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DISTRIBUTIONS OF DISCOUNTED VALUES

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1. The problem

When reinvestment rates are known in advance, there is no problem in discounting future cash flows. Suppose \$1 invested at time $t-1$ is worth \$ $(1+R_t)$ at time t. R_t is then the rate of return in year $(t-1\ ,\ t)$, with corresponding discount factor

$$V_t \;=\; 1/(1+R_t) \quad , \quad t \;=\; 1, 2, ...$$

The discounted values of cash flows $C_1\ , C_2,\ldots$ occurring at times $1, 2,\ldots$ is then

$$Z \;=\; C_1\, V_1 + C_2\, V_1\, V_2 + ...$$

In this paper, the cash flows and discount factors are random variables with given distributions. The problem is then to try to determine the distribution of the discounted value Z. This problem has received some attention in the actuarial literature over the past twenty years, mostly with respect to the calculation of the moments of Z. However, it is sometimes possible to calculate the distribution of Z explicitly. Some general results will be given, as well as three examples. For a more detailed treatment of the subject, the reader is referred to Dufresne (1991a and 1991b) and Frees (1991).

The following assumptions are made in Sections 2, 3 and 4:

A. Cash flows are independent of discount factors, i.e. the sequences (C_k) and (V_k) are independent one from the other.

B. The variables $V_k\ ,\ k\;=\;1, 2,\ldots$ are independent and identically distribued (i.i.d.).

C. The variables C_k , $k = 1, 2, \ldots$ are i.i.d..

(These assumptions are relaxed in Section 5.)

Stated mathematically, the problem is the following: given distributions for (V_k) and (C_k) , what is the distribution of

$$Z = \sum_{k=1}^{\infty} V_1 \ldots V_k C_k \ ?$$

Section 2 discusses the convergence of the above sum. Section 3 applies a technique known as "time reversal" to the calculation of the moments and distribution of Z. Section 4 gives three examples of explicit distribution for Z. One of these examples is explained in great detail, since it has a long history and also has two other interpretations: one in physics, the other in risk theory. Finally, Section 5 looks at what happens when discount factors are no longer assumed to be independent.

2. Conditions for convergence

In the deterministic case $(V_k \equiv v, C_k \equiv c)$, Z boils down to a geometric series with value $cv/(1 - v)$ (if $v < 1$). Observe that the condition $v < 1$ is equivalent to $g = - \log v > 0$.

It turns out that sufficient conditions for convergence of Z, when discount factors and cash flows are random variables satisfying assumptions A, B and C, are simple extensions of the condition $g > 0$.

Theorem 1. (Vervaat, 1979).

If $E \log V_1 < 0$, $E \log |C_1| < \infty$, then Z converges absolutely with probability one.

Proof. Let $G_k = -\log V_k$. Then

$$|V_1 \dots V_n C_n|^{1/n} = \exp\left\{-\frac{1}{n}\sum_{k=1}^{n} G_k + \frac{1}{n}\log|C_n|\right\}$$

If $E\log V_1$ exists, then

$$\frac{1}{n}\sum_{k=1}^{n} G_k \to E\log V_1 \text{ as } n \to \infty$$

by the strong law of large numbers. Similarly

$$\frac{1}{n}\sum_{k=1}^{n} \log|C_k| \to E\log|C|$$

which implies

$$\frac{1}{n}\log|C_n| \to 0\,.$$

Thus the nth root of the absolute value of the nth term of the series converges to

$$\exp(E\log V_1) < 1.$$

On the basis of the nth root test for convergence of series, we conclude that Z converges absolutely. $\square$

The first condition given in the theorem is equivalent to $E\,G_k > 0$, i.e. that the geometric rate of return is positive an average. It is interesting to note that Z may converge even though $P(G_k < 0) > 0$. For instance, the first condition is verified if $G_k \sim \mathbf{N}(\mu, \sigma^2)$ with $\mu > 0$.

The second condition limits the "dispersion" of the distribution of the cash flows. The condition $E\log|C_1| < \infty$ is weaker than $E|C_1|^{\alpha} < \infty$ for any $\alpha > 0$ (this follows from Jensen's inequality). Hence all distributions such that $E|C|^{\alpha} < \infty$ for some $\alpha > 0$ satisfy the second condition. This includes Pareto and Cauchy distributions, for instance.

