

Bounds on Expected Values of Insurance Payments and Option Prices

Samuel H. Cox, F.S.A.
University of Nebraska
311 Burnett Hall
Lincoln, NE 68588-0307

Bounds on Expected Values of Insurance Payments and Option Prices

Abstract

This paper presents best upper and lower bounds on the expected value of a reinsurance payment under the terms of a contract written on a random loss with known moments. The bounds are based on results from Kemperman's survey [13] of moment problems. Bounds on the expected exercise value of a financial option are considered also because the mathematical model used for valuing options is similar to that used in reinsurance net premium calculations.

1. Introduction

Actuaries construct stochastic financial models in order to assist insurance company managers in determining the future financial impact of insured events. Some areas where these models are applied include determination of insurance premiums, calculation of benefit reserves and estimation of insurance fund solvency. There are two sources of errors in these models: (1) mis-specification of the model's statistical distribution and (2) estimation errors in determining the parameters of the distributions involved. As an example, consider a fire insurance policy on a house which is worth b . The value X of fire damage to the house is to be paid to the policyholder, less a deductible, at the end of the policy period. The total loss X is bounded, $0 \leq X \leq b$. The insurance company pays the policyholder the excess (if any) of the loss X over the deductible d which is specified in the policy. Let $h(x) = \max\{0, x - d\}$. The benefit to the policyholder is $h(X) = \max\{0, X - d\}$. When the policy is written, X is modelled as a random variable. For determining premiums, an actuary would be interested in the distribution the discounted value of $Y = h(X)$, perhaps for various values of the deductible d . For this example, suppose that the interest rate variation can be ignored; thus the problem reduces to studying the distribution of Y .

Usually moments of X are known from company or industry experience. In determining premiums, reserves, and so on, an actuary would need estimates of the moments of $Y = h(X)$, but would have available estimates of moments of X . To be

specific, suppose that only two moments are used and estimates of the mean μ and variance σ^2 of X are provided. An example of mis-specification would be to suppose the X is normally distributed, even though it is known that X cannot be normal (because $0 \leq X \leq b$ for example). Occasionally this sort of error is acceptable because it is convenient or because only crude estimates are required. The second type of error is attributed to error in estimating the parameters, μ and σ^2 . Errors arise because of inflation, changes in the risks insured (in the fire insurance example, a change in the method of constructing houses is an example of this phenomenon) or changes in coverage (perhaps mandated by regulation or law). This paper is concerned only with model mis-specification error. The proposed technique of dealing with it in this example is to determine the extremes of $E[h(X)]$ as X varies over all random variables X bounded by 0 and b , having mean μ and variance σ^2 .

The bounds which are developed below follow from results surveyed by Kemperman [13]. Many of these are due to Kemperman. A very special case of the upper bound (two moments and $h(x) = \max\{x - d, 0\}$), using the same fundamental principles (polynomial bounds and contact sets), appears earlier in Scarf [17]. Bowers [2] obtained the upper bound independently, but in a slightly different formulation. We introduce the following notation in order to describe the solution. \mathfrak{M} denotes the set of all cumulative distribution functions F concentrated on $[a, b]$. $\mathfrak{M}(y)$ is the subset of \mathfrak{M} having moments $y = (y_1, y_2, \dots, y_n)$. By this we mean that

$$\int_a^b dF(x) = 1 \quad (F \text{ is concentrated on } [a, b])^1$$

and

$$\int_a^b x^i dF(x) = y_i \quad \text{for each } i = 1, \dots, n \quad (F \text{ has the correct moments}).$$

¹The integral symbol denotes the Riemann-Stieltjes integral. This may be correctly interpreted as the usual integral of introductory calculus with integrand $x^i f(x)dx$ where $F(x) = \int_a^x f(t)dt$ in case the cdf F is differentiable. If F is discrete with probability density f , the Riemann-Stieltjes integral reduces to the sum of $x^i f(x)$ over the countably many values of x for which $f(x)$ is positive. The Riemann-Stieltjes integral generalizes these two types of expectations and handles the mixed distributions correctly as well. See Apostol [1] for a development of Riemann-Stieltjes integration.

The best bounds on $E\{h(X)\}$ are denoted formally by

$$L(h | y) = \inf \left\{ \int_a^b h(x) dF(x) : F \in \mathcal{M}(y) \right\}$$

and

$$U(h | y) = \sup \left\{ \int_a^b h(x) dF(x) : F \in \mathcal{M}(y) \right\}$$

where y denotes the specified vector of moments. In the above example $y = (\mu, \mu^2 + \sigma^2)$. In the next section, the solution for an insurance policy with a deductible is developed. It turns out that the best lower bound corresponding to the function $h(x) = \min\{x, d\}$ is given by

$$L(h | y) = \begin{cases} \frac{d\mu^2}{\sigma^2 + \mu^2} & \text{for } 0 \leq d \leq \frac{\sigma^2 + \mu^2}{2\mu} \\ \frac{1}{2} \left[\mu + d - \sqrt{(\mu - d)^2 + \sigma^2} \right] & \text{for } \frac{\sigma^2 + \mu^2}{2\mu} < d \leq \frac{b^2 - \mu^2 - \sigma^2}{2(b - \mu)} \\ \frac{\mu(b - \mu)^2 + (d + \mu - b)\sigma^2}{(b - \mu)^2 + \sigma^2} & \text{for } \frac{b^2 - \mu^2 - \sigma^2}{2(b - \mu)} < d \leq b \end{cases}$$

This bound is a slight generalization of a result of Scarf [17] and Bowers [2], which can be obtained by letting b tend to infinity. The best upper bound for $h(x) = \min\{x, d\}$ is

$$U(h | y) = \begin{cases} d & \text{for } 0 \leq d \leq \mu - \frac{\sigma^2}{b - \mu} \\ \frac{\mu(b + d) - \mu^2 - \sigma^2}{b} & \text{for } \mu - \frac{\sigma^2}{b - \mu} < d < \mu + \frac{\sigma^2}{\mu} \\ \mu & \text{for } \mu + \frac{\sigma^2}{\mu} < d \leq b \end{cases}$$

Figure 1.1 shows the graphs of $L(h | y)$ and $U(h | y)$ as functions of d for typical values of the problem parameters. Figure 1.2 is the graph of the difference $W = U(h | y) - L(h | y)$ for the same values of d . The interesting shape of the graph W changes in an interesting way when the variance σ and limit b are allowed to vary. An example is given in Figure 1.3 in which σ and d vary.

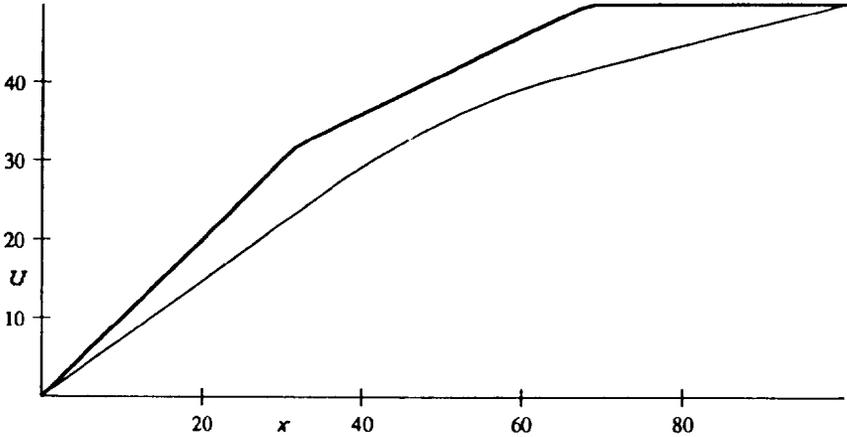


Figure 1.1. Graphs of $L(h | y)$ and $U(h | y)$ for a mean of $\mu = 50$, a standard deviation of $\sigma = 30$, and an upper bound of $b = 100$ as functions of the deductible x , varying from 0 to 100.

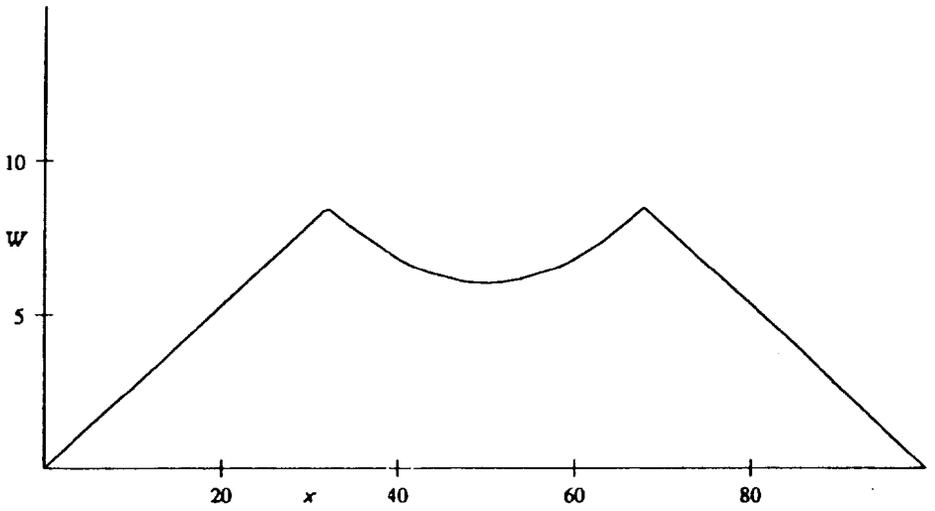


Figure 1.2. Graph of $W = U(h | y) - L(h | y)$ for a mean of $\mu = 50$, a standard deviation of $\sigma = 30$, and an upper bound of $b = 100$ as functions of the deductible x , varying from 0 to 100.

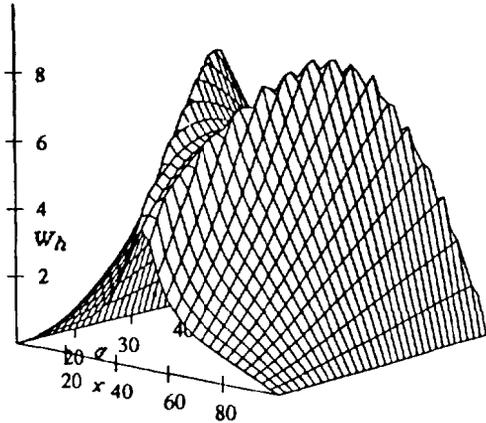


Figure 1.3. Graph of $W = U(h | y) - L(h | y)$ for a mean of $\mu = 50$ and an upper bound of $b = 100$ as functions of the deductible x , varying from 0 to 100, and standard deviation of σ , varying from 10 to 30.

