of coupon bonds, using a cross-sectional sample of bond prices at a given point in time.

Data on US Treasury securities prices were obtained from the Goldman, Sachs & Co. database in August 1986. Those prices were either the trading prices or the mean of the bid and ask price quotations where trading prices were otherwise not available, plus the accumulated interest (for coupon bonds) as of that date. For each price, the corresponding time to maturity as well as (for coupon bonds) the coupon payments, number of payments remaining and
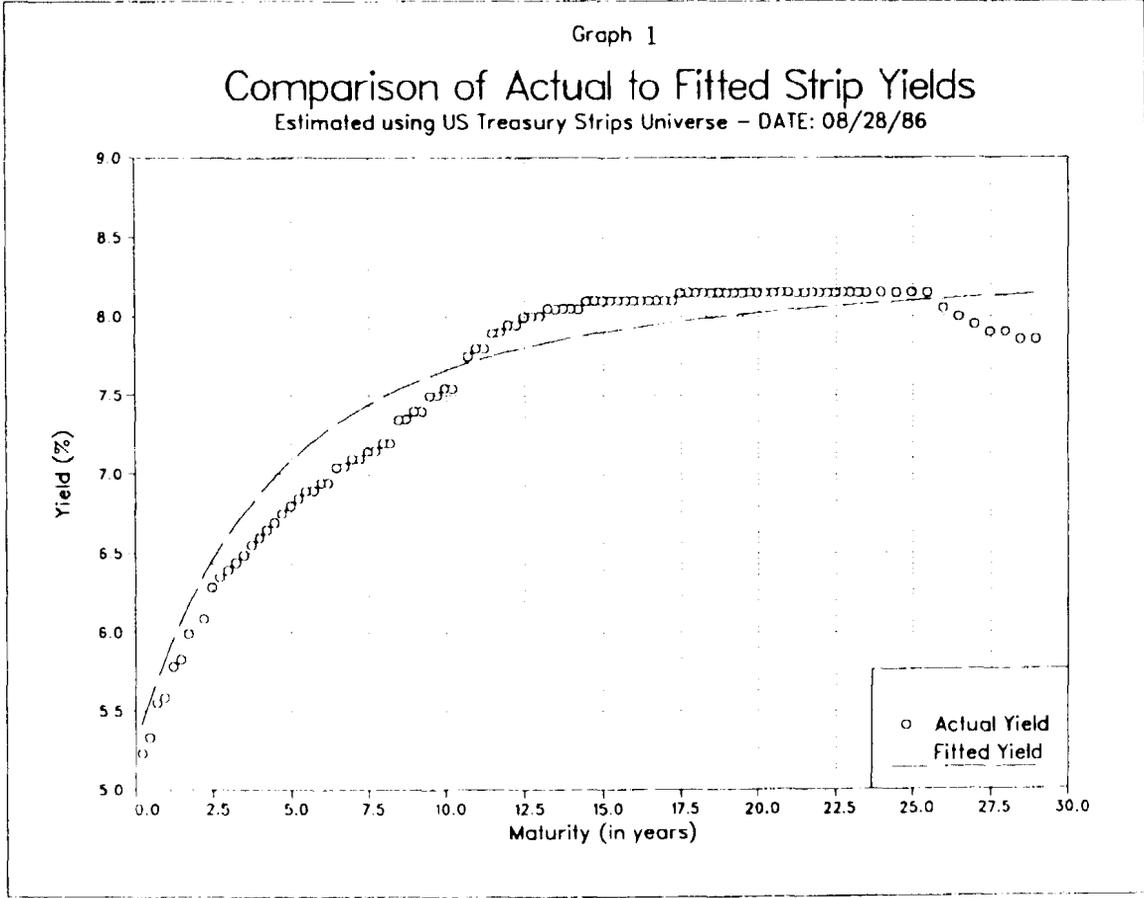
time to next payment were also recorded.

The market for pure discount (zeros or strips) bonds has expanded dramatically over the last years. These pure securities are available at maturities varying between three months and thirty years. So, the universe of U.S. pure discount securities offers a cross- section of between 110 and 120 securities, traded at a given time.[1] For this reason, and because the liquidity problem associated with current versus seasoned issues is not as important an issue (in fact, to some extent, all zeros are equally illiquid), we will use, when available, data from the pure discount securities.[2]

5. EMPIRICAL RESULTS

The estimation of the term structure of interest rates, under the CIR [1985] model involves a two-step procedure, as described in a previous section. In a first step, the yield curve is estimated using the universe of traded US Treasury Strips at a point in time. This yields initial estimates of the parameters of interest $\beta - [r \; \phi_1 \; \phi_2 \; \phi_3]$, as proposed by Brown and Dybvig [1986]. The results of the estimation are shown in Graph 1.

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1. The author recognizes that a nonsynchronous trading problem might exist here. Typically, as in the case of the coupon securities, some bonds are traded more actively than others. In that case, at the end of the day, bonds that did not trade are adjusted up or down before being inputted in the data base. Nevertheless, the pure discount securities offer an analytically tractable way of imposing moment restrictions to identify the parameters of the term structure.