Brandt (1986) has shown that the same result holds when the independence assumptions are replaced with ergodicity (i.e. that the law of large numbers applies to the sequences (V_k) and (C_k)).

3. Time reversal

Let

$$Z_n = V_1 C_1 + V_1 V_2 C_2 + \ldots + V_1 \ldots V_n C_n .$$

There are two ways of obtaining Z_n recursively:

(1) the natural or "forward" way:

$$Z_1 = V_1 C_1 , \; Z_2 = Z_1 + V_1 V_2 C_2 , \ldots$$

or, in general,

$$Z_k = Z_{k-1} + V_1 \ldots V_k C_k;$$

(2) the "backward" way:

$$B_{1,n} = V_n C_n , \; B_{2,n} = B_{n-1} \; (B_{1,n} + C_{n-1}) , \ldots$$

or, in general,

$$B_{k,n} = V_{n-k+1} \, (B_{k-1,n} + C_{n-k+1}) .$$

The second technique is not used often in actuarial work, but turns out to be very fruitful in the present context. Its use will be restated as follows. The vector $\tilde{A} = (V_1, \ldots , V_n , C_1, \ldots , C_n)$ has the same distribution as $\tilde{B} = (V_n, \ldots , V_1, C_n, \ldots , C_1)$, because of the independence assumptions. Consider the mapping from $\mathbb{R}^{2n}$ to $\mathbb{R}$

$$\tilde{x} = (x_1, \ldots , x_{2n}) \longmapsto x_1 x_{n+1} + x_1 \; x_2 \; x_{n+2} + \ldots + x_1 \ldots \; x_n x_{2n} = m(\tilde{x}) \; .$$

We have $Z_n = m(\tilde{A})$. Since $\tilde{A} \overset{\mathcal{D}}{=} \tilde{B}$, we therefore also have

$$Z_n = m(\tilde{A}) \overset{\mathcal{D}}{=} \mathrm{m}(\tilde{B}) = B_n$$

where

$$B_n = V_n C_n + V_n V_{n-1} C_{n-1} + \ldots + V_n \ldots V_1 C_1 \; .$$

The foregoing is a rigorous proof of the fact Z_n and B_n have the same probability distribution (all we need to add is that m $(\bullet)$ is $B(\mathbb{R}^{2n})/B(\mathbb{R})$ measurable!). A more intuitive justification is simply that the V's and C's are independent, and we can therefore take them in any pre-established order without changing the resulting distribution.

The sequence $(B_n \, , n \geq 1)$ satisfies the recursive equation

$$B_n = V_n \, (B_{n-1} + C_n) \;\; , \;\; B_0 = 0 \; . \tag{1}$$

In relation to $(Z_n \, , n \geq 1)$ this means that iterating (1) from $n = 1$ to $n = t$ yields a random variable whose distribution is the same as that of Z_t .Observe that this recursive equation is very different from the one obtained for $(Z_n \, , n \geq 1)$. The sequence $(B_n \, , n \geq 1)$ satisfies the same type of equation that *accumulated* cash flows satisfy : if S_t is the accumulated value, at time t , of $C_0 , \ldots , C_{t-1}$, then

$$S_t = (1 + R_t) \, (S_{t-1} + C_{t-1}) \;\; , \;\; S_0 = 0 \; .$$

This shows that there is a kind of "duality" between accumulating and discounting, at least when the processes describing cash flows and discount factors can be taken in reverse order without affecting their distribution (such processes are called "reversible"). When cash flows are equal to c , and discount factors equal to $v = 1/(1 + i)$, this duality is simply expressed as

$$c\, a_{\overline{n}|i} = c\, \ddot{s}_{\overline{n}|-d} \quad , \quad d = i/(1+i)\,.$$

Eq. (1) yields (i) a way of recursively calculating all the moments of Z_n, and (ii) a functional equation satisfied by the distribution of $Z = \lim_n Z_n$, provided the limit exists.

First, Eq. (1) immediately yields a recursive equation for $E\, B_k$, upon taking expected values on both sides of the equation (since V_n is independent of B_{n-1} and C_n). Squaring Eq. (1) similarly yields a recursive equation for the second moments of B_k, and so on. Since B_k and Z_k have the same distribution, the same recursive equations can be used to compute *all* the moments of $(Z_n\, ,\, n \geq 1)$. The reader is referred to Dufresne (1991a) for more details on this subject.