Analogous bounds are similarly obtainable for closely related functions such as $g(x) = \min\{0, x - d\}$, which is the benefit paid to the policyholder if a loss of x occurs. The lower bound for $E[\min\{X, d\}]$ was obtained by Scarf [17]. Bowers [2] obtained the upper bound for $E[\max\{0, X - d\}]$, which is essentially the same result. Since the policyholder's share, $\min\{d, x\}$, and the insurer's share, $\max\{0, x - d\}$, add up to the total loss x , $\min\{d, x\} + \max\{0, x - d\} = x$, then $L(h | y) + U(g | y) = \mu$ and $U(h | y) + L(g | y) = \mu$. Hence the best bounds for $g(x) = \max\{0, x - d\}$ are determined in terms of the bounds on $h(x) = \min\{x, d\}$:

$$U(g | y) = \mu - L(h | y)$$

and

$$L(g | y) = \mu - U(h | y).$$

Scarf and Bowers considered only the case that $b = +\infty$. Lo [14] used Scarf's lower bound on $\min\{d, X\}$ to obtain the upper bound on $\max\{0, X - d\}$ which he then applied to option pricing. We review option pricing applications later.

The general approach to bounds on $E[h(X)]$ in terms of given moments of X has been investigated also by Brockett and Cox [5], [6], Kaas and Goovaerts [11] and Chang [7]. The major difference is that these methods require h to be very smooth. Kemperman's approach does not require smoothness; h need not be differentiable at all.

However, the strong differentiability requirements yield stronger results. In these papers the function h does not enter into the determination of the probability distributions at which the optimal bounds are obtained. The differentiability requirements can be reduced somewhat without changing the results. See Chang [7] or Brockett and Cox [6] for examples. We review some interesting actuarial applications of these results in Section 2.

In section 3 Kemperman's approach is used to derive the best bounds described above in the case of two known moments of a bounded random variable for the functions $h(x) = \min\{x, d\}$ and $g(x) = \max\{0, x - d\}$. Applying these to insurance and option prices is discussed briefly in sections 4 and 5, respectively.

We briefly review the notion of loss elimination ratio from Hogg and Klugman [10]. The LER is the ratio of the expected loss eliminated (from the insurer's viewpoint) to the expected loss. The LER is then

$$\frac{E[\min(X,d)]}{E[X]} = \frac{E[\min(X,d)]}{\mu}$$

and tight bounds are obtained by dividing $L(h | y)$ and $U(h | y)$ by μ . These are described in section 4.

When the policyholder is also an insurance company, the policy is usually called reinsurance. Reinsurance contracts are usually more complex because they typically pay a portion of the excess of the loss over the deductible, subject to a policy limit. Of course, policies sold to individuals can also be more complex and often have policy maximums in addition to deductibles. Thus a more practical function $h(x)$ describing the policyholder's benefit resulting from a loss of x would be described as follows:

$$h(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq d_1 \\ x - d_1 & \text{for } d_1 \leq x \leq d_2 \\ d_2 - d_1 & \text{for } d_2 \leq x \leq b \end{cases}$$

where $0 < d_1 < d_2 < b$ and $d_2 - d_1$ is the policy limit. An example of this type of policy is a fire insurance policy with a deductible of 5,000 on a dwelling worth 2,000,000, and

maximum payment under the policy of 1,000,000. In this case, $d_1 = 5,000$, $d_2 = 1,005,000$ and $b = 2,000,000$. Bounds for this type of policy are obtained at the end of section 4.

Option contracts are reviewed in section 5. The owner of an option contract has the right to buy (or sell) an asset at time T and at a price d set in the contract. One purpose of option pricing theory is to determine the value of these rights at the time the option is written in terms of market conditions, contract values and the distribution of the future price $X(T)$. The common financial models specify that $X(T)$ is a stochastic process with parameters estimated from past prices. That is, moments of X can be estimated from past price changes and appear in the formula for the current value of the option. Merton [15] established conditions for which the current value of European call and put options are their expected values, discounted for interest. A popular reference for the details is Cox and Rubinstein [8]. In section 5, best bounds on European put and call options are given as a final application.

2. Best Bounds for $E[h(X)]$ based on Kemperman's Approach

The following is a special case of the development of Kemperman [13]. As described earlier, \mathcal{M} denotes the set of all cumulative distribution functions on $[a, b]$ and $\mathcal{M}(y)$ is the subset of \mathcal{M} having moments $y = (y_1, y_2, \dots, y_n)$. It is useful to use the more general notion of moments that Kemperman describes. The polynomials x, x^2, \dots, x^n are replaced by general functions g_i and integrable over $[a, b]$ for $i = 1, \dots, n$. The usual situation is that $g_i(x) = x^i$ for each $i = 1, \dots, n$. Some other interesting examples are given by Kemperman; others appropriate to insurance calculations are discussed below. In general,

$$\mathcal{M}(y) = \{F \in \mathcal{M} : \int_a^b g_i(x) dF(x) = y_i \text{ for all } i = 1, \dots, n\}.$$

We will develop formulas for

$$L(h | y) = \inf \left\{ \int_a^b h(x) dF(x) : F \in \mathcal{M}(y) \right\}$$

and

$$U(h | y) = \sup \left\{ \int_a^b h(x) dF(x) : F \in \mathfrak{M}(y) \right\}$$

for real-valued functions h defined on $[a, b]$ which satisfy mild continuity conditions. The methods yield distributions F_U and F_L in $\mathfrak{M}(y)$ for which the bounds are obtained. That is, there are very special distributions, F_L and F_U , which X might have for which the largest and smallest values of $E[h(X)]$ are actually attained:

$$L(h | y) = \int_a^b h(x) dF_L(x) \quad \text{and} \quad U(h | y) = \int_a^b h(x) dF_U(x)$$

In general, the distributions F_L and F_U depend on the vector of specified moments y , the values of a, b , and the function h . However, if strong geometric conditions are required of h , the distributions do not depend on h . Here is an illustrative example.

Let X be a random variable on $[a, b]$ with mean μ and h a continuous real valued function defined on $[a, b]$. Suppose that h is convex. By convex we mean that, for each x and y in $[a, b]$,

$$h(\lambda x + (1 - \lambda)y) \leq \lambda h(x) + (1 - \lambda)h(y)$$

for all λ in $[0, 1]$. The geometric interpretation is that the chord joining the two points $(x, h(x))$ and $(y, h(y))$ is entirely above the graph of h . For example, if h is twice differentiable and $h''(x) > 0$, then the graph of h is convex. We will use Kemperman's approach to show that in this case

$$h(\mu) \leq E[h(X)] \leq h(a)p + h(b)(1 - p)$$

where $p = \frac{(b - \mu)}{(b - a)}$. If X is the random variable which is always equal to μ (i.e. a degenerate or constant random variable), then $E[h(X)] = h(\mu)$ and the lower bound is obtained. That is, in this case F_L is the discrete distribution with all of its probability at μ . If X is the random variable which takes the value a with probability p and the value b with

probability $1 - p$, then its distribution function F_U is in $\mathcal{M}(y)$ because $p = \frac{(b - \mu)}{(b - a)}$ and the upper bound is $E[h(X)] = h(a)p + h(b)(1 - p)$. Note that neither $L(h | y)$ nor $U(h | y)$ depend on h , only on its convexity. This contrasts with the example given earlier, $h(x) = \min\{d, x\}$, for which $L(h | y)$ and $U(h | y)$ depend on d . Note also that the first inequality is Jensen's inequality. Usually Jensen's inequality is stated with the hypothesis that h is twice differentiable and $h^{(2)}(x) \geq 0$ on $[a, b]$. See Bowers *et al* ([3], page 9) for example. Kemperman's approach does not require that h be differentiable.

Before establishing the bounds corresponding to a convex function with one known moment, we continue with the description of Kemperman's method. Suppose there is a

polynomial of degree n , $q(x) = \sum_{j=1}^n d_j x^j$, for which $h(x) \geq q(x)$ for all x in $[a, b]$. Let Z denote the contact set of q , which is defined by

$$Z = \{x \in [a, b] : h(x) = q(x)\}.$$

Assume that there is a cumulative distribution F in $\mathcal{M}(y)$ with its support entirely within the contact set Z , that is,

$$\int_Z dF(x) = 1.$$

In other words, if X has the distribution specified by F , then $\text{Prob}[X \in Z] = 1$. Then, for every cumulative distribution G in $\mathcal{M}(y)$,

$$\begin{aligned} \int_a^b h(x) dF(x) &= \int_Z h(x) dF(x) && \text{(because } F \text{ is concentrated on } Z) \\ &= \int_Z q(x) dF(x) && \text{(because } h = q \text{ on } Z) \end{aligned}$$

$$= \sum_{j=1}^n d_j \int_Z x^j dF(x) \quad (\text{because } q(x) = d_1x + d_2x^2 + \dots + d_nx^n)$$

$$= \sum_{j=1}^n d_j \int_{\cdot} x^j dF(x) \quad (\text{because } F \text{ is concentrated on } Z)$$

$$= \sum_{j=1}^n d_j y_j \quad (\text{because } F \text{ is in } \mathcal{M}(y))$$

$$= \sum_{j=1}^n d_j \int_{\cdot} x^j dG(x) \quad (\text{because } G \text{ is in } \mathcal{M}(y))$$

$$= \int_{\cdot} q(x) dG(x) \quad (\text{because } q(x) = d_1x + d_2x^2 + \dots + d_nx^n)$$

$$\leq \int_{\cdot} h(x) dG(x) \quad (\text{because } q(x) \leq h(x) \text{ on } [a, b])$$

This establishes that $E[h | F]$ is equal to the greatest lower bound, i.e., $E[h | F] = L(h | y)$. Kemperman shows ([13], page 36) that in the circumstances considered here (h and g_1, \dots, g_n continuous functions), such a polynomial q with contact set Z and distribution F concentrated on Z always exist. This means that to determine the best lower bound on $E[h(X)]$ for a given h we need only (if we can) determine q , Z and F . Similarly, to determine $U(h | y)$ we need only study polynomials $q(x)$ of degree n for which $q(x) \geq h(x)$ on $[a, b]$ and distributions with support contained in the contact set $\{x \in [a, b] : h(x) = q(x)\}$. We note that this is essentially the method used by Scarf and Bowers, although they considered only very special cases.

As an illustration, we return to the earlier example concerning a convex function h .