2. Gibbons and Ramaswamy [1986] observed the presence of these stripped, single payment certificates, derived from coupon- bearing Treasuries, but felt that, for their 1964-1983 period of observation, they lacked a sufficient history. Our approach, while raising the problems associated with daily data, does not require a long history of stripped securities.

Graph 1

Comparison of Actual to Fitted Strip Yields

Estimated using US Treasury Strips Universe — DATE: 08/28/86

The point estimates of the cross-sectional estimation were

$$r^* = 0.05278, \quad \phi_1{}^* = 0.41436, \quad \phi_2{}^* = 0.40736, \quad \phi_3{}^* = 11.998 \ .$$

As Brown and Dybvig noted, it is impossible to separately estimate the five parameters of interest $\beta = [r \ \kappa \ \lambda \ \sigma \ \theta]$, using only a cross-section, unless the Local Expectations Hypothesis (LEH) assumption is used. Using the transformation mentioned in Brown and Dybvig [1986], mainly

(7c) $\phi_1 = \{ \ (\kappa+\lambda)^2 + 2\sigma^2 \ \}^{1/2}$

(7d) $\phi_2 = (\kappa+\lambda + \phi_1 \ ) \ / \ 2$

(7e) $\phi_3 = 2\kappa\theta \ / \ \sigma^2$,

the identification problem is highlighted by the following system of 2 equations with three unknowns

(12a) $\kappa\theta = 0.03378$

(12b) $\kappa+\lambda = 0.40036$.

Hence, in a second-step, the yield <u>volatility</u> curve is estimated using a time-series of daily yields on the universe of traded US Treasury Strips for the past 60 trading days prior to the valuation date. The volatilities were adjusted for the econometric problems mentioned in a previous chapter. The results of the volatility estimation are shown in the first fitted curve (Fitted Volatility-1) of Graph 2.

Graph   2

Comparison of Actual to Fitted Strip Yield Volatilities

Estimated using US Treasury Strips Universe – DATE: 08/28/86
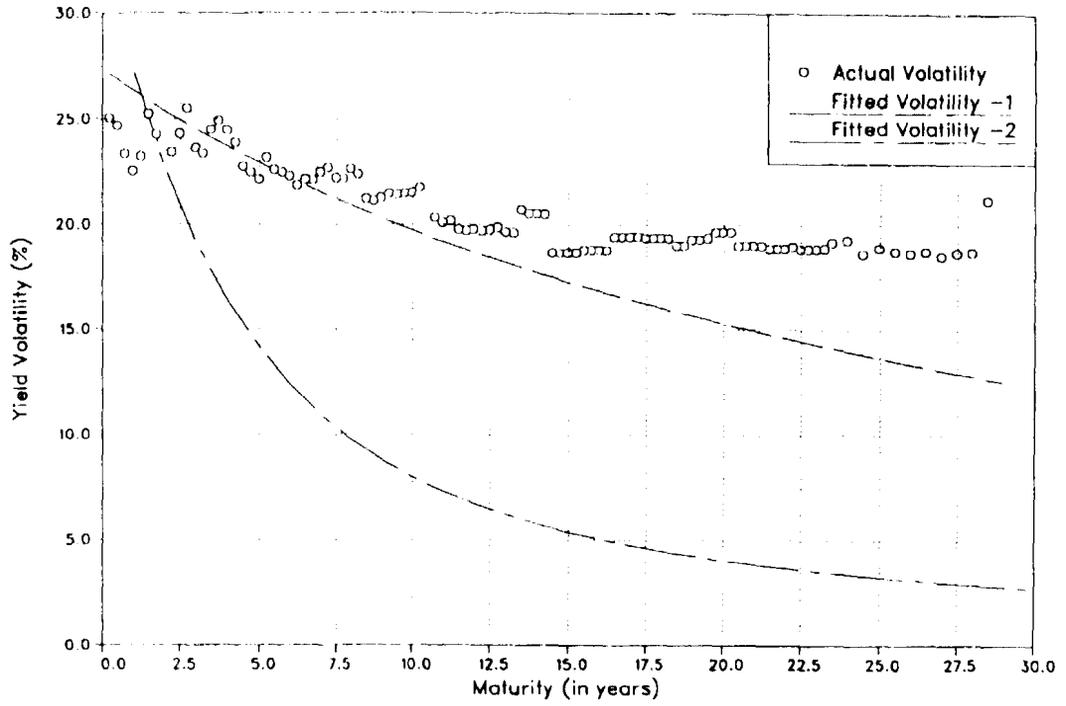
Note that the estimates are biased. Several initial estimation points were tried, but, in all cases, the resulting likelihood estimates were found to be downward biased. This is perceived to be due to the intrinsic limitation of the one-factor model used in this research. Nevertheless, the corresponding "underidentified" system in that case was

(13a) $\kappa\theta = 0.00157$

(13b) $\kappa+\lambda = -0.01464$.