Proposition 2.1: Let X is a random variable on $[a, b]$ with known mean μ . If h is a continuous function which is convex over $[a, b]$, then

$$h(\mu) \leq E[h(X)] \leq h(a)p + h(b)(1 - p) \text{ where } p = \frac{b - \mu}{b - a}.$$

Proof: In this case, there is one moment: $n = 1$ and $g_1(x) = x$. By Kemperman's results, the lower bound is obtained by a distribution F with mean μ concentrated on the contact set Z of a polynomial $q(x)$ of degree $n = 1$. That is, we may assume $L(h | y) = E[h | F]$. The graph of q is a straight line, lying below the graph of h , which touches the graph of h at the points of Z . If the graph of h is strictly convex, Z consists of single point, c . Then F is a degenerate distribution concentrated at c and so $c = \mu$ and $L(h | y) = E[h | F] = h(\mu)$. In general, the convexity of h and linearity of g , imply that Z is at most a sub-interval of $[a, b]$. Moreover, since $q = h$ on Z , h is a linear function when restricted to Z . Hence, $E[h | F] = h(E[X])$ by the linearity of h and the fact that F is concentrated on Z . In either case, $L(h | y) = h(\mu)$.

For the upper bound, q is above h over the interval $[a, b]$. We can suppose h is not linear. Since h is convex, the contact set is at most $Z = \{a, b\}$. Because F is concentrated on Z , its probability density f satisfies $f(a) = p$ and $f(b) = 1 - p$ where $ap + b(1 - p) = \mu$. This implies $p = \frac{b - \mu}{b - a}$. Hence

$$U(h | y) = E[h | F] = h(a)p + h(b)(1 - p)$$

which completes the proof.

The proposition applies to functions h which are twice differentiable and $h^{(2)}(x) \geq 0$ on $[a, b]$. This was developed by Brockett and Cox [5] using results from Karlin and Studden [12]. Here are several actuarial applications. Let $T = T(x)$ be the complete lifetime of a person age (x) . Suppose that the expected lifetime $\mu = E[T]$ is known. In this case, $a = 0$ and $b = \omega - x$. Consider the function $h(t) = (1 + i)^{-t} = v^t$ for $0 \leq t \leq \omega - x$ where i is the annual interest rate. Then $\bar{A}_x = E[h(T)]$ is the net single premium for a life insurance of 1 paid at the moment of death. Since $h^{(2)}(t)$ is positive, we can apply the inequalities just developed:

$$v\mu \leq \bar{A}_x \leq \frac{\omega - x - \mu}{\omega - x} \leq v^{\omega-x} \frac{\mu}{\omega - x}$$

A similar application with $h(t) = \frac{1 - v^t}{\delta}$ where $\delta = \log(1 + i)$ gives bounds on

$E[h(K)] = \bar{a}_x$, the life annuity of 1 per year paid continuously. In this example, the function $-h$ is convex so the inequalities are reversed:

$$\frac{\omega - x - \mu}{\omega - x} + \frac{1 - v^{\omega-x}}{\delta} \frac{\mu}{\omega - x} \leq \bar{a}_x \leq \frac{1 - v^\mu}{\delta}$$

The next example is the expected value of the benefit payment of an insurance policy with a deductible d . Suppose the policy covers a random loss X with $0 \leq X \leq b$ and mean μ and that the policy has a deductible d , $0 < d < b$. In this case $h(x) = \max\{0, x - d\}$ is convex. Hence, we can write that

$$\max\{0, \mu - d\} \leq E[\max\{0, X - d\}] \leq (b - d) \frac{\mu}{b}$$

The loss not covered by the policy, the retention, is $\min\{d, X\}$. Bounds on the expected retention are obtained by the relation $\min\{d, X\} = X - \max\{0, X - d\}$ and the bounds on $E[h(X)]$. In the next section, we consider these functions and two given moments.

The last example gives crude bounds on the price of an option. Suppose that the option valuation model assumptions are such that option prices are discounted expected exercise values (discussed in greater detail later). Consider a call option which conveys the right to buy at time T a stock at a price of K , when the market price is $S(T)$. The current stock price is S and the risk free force of interest is r , under the assumptions of such valuation models. The exercise value is $h(S(T))$ where $h(s) = \max\{0, s - K\}$. The price, which we denote by $C(S, T, K)$, is equal to $e^{-rT}E[h(S(T))]$, where r is the force of interest, since we are assuming prices are discounted expected values. And for the same reason $E[S(T)] = Se^{rT}$. By Proposition 2.1, we have

$$e^{-rT}\max\{0, Se^{rT} - K\} \leq C(S, T, K) \leq e^{-rT}(b - K) \frac{Se^{rT}}{b}$$

$$\max\{0, S - Ke^{-rT}\} \leq C(S, T, K) \leq \frac{b - K}{b} S$$

Simple bounds such as these can also be derived from economic considerations. For the case $b = +\infty$, these bounds are derived on an economic basis by Cox and Rubinstein ([8], page 154).

We derive bounds based on two moments in section 5.

3. Best bounds for $E[\min\{d, X\}]$, given μ and σ^2

The function $h(x) = \min\{x, d\}$ describes the loss retained by the policyholder who buys a policy with a deductible d to cover a random loss X , conditional on $X = x$. The exercise value of a put option is closely related to h : The exercise value of a put option on an asset with price X and having exercise price d is $d - X$ if X is less than d ; it is 0 if X is greater than d . Hence the exercise value is $d - h(X)$.

In this section we develop best bounds on $E[h(X)]$ for a random variable X on $[a, b]$ with two known moments, $\mu = E[X]$ and $\sigma^2 + \mu^2 = E[X^2]$. For most applications $a = 0$. The results are easier to state when $a = 0$ and the general case is obtainable from the case $a = 0$ by a change of variables. Hence, we assume that $a = 0$.

Proposition 3.1: Let $h(x) = \min\{x, d\}$ for $0 \leq x \leq b$, where $0 < d < b$ are given constants. Let X be a random variable with mean μ variance $\sigma^2 > 0$ for which $\Pr\{0 \leq X \leq b\} = 1$. Then the best lower bound on $E[h(X)]$ is

$$L(h | y) = \begin{cases} \frac{d\mu^2}{\sigma^2 + \mu^2} & \text{for } 0 \leq d \leq \frac{\sigma^2 + \mu^2}{2\mu} \\ \frac{1}{2}[\mu + d - \sqrt{(\mu - d)^2 + \sigma^2}] & \text{for } \frac{\sigma^2 + \mu^2}{2\mu} < d \leq \frac{b^2 - \mu^2 - \sigma^2}{2(b - \mu)} \\ \frac{\mu(b - \mu)^2 + (d + \mu - b)\sigma^2}{(b - \mu)^2 + \sigma^2} & \text{for } \frac{b^2 - \mu^2 - \sigma^2}{2(b - \mu)} < d \leq b \end{cases}$$

The best upper bound on $E[h(X)]$ is

$$U(h | y) = \begin{cases} d & \text{for } 0 \leq d \leq \mu - \frac{\sigma^2}{b - \mu} \\ \frac{\mu(b + d) - \mu^2 - \sigma^2}{b} & \text{for } \mu - \frac{\sigma^2}{b - \mu} < d < \mu + \frac{\sigma^2}{\mu} \\ \mu & \text{for } \mu + \frac{\sigma^2}{\mu} < d \leq b \end{cases}$$

where $y = (\mu, \mu^2 + \sigma^2)$.

Proof: From Kemperman's survey ([13], Condition 4.7 on page 36 for example), we know that $L(h | y) = E[h | F]$ where F is a distribution concentrated on $[0, b]$ with support contained in a contact set

$$Z = \{x \in [0, b] : q(x) = h(x)\}$$

for a polynomial $q(x)$ of degree two or less for which $q(x) \leq h(x)$ for all $x \in S$. If $\xi \in Z$ is an interior point and h is differentiable at ξ , then $q'(\xi) = h'(\xi)$ since $q(x)$ touches but does not cross $h(x)$ at ξ . Since the variance is positive, it is only polynomials which touch $h(x)$ at least twice that are worth considering. Suppose $\xi < \eta$ both lie in Z . Consider first the case that both are interior points. It is not possible that $\eta > \xi > d$ because this would require $q'(\xi) = h'(\xi) = 0$ and $q'(\eta) = h'(\eta) = 0$. But $q'(x)$ is a one-to-one function so this cannot happen. If $d > \eta > \xi$, the $q'(\xi) = h'(\xi) = 1$ and $q'(\eta) = h'(\eta) = 1$, which is similarly ruled out. Hence, we must have $0 < \xi < d < \eta < b$. We can expand $q(x)$ in a power series at η and apply the conditions $q(\xi) = \xi$, $q'(\xi) = 1$, $q(\eta) = d$, and $q'(\eta) = 0$:

$$q(x) = h(\eta) + h'(\eta)(x - \eta) + \lambda(x - \eta)^2$$

$$q(x) = d + \lambda(x - \eta)^2 \quad \text{and hence } \xi = d + \lambda(\xi - \eta)^2.$$

$$\text{Thus } q(x) = d - \frac{(d - \xi)(x - \eta)^2}{(\xi - \eta)^2}.$$

In order that $q'(\xi) = 1$ we must have $\xi + \eta = 2d$. So $Z = \{\xi, \eta\}$ is the contact set for $q(x)$

provided $0 < \xi < d$ and $d < 2d - \xi < b$. To discover the probability distribution having mean μ and variance σ^2 which is supported by Z we let p be the probability of $X = \xi$. Solving the moment equations

$$\xi p + \eta(1 - p) = \mu$$

$$\xi^2 p + \eta^2(1 - p) = \mu^2 + \sigma^2$$

after substituting $2d - \eta$ for ξ yields a quadratic equation for η :

$$\eta^2 - 2d\eta + 2d\mu - \mu^2 - \sigma^2 = 0$$

and hence

$$\eta = d + \sqrt{(d - \mu)^2 + \sigma^2}$$

and

$$\xi = d - \sqrt{(d - \mu)^2 + \sigma^2}$$

Now we can calculate

$$L(h | y) = h(\xi)p + h(\eta)(1 - p)$$

$$L(h | y) = \xi p + d(1 - p)$$

$$L(h | y) = d - p(d - \xi)$$

$$L(h | y) = \frac{1}{2}(\xi - \mu)$$

$$L(h | y) = \frac{1}{2}[d + \mu - \sqrt{(d - \mu)^2 + \sigma^2}]$$

The condition that $2d - \xi < b$ is equivalent to the condition $d < \frac{b^2 - \sigma^2 - \mu^2}{2(b - \mu)}$ and the

condition that $\xi > 0$ is equivalent to $d > \frac{\sigma^2 + \mu^2}{2\mu}$. This determines $L(h | y)$ in the range

$\frac{\sigma^2 + \mu^2}{2\mu} < d < \frac{b^2 - \sigma^2 - \mu^2}{2(b - \mu)}$. For other values of d we must consider contact sets which

include an end point of $[0, b]$. The remaining possibilities are sets of the form $Z = \{0, \eta\}$ with $d < \eta < b$ and $Z = \{\xi, b\}$ with $0 < \xi < d$. In the first case, since η is an interior point, the polynomial must have the form $q(x) = d + \lambda(x - \eta)^2$. Then, since $q(0) = 0$, we

must have $q(x) = d - d \frac{(x - \eta)^2}{\eta^2}$. In order that $q(x)$ does not cross $h(x)$ it is necessary that the slope of $q(x)$ at $x = 0$ be less than that of $h(x)$. Hence $q'(0) \leq 1$ insures that $Z = \{0, \eta\}$ is the contact set of $q(x)$. The condition $q'(0) \leq 1$ is equivalent to $2d \leq \eta$. To determine the distribution with support in $\{0, \eta\}$, let p be the probability of 0. Then the moment

equations $(1 - p)\eta = \mu$, $(1 - p)\eta^2 = \mu^2 + \sigma^2$ can be solved for $p = \frac{\sigma^2}{\mu^2 + \sigma^2}$ and

$\eta = \frac{\sigma^2 + \mu^2}{\mu}$. This leads to $L(h | y) = h(0)p + h(\eta)(1 - p) = d(1 - p) = \frac{d\mu^2}{\mu^2 + \sigma^2}$. The

condition that $2d \leq \eta$ is equivalent to $d \leq \frac{\sigma^2 + \mu^2}{2\mu}$. The remaining possible contact set is

$Z = \{\xi, b\}$ gives rise to the polynomial $q(x) = x - \frac{(b - d)(x - \xi)^2}{(b - \xi)^2}$ subject to the condition

that $q'(b) \geq 0$ so that $q(x)$ does not cross $h(x)$. $q'(b) \geq 0$ is equivalent to $\xi \leq 2d - b$. Let p

be the probability of ξ . Solving the moment equations gives $p = \frac{(b - \mu)^2}{\sigma^2 + (b - \mu)^2}$ and

$\xi = \mu - \frac{\sigma^2}{b - \mu}$. Then

$$L(h | y) = h(\xi)p + h(b)(1 - p) = \xi p + d(1 - p) = \frac{(d + \mu - b)\sigma^2 + \mu(b - \mu)^2}{\sigma^2 + (b - \mu)^2}.$$

The condition that $\xi \leq 2d - b$ is equivalent to $\frac{b^2 - \mu^2 - \sigma^2}{2(b - \mu)} \leq d$. This completes the derivation of the best lower bound.

The best upper bound for $h(x) = \min\{x, d\}$ is obtained by considering polynomials $q(x)$ of degree two or less for which $q(x) \geq h(x)$ on $[0, b]$ and their corresponding contact sets. The possible contact sets are

$$Z = \{0, d, b\} \text{ corresponding to } q(x) = d - \frac{(x - b)(x - d)}{b},$$

$$Z = [0, d] \text{ corresponding to } q(x) = x \text{ and}$$

$$Z = [d, b] \text{ corresponding to } q(x) = d.$$

The case of $Z = \{0, d, b\}$ requires $\mu - \frac{\sigma^2}{b - \mu} < d < \mu + \frac{\sigma^2}{\mu}$ and leads in a straightforward

way to $U(h | y) = \frac{\mu(b + d) - \sigma^2 - \mu^2}{b}$. For $d \geq \mu + \frac{\sigma^2}{\mu}$ the contact set must be a subset of

$[0, d]$ because of the two remaining possibilities this is the one with mean μ less than d . In order that a probability distribution with support Z contained in $[0, d]$ have moments $y = (\mu, \mu^2 + \sigma^2)$ it is necessary and sufficient that y be in the convex hull Y of the set $\{(x, x^2) : 0 \leq x \leq d\}$. This is equivalent to the inequalities $\mu^2 \leq \mu^2 + \sigma^2 \leq \mu d$ because Y is the set of points between the graphs of $y = x^2$ and $y = dx$, $0 \leq x \leq d$. Thus we see that the condition

$d \geq \mu + \frac{\sigma^2}{\mu}$ is precisely what is needed to insure that there is a probability distribution F with support in $[0, d]$ and moments y . Then

$$U(h | y) = E[h | F] = \int_0^d h(x) dF(x) = \int_0^d x dF(x) = \mu. \text{ Similarly the condition}$$

$d \leq \mu + \frac{\sigma^2}{\mu}$ is exactly what is needed for the existence of a probability distribution G on

$[d, b]$ with moments y . Such a measure gives $U(h | y) = \int_0^b h(x) dG(x) = \int_0^d d dG(x) = d$.

This finishes the derivation of the best upper bound and completes the proof of the proposition.

Proposition 3.2: Let $g(x) = \max(0, x - d)$ for $0 \leq x \leq b$, where $0 < d < b$ are given constants. Let X be a random variable with mean μ variance $\sigma^2 > 0$ for which $\Pr[0 \leq X \leq b] = 1$. Then the best lower bound on $E[g(X)]$ is

$$L(g | y) = \begin{cases} \mu - d & \text{for } 0 \leq d \leq \mu - \frac{\sigma^2}{b - \mu} \\ \frac{\mu^2 + \sigma^2 - \mu d}{b} & \text{for } \mu - \frac{\sigma^2}{b - \mu} < d < \mu + \frac{\sigma^2}{\mu} \\ 0 & \text{for } \mu + \frac{\sigma^2}{\mu} < d \leq b \end{cases}$$

The best upper bound on $E[g(X)]$ is

$$U(g | y) = \begin{cases} \frac{\mu(\sigma^2 + \mu^2 - d\mu)}{\sigma^2 + \mu^2} & \text{for } 0 \leq d \leq \frac{\sigma^2 + \mu^2}{2\mu} \\ \frac{1}{2}[\mu - d + \sqrt{(\mu - d)^2 + \sigma^2}] & \text{for } \frac{\sigma^2 + \mu^2}{2\mu} < d \leq \frac{b^2 - \mu^2 - \sigma^2}{2(b - \mu)} \\ \frac{(b - d)\sigma^2}{(b - \mu)^2 + \sigma^2} & \text{for } \frac{b^2 - \mu^2 - \sigma^2}{2(b - \mu)} < d \leq b \end{cases}$$

where $y = (\mu, \mu^2 + \sigma^2)$.

Proof: Consider the lower bound $L(g | y)$. Since $g(x) + h(x) = x$, then for all $F \in \mathcal{M}(y)$, $E[g | F] + E[h | F] = \mu$. Hence, $L(g | y) = \inf\{E[g | F] : F \in \mathcal{M}(y)\} = \inf\{\mu - E[h | F] : F \in \mathcal{M}(y)\} = \mu - \sup\{E[h | F] : F \in \mathcal{M}(y)\} = \mu - U(h | y)$. The best lower bounds are obtained by subtracting the best upper bounds of Proposition 3.1 from the mean. Similarly $U(g | y) = \mu - L(h | y)$. This completes the proof.

Both of these results could be extended to the case of three given moments. This is important for insurance applications for which the third moment may measure the skewness of the distribution. The calculations are conceptually straightforward, but rather challenging. The analogous results for unimodal distributions would also be useful. Evidently the same method used in Brockett and Cox [5] to transfer (via Khinechin's characterization of unimodality) from the unimodal setting can be applied here.

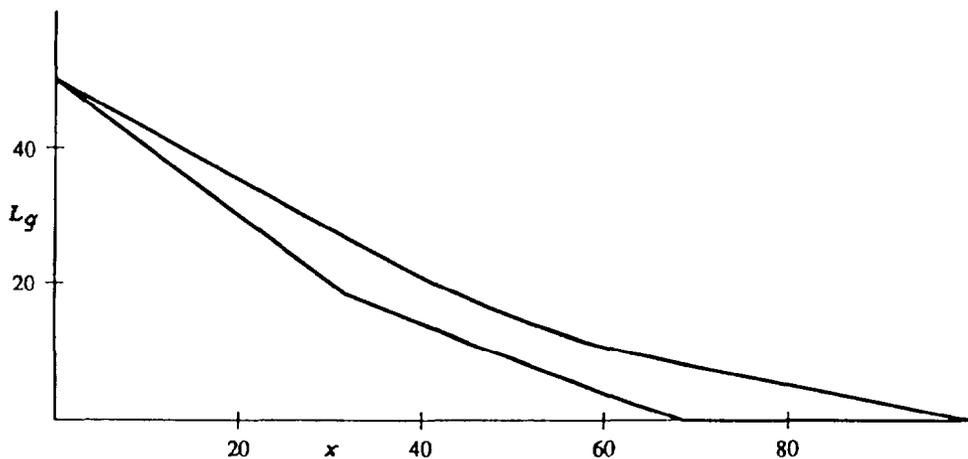


Figure 3.1. Graphs of $U(g|y)$ and $L(g|y)$ for a mean of $\mu = 50$, a standard deviation of $\sigma = 30$, and an upper bound of $b = 100$ as functions of the deductible x , varying from 0 to 100. In this example, $g(s) = \max\{0, s - d\}$.

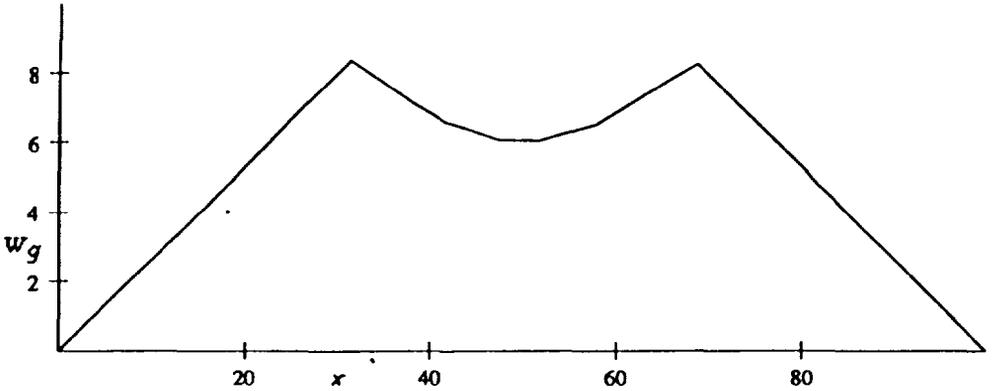


Figure 3.2. Graph of $W_g = U(g | y) - L(g | y)$ for a mean of $\mu = 50$, a standard deviation of $\sigma = 30$, and an upper bound of $b = 100$ as functions of the deductible x , varying from 0 to 100. Here $g(t) = \max\{0, t - d\}$. Note that this is the same graph as that given in Figure 1.2. This is because of the relations $U(g | y) = \mu - L(h | y)$ and $L(g | y) = \mu - U(h | y)$, which imply that $U(g | y) - L(g | y) = U(h | y) - L(h | y)$.