Note also that the long-term mean reversion parameter, θ, is a parameter of interest only in the yield curve matching, and not in the yield volatility curve matching. Since the long-term mean reversion parameter, θ, was found, in the yield curve matching, to be of the order of 8.5%, this would imply the following point estimates for the speed of adjustment and the risk premium parameter in the volatility matching: $\kappa^* = 0.0185$, $\lambda^* = -0.0331$. If this point estimate were to be significant and correct, the excess return on a strip of maturity $\tau = T - t$ would be $-\lambda^* \tau$. This is obtained from the equilibrium rate of return relationship shown previously in the risk premium section. Recalling that

(14) $\alpha(r,t,T) = \alpha(\tau) = r + \lambda^*(r,t)\ P_r/P = r + \lambda^*\ r\ P_r/P = r + \lambda^*\ D_{mod}$,

and since the duration of a zero is equal to its maturity, this would have mean that the expected return on a 30-year zero would have been of the order of $5+(3.31)30 = 104\%$. Since there is no reason a priori to expect the excess return on a 30-year zero to be ten times as much as the one on a 3-year zero, and 30 times as much as the 3.3% excess return on a 1-year zero, the joint estimation of the yield curve and yield curve volatilities was redone, using the initial estimates of the yield curve only estimation and the LEH assumption. The results are found in Table 1

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Table 1   Parameter Estimates of Cox, Ingersoll and Ross Model of
           Term Structure, August 1986 (Standard errors in
           parentheses)

Point Estimates

r^*	κ^*	σ^*	θ^*
0.05278	0.40037	0.07551	0.08544
(0.01065)	(0.00517)	(0.02767)	(0.05539)

Variance-Covariance/ Correlation Matrix (lower- correlation)

r^*	κ^*	σ^*	θ^*
0.0001134	0.0000013	0.0001577	0.0002462
0.023	0.0000267	-0.0001103	-0.0002412
0.53	-0.77	0.0007656	0.0015136
0.42	-0.84	0.98	0.0030681

Long-run yield $(r^*_L) - 8.40\%$

Short-Rate Volatility $- \sigma^*/\sqrt{r^*} - 32.9\%$

The yield curve matching was left almost unchanged. But the yield curve volatility matching is found to be much worse. This is shown in the second fitted curve (Fitted Volatility-2) of Graph 2, where the second fitted volatility is obtained from the final maximum likelihood estimation. The downward sloping volatility curve was observed by other researchers, such as Black, Derman, and Toy [1987] and is often associated with the mean reverting behavior of the short term interest rate:

> "In our model, today's long rate reflects expected future short rates, and today's long-rate volatility reflects expected future short-rate volatilities. Therefore, when we match today's term structure to expected future short rates, our model's future short rate volatility must also decrease with time. In our model... the expected short rate volatility depends only on time and not on the short rate itself. If future short rate volatilities decrease with time, then high future short rates become less likely as time goes by. This damping out of fluctuations in high short rates is equivalent to mean reversion. So, in our model the declining volatility curve is equivalent to mean reversion. [p. 14]

Even if, as in the CIR model, the expected short rate volatility depends not only on time, but also on the short rate itself, these conclusions somewhat remain the same. Specifically, if λ and θ are positive, then one can show that

$$(8.5) \qquad \lim_{\tau \to \infty} \tau^{-1} B(\tau) = 0$$

i.e. in a one-factor model, the yield variability of the long-term zero coupon bond tends to zero, implying a declining yield volatility curve.

This paper examined the justification of using the one-factor general equilibrium model of Cox, Ingersoll, and Ross (CIR) [1985] to model both the term structure of interest rates and its associated volatility. A maximum likelihood approach attempted to match both the yield curve and the yield volatilities, using a short history of zero coupon bonds (strips) from the U.S. government securities market. Although the CIR model appears to match fairly well the yield curve, matching simultaneously both the yield levels and their volatilities is found to be more difficult.

The evidence provided here raised objections in using a one-factor model to model yield curve volatility behavior. In a related study to this paper, Stambaugh [1986] tested the number of latent variables in expected returns on U.S. Treasury bills, using a Generalized Method of Moments (GMM) framework and rejected a single-variable specification of the term structure. But Stambaugh provided some evidence that two or three latent variables appear to describe expected returns on bills of all maturities. Nevertheless, Stambaugh and most researchers agree that expected returns on U.S Treasury bills appear to change over time in a manner that is consistent with parsimonious models of the term structure, such as models developed by Cox, Ingersoll, and Ross [1985].

We have discussed here one limitation of the Cox, Ingersoll, and Ross [1985] model, which imposes restrictions on the joint determination of yield curves and yield volatilities. Because the same parameters determine both the yield curve and the yield curve volatilities, subtle combinations of term structure and volatilities are impossible. For instance, a model where mean reversion occur could not possibly have a rising yield volatility curve. Nevertheless, one- and two-factor models of the term structure remain popular because of their tractability.

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