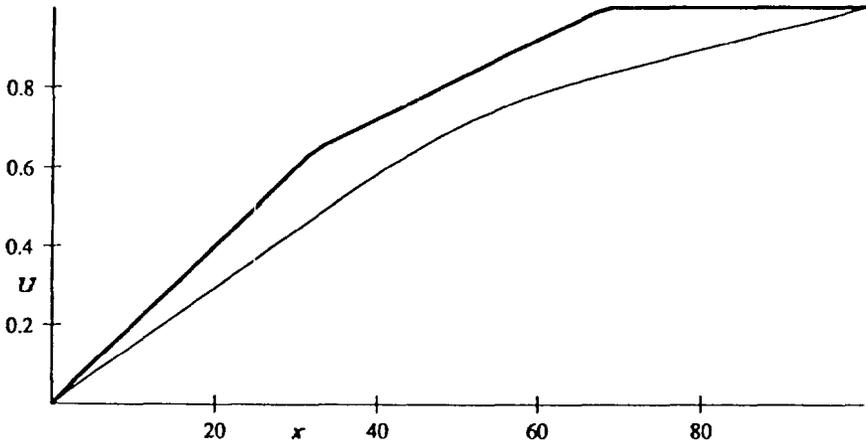
4. Bounds on Insurance Policy Values

In this section we consider applications to insurance in greater detail. Consider the problem of determining premiums after a change in deductible. The concepts are discussed in detail by Hogg and Klugman [10], which is summarized briefly as follows. The expected value of benefits paid under a policy is often referred to as the pure premium. Let p be the frequency of loss for the policy period. This means that the probability of an insured loss (of some size) occurring is p . The random loss is X . We assume that its mean μ and variance σ^2 are known, and that $0 \leq X \leq b$ with probability 1 where b is a given upper bound. The pure premium for a full coverage policy then is $pE[X]$. When the policy deductible of $d > 0$ is introduced, then the benefit to the policyholder changes from X to $g(X)$ where $g(x) = \max\{0, x - d\}$. The new pure premium is $pE[g(X)]$, the

frequency times the new expected benefit². The loss elimination ratio is $\frac{E[X;d]}{\mu} = \frac{E[h(X)]}{E[X]}$

where $h(x) = \min\{d, x\}$. The excess pure premium ratio is the ratio of the pure premium with deductible d to the pure premium without the deductible. It is equal to $\frac{E[g(X)]}{E[X]}$ and, since $g(x) = x - h(x)$, this reduces to $1 - \frac{E[X;d]}{\mu}$. Propositions 3.1 and 3.2 above apply to

give bounds on loss elimination ratios and excess pure premiums. Graphs of the resulting of upper and lower bounds on the loss elimination ratios are given in Figure 4.1.



²The frequency of (non-zero) payments, but not loss occurrences, would change to $p\Pr[X > d]$. Best upper and lower bounds on $\Pr[X > d]$ subject to the moment constraints can be obtained by applying the method recommended by Kemperman to the function $h(x) = 1$ for $x > d$ and $h(x) = 0$ for $0 \leq x \leq d$.

Figure 4.1: Graphs of the upper and lower bounds on $\frac{E[X;d]}{\mu}$ for a mean of $\mu = 50$, a standard deviation of $\sigma = 30$, and an upper bound of $b = 100$ as functions of the deductible d , varying from 0 to 100.

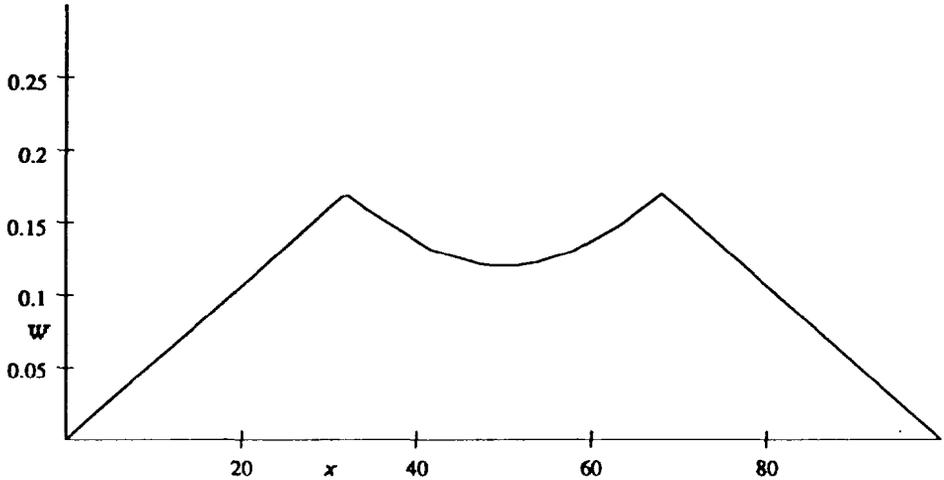


Figure 4.2. Graph of the difference of the upper and lower bounds of $\frac{E[X;d]}{\mu}$ for a mean of $\mu = 50$, a standard deviation of $\sigma = 30$, and an upper bound of $b = 100$ as functions of the deductible d , varying from 0 to 100.

The methods used here can be applied also to determine bounds applicable to franchise deductibles and policies subject to a deductible and a maximum payment. The franchise deductible is described by Hogg and Klugman [10] as follows. A policy which specifies a franchise deductible of d pays nothing for losses $X = x$ if $x \leq d$ but pay the full loss $X = x$ if $x > d$. In this case, we define $h(x) = 0$ if $0 \leq x \leq d$ and $h(x) = x$ if $d < x \leq b$. The Kemperman approach to bounding $E[h(X)]$ applies; it does not require that h be continuous.

Reinsurance contracts are the insurance policies which one company buys from another. The model we are using applies to these as well of course. However, these contracts (as well as some contracts sold to individuals) usually require a more complex definition of the function h than the simple deductible policy (to which Propositions 3.1 and 3.2) apply. The next level of complexity is a policy which specifies a maximum benefit. Propositions 4.1 provides the bounds for such policies.

Proposition 4.1: Let

$$h(x) = \begin{cases} x & \text{for } 0 \leq x \leq d_1 \\ d_1 & \text{for } d_1 \leq x \leq d_2 \\ x - d_2 + d_1 & \text{for } d_2 \leq x \leq b \end{cases}$$

where $0 \leq d_1 \leq d_2 \leq b$ are given constants. Let X be a random variable with mean μ and variance $\sigma^2 > 0$ for which $\Pr(0 \leq X \leq b) = 1$. Then $L(h | y)$, the best lower bound on $E[h(X)]$, is a function of the two variables, defined on the triangle $\{(d_1, d_2) | 0 \leq d_1 \leq d_2 \leq b\}$, and is described as follows:

For values of d_2 satisfying $0 \leq d_2 \leq \mu - \frac{\sigma^2}{b - \mu}$,

$$L(h | y) = \mu - d_2 + d_1$$

For values of d_2 satisfying $\mu - \frac{\sigma^2}{b - \mu} < d_2 \leq \mu + \frac{\sigma^2}{\mu}$,

$$L(h | y) = \frac{bd_1(b\mu - \mu^2 - \sigma^2) + d_2(b - d_2 + d_1)(\mu^2 + \sigma^2 - d_2\mu)}{bd_2(b - d_2)}$$

For values of d_2 satisfying $\mu + \frac{\sigma^2}{\mu} < d_2 \leq b$,

$$L(h | y) = \begin{cases} \frac{d_1 \mu^2}{\sigma^2 + \mu^2} & \text{for } 0 \leq d_1 \leq \frac{\mu}{2} + \frac{\sigma^2}{2\mu} \\ \frac{1}{2} [\mu + d_1 - \sqrt{(\mu - d_1)^2 + \sigma^2}] & \text{for } \frac{\mu}{2} + \frac{\sigma^2}{2\mu} < d_1 \leq \frac{d_2^2 - \mu^2 - \sigma^2}{2(d_2 - \mu)} \\ \frac{\mu(d_2 - \mu)^2 + (d_1 + \mu - d_2)\sigma^2}{(d_2 - \mu)^2 + \sigma^2} & \text{for } \frac{d_2^2 - \mu^2 - \sigma^2}{2(d_2 - \mu)} < d_1 \leq d_2 \end{cases}$$

Similarly, $U(h | y)$, the best upper bound on $E[h(X)]$, is described as follows:

For values of d_1 satisfying $0 \leq d_1 \leq \frac{b\mu - \mu^2 - \sigma^2}{b - \mu}$,

$$U(h | y) = \begin{cases} \frac{(\mu - d_2 + d_1)(\mu - d_1)^2 + \mu\sigma^2}{(\mu - d_1)^2 + \sigma^2} & \text{for } d_1 \leq d_2 \leq \frac{\sigma^2 + \mu^2 - d_1^2}{2(\mu - d_1)} \\ d_1 + \frac{1}{2} [\mu - d_2 + \sqrt{(d_2 - \mu)^2 + \sigma^2}] & \text{for } \frac{\sigma^2 + \mu^2 - d_1^2}{2(\mu - d_1)} < d_2 < \frac{b^2 - \sigma^2 - \mu^2}{2(b - \mu)} \\ \frac{d_1(b - \mu)^2 + (b - d_2 + d_1)\sigma^2}{(b - \mu)^2 + \sigma^2} & \text{for } \frac{b^2 - \sigma^2 - \mu^2}{2(b - \mu)} \leq d_2 \leq b \end{cases}$$

For values of d_1 satisfying $\frac{b\mu - \mu^2 - \sigma^2}{b - \mu} \leq d_1 \leq \frac{\mu^2 + \sigma^2}{\mu}$, and values of $d_1 \leq d_2 \leq b$, we have

$$U(h | y) = \frac{b(b\mu - \sigma^2 - \mu^2) + (b - d_2 + d_1)(\mu^2 + \sigma^2 - d_1\mu)}{b(b - d_1)}$$

For values of d_1 and d_2 satisfying $d_1 \geq \frac{\mu^2 + \sigma^2}{\mu}$ and $d_1 \leq d_2 \leq b$, we have

$$U(h | y) = \mu .$$

Proof: The argument is similar to that given for Proposition 3.1. We only give the main points. The contact sets which come into play, the corresponding contact polynomials, probability distributions and resulting lower bounds arise as one of the following five cases:

I. $Z = [d_2, b]$ $q(x) = x - d_2 + d_1$

$$L(h | y) = \mu - d_2 + d_1 \text{ for } 0 \leq d_2 \leq \frac{b\mu - \mu^2 - \sigma^2}{b - \mu} \text{ and } 0 \leq d_1 \leq d_2$$

For Case I any distribution concentrated on Z will do for F_L ; the distribution for which the lower bound is assumed is not unique.

II. $Z = \{0, d_2, b\}$ $q(x) = \frac{d_1 x(b - x)}{d_2(b - d_2)} + \frac{(b - d_2 + d_1)x(x - d_2)}{b(b - d_2)}$

$$L(h | y) = \frac{bd_1(b\mu - \sigma^2 - \mu^2) + (b - d_2 + d_1)d_2(\mu^2 + \sigma^2 - d_2\mu)}{bd_2(b - d_2)} \text{ for}$$

$$\frac{b\mu - \mu^2 - \sigma^2}{b - \mu} \leq d_2 \leq \frac{\mu^2 + \sigma^2}{\mu} \text{ and } 0 \leq d_1 \leq d_2$$

For Case II the distribution for which the lower bound is attained is unique. It is discrete and has the probability density function f_L where

$$f_L(0) = \frac{bd_2 - (b + d_2)\mu + \mu^2 + \sigma^2}{bd_2}$$

$$f_L(d_2) = \frac{b\mu - \mu^2 - \sigma^2}{(b - d_2)d_2}$$

$$f_L(b) = \frac{\mu^2 + \sigma^2 - d_2\mu}{(b - d_2)b}$$

III. $Z = \{\xi, d_2\}$ where $\xi = \frac{d_2\mu - \mu^2 - \sigma^2}{d_2 - \mu}$ and $0 < \xi < d_1$

$$q(x) = x - \lambda(x - \xi)^2 \text{ where } \lambda = \frac{d_2 - d_1}{(d_2 - \xi)^2}$$

$$L(h|y) = \mu - \frac{\sigma^2(d_2 - d_1)}{(d_2 - \mu)^2 + \sigma^2} \text{ for } \frac{d_2^2 - \mu^2 - \sigma^2}{2(d_2 - \mu)} \leq d_1 \leq d_2 \text{ and } \frac{\mu^2 + \sigma^2}{\mu} \leq d_2$$

$\leq b$.

For Case III the distribution for which the lower bound is attained is unique. It is discrete and has the probability density function f_L where

$$f_L(\xi) = \frac{(d_2 - \mu)^2}{(d_2 - \mu)^2 + \sigma^2} \quad \text{and} \quad f_L(d_2) = \frac{\sigma^2}{(d_2 - \mu)^2 + \sigma^2}.$$

IV. $Z = \{\xi, \eta\}$ where $\xi = d_1 - \sqrt{(d_1 - \mu)^2 + \sigma^2}$ and $\eta = 2d_1 - \xi$ and $0 < \xi < d_1 < \eta < d_2$.

$$q(x) = d_1 - \lambda(x - \eta)^2 \text{ where } \lambda = \frac{1}{2(\eta - \xi)}$$

$$L(h|y) = \frac{1}{2} \left(\mu + d_1 - \sqrt{(d_1 - \mu)^2 + \sigma^2} \right)$$

$$\text{for } \frac{\mu^2 + \sigma^2}{2\mu} \leq d_1 \leq \frac{d_2^2 - \mu^2 - \sigma^2}{2(d_2 - \mu)} \text{ and } \frac{\mu^2 + \sigma^2}{\mu} \leq d_2$$

$\leq b$.

For Case IV the distribution for which the lower bound is attained is unique. It is discrete and has the probability density function f_L where

$$f_L(\xi) = \frac{1}{2} + \frac{1}{2} \left(\frac{d_1 - \mu}{\sqrt{(d_1 - \mu)^2 + \sigma^2}} \right) \quad \text{and} \quad f_L(\eta) = \frac{1}{2} - \frac{1}{2} \left(\frac{d_1 - \mu}{\sqrt{(d_1 - \mu)^2 + \sigma^2}} \right)$$

V. $Z = \{0, \eta\}$ where $\eta = \frac{\mu^2 + \sigma^2}{\mu}$ and $d_1 < \eta < d_2$

$$q(x) = d_1 - \lambda(x - \eta)^2 \text{ where } \lambda = \frac{d_1}{\eta^2}$$

$$L(h | y) = \frac{d_1 \mu^2}{\mu^2 + \sigma^2} \text{ for } 0 \leq d_1 \leq \frac{\mu^2 + \sigma^2}{2\mu} \text{ and } \frac{\mu^2 + \sigma^2}{\mu} \leq d_2 \leq b.$$

For Case V the distribution for which the lower bound is attained is unique. It is discrete and has the probability density function f_L where

$$f_L(0) = \frac{\sigma^2}{\mu^2 + \sigma^2} \text{ and } f_L(\eta) = \frac{\mu^2}{\mu^2 + \sigma^2}$$

The development of the upper bound $U(h | y)$ is similar. Again we present only an outline.

$$\text{I. } Z = [0, d_1] \quad q(x) = x$$

$$U(h | y) = \mu \text{ for } \frac{\mu^2 + \sigma^2}{\mu} \leq d_1 \leq b \text{ and } d_1 \leq d_2 \leq b$$

For Case I any distribution concentrated on Z will do for F_U ; the distribution for which the lower bound is assumed is not unique.

$$\text{II. } Z = (0, d_1, b) \quad q(x) = \frac{d_1 x(b - x)}{d_1(b - d_1)} + \frac{(b - d_2 + d_1)x(x - d_1)}{b(b - d_1)}$$

$$U(h | y) = \frac{b(b\mu - \sigma^2 - \mu^2) + (b - d_2 + d_1)(\mu^2 + \sigma^2 - d_1\mu)}{b(b - d_1)} \text{ for}$$

$$\frac{b\mu - \mu^2 - \sigma^2}{b - \mu} \leq d_1 \leq \frac{\mu^2 + \sigma^2}{\mu} \text{ and } d_1 \leq d_2 \leq b.$$

For Case II the distribution for which the lower bound is attained is unique. It is discrete and has the probability density function f_U where

$$f_U(0) = \frac{bd_1 - (b + d_1)\mu + \mu^2 + \sigma^2}{bd_1}$$

$$f_U(d_1) = \frac{b\mu - \mu^2 - \sigma^2}{(b - d_1)d_1}$$

$$f_U(b) = \frac{\mu^2 + \sigma^2 - d_1\mu}{(b - d_1)b}$$

III. $Z = \{d_1, \eta\}$ where $\eta = \frac{\mu^2 + \sigma^2 - d_1\mu}{\mu - d_1}$ and $d_2 < \eta < b$

$$q(x) = x - d_2 + d_1 + \lambda(x - \eta)^2 \text{ where } \lambda = \frac{d_2 - d_1}{(\eta - d_1)^2}$$

$$U(h | y) = \mu - \frac{(\mu - d_1)^2(d_2 - d_1)}{(\mu - d_1)^2 + \sigma^2}$$

$$\text{for } d_1 \leq d_2 \leq \frac{\mu^2 + \sigma^2 - d_1^2}{2(\mu - d_1)} \text{ and } 0 \leq d_1 \leq \frac{b\mu - \mu^2 - \sigma^2}{b - \mu}.$$

For Case III the distribution for which the lower bound is attained is unique. It is discrete and has the probability density function f_U where

$$f_U(d_1) = \frac{\sigma^2}{(d_1 - \mu)^2 + \sigma^2} \quad \text{and} \quad f_U(\eta) = \frac{(d_1 - \mu)^2}{(d_1 - \mu)^2 + \sigma^2}$$

IV. $Z = \{\xi, \eta\}$ where $\xi = d_2 - \sqrt{(d_2 - \mu)^2 + \sigma^2}$ and $\eta = 2d_2 - \xi$ and $d_1 < \xi < d_2 < \eta < b$

$$q(x) = d_1 + \lambda(x - \xi)^2 \text{ where } \lambda = \frac{1}{2(\eta - \xi)}$$

$$U(h | y) = \frac{1}{2} \left(\mu + 2d_1 - d_2 + \sqrt{(d_2 - \mu)^2 + \sigma^2} \right)$$

$$\text{for } \frac{\mu^2 + \sigma^2 - d_1^2}{2(\mu - d_1)} \leq d_2 \leq \frac{b^2 - \mu^2 - \sigma^2}{2(b - \mu)} \text{ and } 0 \leq d_1 \leq \frac{b\mu - \mu^2 - \sigma^2}{b - \mu}.$$

For Case IV the distribution for which the lower bound is attained is unique. It is discrete

and has the probability density function f_U where

$$f_U(\xi) = \frac{1}{2} + \frac{1}{2} \left(\frac{d_2 - \mu}{\sqrt{(d_2 - \mu)^2 + \sigma^2}} \right) \quad \text{and} \quad f_U(\eta) = \frac{1}{2} - \frac{1}{2} \left(\frac{d_2 - \mu}{\sqrt{(d_2 - \mu)^2 + \sigma^2}} \right)$$

V. $Z = \{\xi, b\}$ where $\xi = \frac{b\mu - \mu^2 - \sigma^2}{b - \mu}$ and $d_1 < \xi < d_2$

$$q(x) = d_1 + \lambda(x - \xi)^2 \quad \text{where} \quad \lambda = \frac{b - d_2}{(b - \xi)^2}$$

$$U(h | y) = \frac{d_1(b - \mu)^2 + (b - d_2 + d_1)\sigma^2}{(b - \mu)^2 + \sigma^2}$$

$$\text{for } \frac{b^2 - \mu^2 - \sigma^2}{2(b - \mu)} \leq d_2 \leq b \text{ and } 0 \leq d_1 \leq \frac{b\mu - \mu^2 - \sigma^2}{b - \mu}$$

For Case V the distribution for which the upper bound is attained is unique. It is discrete and has the probability density function f_U where

$$f_U(\xi) = \frac{(b - \mu)^2}{b^2 - \mu^2 - \sigma^2} \quad \text{and} \quad f_L(b) = 1 - \frac{(b - \mu)^2}{b^2 - \mu^2 - \sigma^2}$$

For the function $g(x) = x - h(x)$, best upper and lower bounds are obtained directly from Proposition 4.1. $L(g | y) = \mu - U(h | y)$ and $U(g | y) = \mu - L(h | y)$. This completes the proof of Proposition 4.1.

The bounds for both $h(x)$ and $g(x)$ are quite a bit simpler in the limit as b tends to infinity. We restate this special case as a corollary.

Corollary 4.1: Let

$$h(x) = \begin{cases} x & \text{for } 0 \leq x \leq d_1 \\ d_1 & \text{for } d_1 \leq x \leq d_2 \\ x - d_2 + d_1 & \text{for } d_2 \leq x < \infty \end{cases}$$

where $0 \leq d_1 \leq d_2$ are given constants. Let X be a random variable with mean μ and variance $\sigma^2 > 0$ for which $\Pr[X \geq 0] = 1$. Then $L(h | y)$, the best lower bound on $E[h(X)]$, is a function of the two variables, defined on the triangle $\{(d_1, d_2) | 0 \leq d_1 \leq d_2 < \infty\}$, and is described as follows:

For values of d_2 satisfying $0 \leq d_2 \leq \mu$,

$$L(h | y) = \mu - d_2 + d_1$$

For values of d_2 satisfying $\mu \leq d_2 \leq \mu + \frac{\sigma^2}{\mu}$,

$$L(h | y) = \frac{\mu d_1}{d_2}$$

For values of d_2 satisfying $d_2 \geq \mu + \frac{\sigma^2}{\mu}$,

$$L(h | y) = \begin{cases} \frac{d_1 \mu^2}{\sigma^2 + \mu^2} & \text{for } 0 \leq d_1 \leq \frac{\mu}{2} + \frac{\sigma^2}{2\mu} \\ \frac{1}{2}[\mu + d_1 - \sqrt{(\mu - d_1)^2 + \sigma^2}] & \text{for } \frac{\mu}{2} + \frac{\sigma^2}{2\mu} < d_1 \leq \frac{d_2^2 - \mu^2 - \sigma^2}{2(d_2 - \mu)} \\ \frac{\mu(d_2 - \mu)^2 + (d_1 + \mu - d_2)\sigma^2}{(d_2 - \mu)^2 + \sigma^2} & \text{for } \frac{d_2^2 - \mu^2 - \sigma^2}{2(d_2 - \mu)} < d_1 \leq d_2 \end{cases}$$

Similarly, $U(h | y)$, the best upper bound on $E[h(X)]$, is described as follows:

For values of d_1 satisfying $0 \leq d_1 < \mu$,

$$U(h | y) = \begin{cases} \frac{(\mu - d_2 + d_1)(\mu - d_1)^2 + \mu\sigma^2}{(\mu - d_1)^2 + \sigma^2} & \text{for } d_1 \leq d_2 \leq \frac{\sigma^2 + \mu^2 - d_1^2}{2(\mu - d_1)} \\ d_1 + \frac{1}{2}[\mu - d_2 + \sqrt{(d_2 - \mu)^2 + \sigma^2}] & \text{for } \frac{\sigma^2 + \mu^2 - d_1^2}{2(\mu - d_1)} < d_2 \end{cases}$$

For values of d_1 satisfying $d_1 \geq \mu$,

$$U(h | y) = \mu.$$

Figure 4.2 shows the graph of $W_h = U(h | y) - L(h | y)$ where $L(h | y)$ and $U(h | y)$ given by Proposition 4.1 for a loss random variable having mean $\mu = 50$ and variance $\sigma^2 = 900$. The values of d_1 and d_2 satisfy $0 \leq d_1 \leq d_2 \leq b = 100$. d_1 is on the x axis and d_2 is on the y axis.

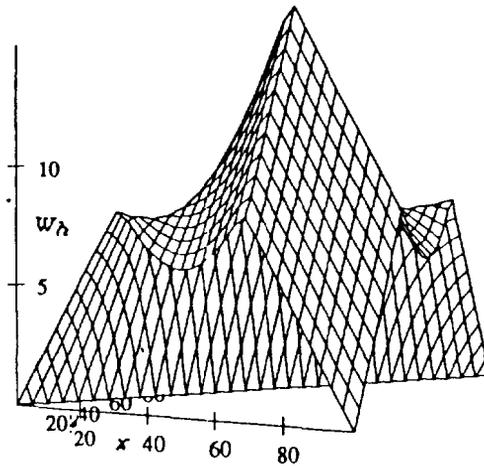


Figure 4.2. Graphs of the upper and lower bounds on $E[h(X)]$ given by Proposition 4.1 when X has mean $\mu = 50$, variance $\sigma^2 = 900$ and is bounded by 0 and 100.

The analog of Proposition 4.1 for the function $g(x) = x - h(x)$ is established by subtracting the bounds of 4.1 from the expected value of X . If h describes the losses which the policyholder, then g describes the benefits. The bounds on expected values found by the relations $U(g | y) = \mu - L(h | y)$ and $L(g | y) = \mu - U(h | y)$ and Proposition 4.1.

5. Bounds on Option Prices

An option is a contract between two people that conveys the right to buy or sell specified property at a specified price for a specified period of time. The person offers the contract for sale is called the writer or seller. The other party is called the owner or buyer. When the contract is made the buyer pays for the right to buy or sell at a known price, which removes some risk in a future transaction. The seller is paid up front to accept the risk of having to buy the asset later at a price which is below the market value. The owner

of the option contract has the right to buy or sell the underlying asset but has no obligation to do so.

In modern financial and commodity markets, exchanges arrange all of the contracts. Traders deal only with the exchange (perhaps through brokers, but we are ignoring them). The exchange assures that the terms of the contract will be met. For example, consider an option owner with the right to buy 100 shares of stock at a contract value of 20 dollars per share, at a time that the stock market price is 25 dollars per share. If the option is exercised, the exchange credits the owner's account with $100(25 - 20) = 500$ dollars (less commissions), and collects that amount from the other party's account. The exchange stands behind the contract for the full exercise value even if the seller's account is short of the required \$500. In order to avoid having to covering a trader's loss, the exchange requires cash margins, in this case of the seller.

For contracts which are settled in cash, the "assets or property" on which the options are written need not be concrete (i.e., the American Stock Exchange offers options on the Major Market Index). However, it is necessary that all parties agree as to what the market price of the asset is. For example, the "asset" could be the value of a stock index, a futures price or foreign currency. These are values which are published widely and which can not be controlled by any of the market participants.

By valuing a contract we mean calculating the market value $C(S, T, K)$ in terms of parameters of the contract, market price statistics of the asset on which the contract is written and interest rates. Two widely used models are the binomial option pricing model and the Black/Scholes option pricing model.

Each of the models applies to options on tangible assets as well as intangibles such as stock indices, futures contracts and other option contracts. The distinguishing feature of the models is the asset price distribution. The asset price movement in the binomial option pricing model is assumed to be binomial. On the other hand, for the Black/Scholes model the asset price is lognormally distributed. Although each model can be generalized to fit more realistic conditions, in their usual context the most important feature is constant interest rates. In this form they are applicable only to short term contracts written on assets whose prices change even though interest rates do not. For example, the models are not applicable to pricing options on bonds.

There are five elements which describe an option contract:

- a) the type of option -- Put or Call.
- b) the underlying asset -- the particular common stock, tract of land, or contract rights *etc.*, which the owner of the option contract buys or sells if the contract is exercised.

We say the option is written on the underlying asset or contract. Some times this it is called the spot asset.

- c) the expiration date of the option.
- d) the exercise (or striking) price.
- e) the rule describing exercise -- American or European.

A call option is a contract which permits the holder to buy an asset during a specified time interval and for a specified price, and requires the seller of the option to sell. Sometimes the seller is called the writer, and selling an option is called writing an option. At the time the contract is written both parties agree as to the price at which the purchase can be made and the time period within which the option can be exercised. A put option differs from a call option in that it allows the holder to sell an asset rather purchase it.

Option contracts are also described by the type of restriction placed on the time the contract may be exercised. A European option contract can be exercised only at the termination of the contract, i.e., on its expiration date, whereas an American option can be exercised at any time up to and including the exercise date. The American option obviously offers the holder greater flexibility which, apparently, makes its valuation more difficult. The American style option is by far the more common throughout the world, but there are a few exchanges in which European options traded. An example, is the Philadelphia exchange which offers both American and European style currency options.

The underlying asset's market price is denoted by S . In option pricing models, S is a random variable, usually with a specified distribution. In all cases, $S \geq 0$. In some cases, such as the binomial option pricing model, S is bounded above as well. In general, the asset could be stock, a bond, a stock index, a bond index, a foreign currency or a commodity such as frozen pork bellies or gold. The contract price at which the owner has the right to exercise the contract is denoted by K . When S exceeds K , the owner of a call option can buy the asset for K under the terms of the contract and then sell it in the market for S , making a gain of $S - K$. The right to do this should such an opportunity occur is what the owner purchased when the contract was written. The seller suffers a loss of $S - K$ when the contract is exercised in this way. Thus, the exercise value of a call option is $g(S)$ where $g(s) = \max\{0, s - K\}$.

The Black/Scholes model [4], [8] results in the following formula for European call options. We present the version applicable to a European call option on a stock which pays no dividends during the term of the option. The risk-free annual, continuously compounded interest rate is denoted by r . In this model the stock price S satisfies a stochastic differential equation $dS = \mu Sdt + \delta Sdw$ where μ and δ are constants and $w(t)$ is

a standard Brownian motion. Thus in this model option market value is calculated as if the asset price is lognormally distributed with known mean and variance. The resulting market value of a European call option which matures in T years is $e^{-rT}E[\max\{0, S(T) - K\}]$. See Merton [15] for a derivation based on Ito's formula.

Black/Scholes Call Option Formula:

$$C(S, T, K) = S\Phi(z) - Ke^{-rT}\Phi(z - \sigma\sqrt{T})$$

where $z = \frac{\log(S) + rT - \log(K) + \frac{1}{2}\sigma^2T}{\sigma\sqrt{T}}$, S is the current price, r is the risk free

force of interest, T is the maturity of the option, K is the exercise price and σ^2 is the volatility of the log-return (this means $\sigma^2T = \text{Var}[\log\{S(T)/S\}]$).

The binomial option pricing model also yields a formula for the option price of European call option of the form $e^{-rT}E[g(S(T))]$. In this model $\log(S(T))$ is binomially distributed with known mean and variance. Such formulas (discounted expected values) for European options follow in very general circumstances. Follmer and Schweizer [9] discuss this in detail. In all of these models, only one parameter, rather than both mean and variance, is needed because the valuation distribution specifies that the mean price at time T is the current price accumulated T years at the risk free rate. The second moment of $S(T)$ about Se^{rT} is a parameter of the valuation distribution. That is, the model assumptions allow us to assume that $E[S(T)] = Se^{rT}$ and hence we need only determine $E\{(S(T) - Se^{rT})^2\}$ in order to apply Proposition 3.2.

When S is less than K, the owner of a put option can buy the assets in the market for S and sell it to the option writer for K, making a gain of $K - S$. Thus the exercise value is $f(S)$ where $f(s) = \max\{0, K - s\}$. Note that $f(s) + s - d$ is equal 0 if $0 \leq s \leq K$ and is equal to $s - K$ if $s \geq K$. That is $f(s) + s - K = g(s)$. As a result, European put option prices can be determined from European call option prices: $Se^{-rT}E[f(S(T))] = Se^{-rT}E[g(S(T))] + S - e^{-rT}K$. This carries over to bounds on European prices as well. Thus whenever the model assumptions are sufficient to imply that the market value of an option, either put or call, is the discounted value the expected payoff to the owner, Proposition 3.2 can be applied to determine optimal bounds on the option's price.

Proposition 5.1. Suppose that the European call option is given by the formula:

$$C(S, T, K) = e^{-rT}E[\max\{0, S(T) - K\}]$$

where S is the current asset price, $S(T)$ is the asset price when the option matures, K is the exercise price, r the risk-free annual interest rate. Let $V = \frac{E[\{S(T) - Se^{rT}\}^2]}{S^2}$. Then the

best lower bound on $C(S, T, K)$ is $\underline{C}(S, T, K)$ given by

$$\underline{C}(S, T, K) = \begin{cases} S - Ke^{-rT} & 0 \leq K \leq Se^{rT} - \frac{V}{b - Se^{rT}} \\ \frac{S^2e^{rT} + V - SK}{b} & Se^{rT} - \frac{V}{b - Se^{rT}} < K < Se^{rT} + \frac{V}{Se^{rT}} \\ 0 & Se^{rT} + \frac{V}{Se^{rT}} < K \leq b \end{cases}$$

The best upper bound on $C(S, T, K)$ is $\bar{C}(S, T, K)$ where

$$\bar{C}(S, T, K) = \frac{VS + Se^{2rT} - Ke^{rT}}{V + e^{2rT}} \text{ when } 0 \leq K \leq \frac{VS + Se^{2rT}}{2e^{rT}}$$

$$\bar{C}(S, T, K) = \frac{1}{2}[S - Ke^{-rT} + \sqrt{(S - Ke^{-rT})^2 + S^2Ve^{-2rT}}] \text{ when } \frac{VS + Se^{2rT}}{2e^{rT}} < K \leq \frac{b^2 - S^2e^{2rT} - S^2V}{2(b - Se^{rT})}$$

$$\bar{C}(S, T, K) = \frac{(b - K)S^2V}{(b - Se^{rT})^2 + S^2V} \text{ when } \frac{b^2 - S^2e^{2rT} - S^2V}{2(b - Se^{rT})} < K \leq b$$

The lower bound on the corresponding put option is equal to $\underline{C}(S, T, K) - S + Ke^{-rT}$ and the upper bound is equal to $\bar{C}(S, T, K) - S + Ke^{-rT}$.

Many of the popular option pricing formulas are discounted expected value formulas. As such they must all give values between the bounds described above. This includes the Black/Scholes formula, the binomial formula and the formula Merton [16] developed using a mixed diffusion-jump process. Just as Lo did, we are using V for the variance of $\frac{S(T)}{S}$ rather than σ^2 . The reason is that in working with options the symbol σ^2 is often reserved for the volatility, which is defined to be the variance per unit time (e.g., the variance per year) of the log-return on the underlying asset. That is, $\sigma^2T =$

$\text{Var}[\log(\frac{S(T)}{S})]$. For a given model, we can determine the relation between σ^2 and V explicitly. For example, in the Black/Scholes setting, the asset price $S(T)$ is log-normal. Hence, the random variable $X = \log(\frac{S(T)}{S})$ is normal, and its variance is $\sigma^2 T$; let μT denote its mean. Then the moment generating function of X is $M_X(s) = \exp(s\mu T + s^2\sigma^2 T/2)$ and hence $E[S(T)] = SE[\exp(\log(\frac{S(T)}{S}))] = SE[e^X] = SM_X(1) = S\exp(\mu T + \sigma^2 T/2)$ and similarly $E[S(T)^2] = S^2 M_X(2) = S^2 \exp(2\mu T + 2\sigma^2 T)$. Since the valuation distribution has the same mean as a risk free investment, $E[S(T)] = Se^{rT}$. This finally gives $V = \exp(2\mu T + 2\sigma^2 T) - \exp(2\mu T + \sigma^2 T) = \exp(2[\mu T + \sigma^2 T/2]) (\exp(\sigma^2 T) - 1) = e^{2rT} (\exp(\sigma^2 T) - 1)$. The relationship between $\sigma^2 T$ and V must be considered when calculating the bounds (for which V is a parameter) and the data available includes price volatilities.

As an illustration, we present the graphs of Black/Scholes option prices and the corresponding bounds. Similar comparisons can be made of the other popular models which yield discounted expected value formulas such as the binomial and jump-diffusion models (see [8] for the formulas and discussions of these models).

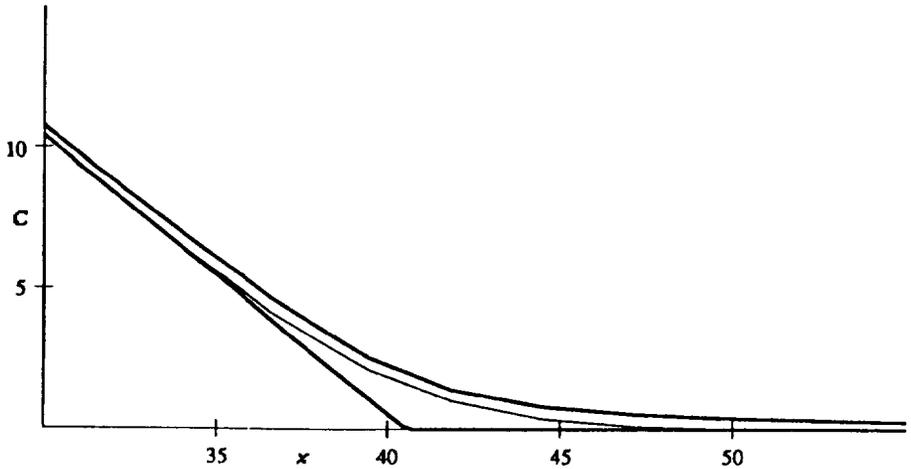


Figure 5.1. The heavy lines are the graphs of $\underline{C}(S, T, x)$ and $\bar{C}(S, T, x)$ for $r = 0.06$, $S = 40$, $\sigma = 0.20$ (and the corresponding V calculated as described above), $T = 12$ weeks = $\frac{12}{52}$ years and exercise price x varying from 0 to 60. The lighter line is the graph of $C(S, T, x)$ given by the Black/Scholes formula:

$$C(S, T, x) = S\Phi(z) - xe^{-rT}\Phi(z - \sigma\sqrt{T})$$

$$\text{where } z = \frac{\log(S) + rT - \log(x) + \frac{1}{2}\sigma^2T}{\sigma\sqrt{T}}$$

6. Conclusion

The upper and lower bounds for expected values of insurance benefits and European option prices have been derived here using methods presented by Kemperman [13]. The option price bounds generalize results of Lo [14]. There are several other areas in which these techniques might be used, but the details have not been worked out. One interesting situation arises when the insurance contract is a function of two (or more) loss random variables. For example, a homeowners policy covers both property losses and liability losses. The moment problem would yield bounds on the expected value of $h(X, Y)$ where X is the property loss and Y is liability loss during a given policy period, and $h(x, y)$ is the policyholders benefit; the bound would be functions of the moments of the joint distribution of X and Y . The same sort of problem arises for health insurance policies which cover hospitalization per day costs as well as costs of treatment.

References

1. Apostol, T. M. (1975). *Mathematical Analysis* Addison-Wesley.
2. Bowers, Jr., N. L. (1969). An upper bound on the stop-loss net premium - actuarial note. *Transactions of the Society of Actuaries* XXI Part I, 211-217.
3. Bowers, Jr., N. L., H. Gerber, J. Hickman, D. Jones, & C. Nesbitt (1985). *Actuarial Mathematics* Society of Actuaries.
4. Black, F. & M. Scholes (1973). The pricing of options and corporate liabilities. *Journal of Political Economy* 81, 637-654.
5. Brockett, P. L. & S. H. Cox (1985). Insurance calculations using incomplete information. *Scandinavian Actuarial Journal*, 94-108.
6. Brockett, P. L. & S. H. Cox (1989). Bounds on the variance and absolute deviation of a unimodal random variable. Working paper.

7. Chang, ~~Y.-C.~~ (1989). University of Texas - Austin doctoral dissertation.
8. Cox, J. C. & M. Rubinstein (1985). *Option Markets*. Prentice-Hall, Englewood Cliffs, New Jersey.
9. Follmer, H. & M. Schweizer (1988). Hedging by sequential regression: an introduction to the mathematics of option trading. *Astin Bulletin* 18, 147-160.
10. Hogg, R. V. & S. A. Klugman (1984). *Loss Distributions*. Wiley, New York.
11. Kaas, R. & M. J. Goovaerts (1985). Bounds on distribution functions under integral constraints. *Bulletin of the Association Royale des Actuairees Belges*, 45-59.
12. Karlin, S. & W. Studden (1966). *Tchebyceff systems with applications in analysis and statistics*. Interscience, New York.
13. Kemperman, J. H. B. (1987). Geometry of the moment problem. *Proceedings of Symposia in Applied Mathematics* 37, 16-53.
14. Lo, Andrew W. (1987). Semi-parametric upper bounds for option prices and expected payoffs. *Journal of Financial Economics* 19, 373-387.
15. Merton, R. C. (1973). Theory of rational option pricing. *Bell Journal of Economics and Management Science* 4, 141-183.
16. Merton, R. C. (1976). Option pricing when underlying stock returns are discontinuous, *Journal of Financial Economics* 3, 144-152.
17. Scarf, H. (1958). A min-max solution of an inventory problem, in: K. J. Arrow, S. Karlin & H. Scarf, eds. *Studies in the mathematical theory of inventory and production* (Stanford University Press, Stanford, CA) 201-209.