Second, Eq. (1) has an important consequence when looked upon as a relationship between the distributions of B_n , V_n , B_{n-1} and C_n. Observe that on the right hand side the random variables are independent. Furthermore, the distribution of V_n is the same for all n ; same thing for C_n. Letting n go to infinity we obtain the following theorem.

Theorem 2. If $\lim_{n \to \infty} Z_n = Z$ exists, then

$$Z \stackrel{\mathcal{D}}{=} V_1\, (Z + C_1) \tag{2}$$

where V_1 , Z and C_1 are independent on the right hand side.

Formula (2) is not an ordinary equation: it relates not numbers but probability distributions. In the case at hand the known distributions are those of V_1 and C_1 , and the unknown is the distribution of Z. Formula (2) is equivalent to a certain integral equation, but the latter has not been solved in general. This far all known explicit solutions of (2) have been found by ad hoc methods.

4. Examples

I will give three examples. The first one has its origin in physics, and its derivation is relatively simple. The second one is in a way a generalization of the first one, and is taken from Dufresne (1991a). The third one is particularly interesting, because (i) the cash flows are constant, and (ii) the distribution of Z is of the same type as the distribution of the discount factors.

Example 1. (Takacs, 1954, 1955; Vervaat, 1979; see also Karlin and Taylor, 1975, Section 4.3). Takacs considered the following problem. Signals (electronic pulses) arrive according to a Poisson process. The signals are recorded, and each leaves an impression (on the recording apparatus) which decreases exponentially over time. Suppose the recording is started at time 0, and define the following variables: S_t is the total impression at time t ; N_t is the number of arrivals in $[0, t]$; X_i is the amplitude of the ith signal; α is the rate at which impressions decrease; and T_i is the time at which the ith arrival occurs. Then

$$S_t = \sum_{i=1}^{N_t} X_i \; e^{-\alpha(t-T_i)} \;, \; t \geq 0 . \tag{3}$$

One way of dealing with Eq. (3) is to look at what happens in an interval $[t, t+dt]$ and then derive an integral equation for (say) the distribution function of S_t ; this approach is often used in risk theory, and is the one chosen by Takacs in his 1954 paper. In his 1955 paper a slightly different approach is taken. Let

$$U_n = \lim_{t \uparrow T_n} S_t \;.$$

U_n is the total amplitude just before the nth arrival. Then

$$U_{n+1} = e^{-\alpha W_{n+1}} (U_n + X_n)$$

$$= Y_{n+1} (U_n + X_n)$$

where $W_{n+1} = T_{n+1} - T_n$. Thus the process $((U_n, n \geq 1)$ has the same evolution as the sequence of accumulated values of (X_n) if growth factors are (Y_n) . S_t in Eq. (3) can also be interpreted as the accumulated value, at rate $-\alpha$, of amounts (X_i) invested at times (T_i) .

Vervaat (1979) refines this point of view. Let Q_1 the time elapsed since the last arrival, $Q_1 + Q_2$ the time elapsed since the previous arrival, etc. Similarly relabel the impulses C_1 , C_2 ,... starting with the latest one. Then

$$S_t = e^{-\alpha Q_1} C_1 + e^{-\alpha (Q_1 + Q_2)} C_2 + \dots$$

$$= V_1 C_1 + V_1 V_2 C_2 + \dots .$$

(These sums terminate with the first recorded impulse.)

Now suppose the counter has been in operation for a "long time", i.e. consider the steady-state distribution of S. From the properties of Poisson processes, the variables (V_n) are independent with common density

$$f_V(x) = a\, x^{a-1} \quad , \quad 0 < x < 1 \, ,$$

where $a = \lambda/\alpha$, λ the intensity of the Poisson process.

From Theorem 2,

$$S \overset{\mathcal{D}}{=} V_1 \ (S + C_1)$$

By conditioning on the values of V_1 and C_1 (on the right hand side), it can be shown that

$$\varphi_S(t) = \exp\left[a \int_0^t \frac{\varphi_C(u) - 1}{u}\, du\right] \tag{4}$$

where φ_S and φ_C are the characteristic functions of S and C , respectively. For example, if $C_k \sim \exp(m)$, then $S \sim \Gamma(a, m)$.

Reinterpreting this example in terms of discounted values, we have the following: if the geometric rates of return (G_k) are independent and exponentially distributed with mean $1/a$, then the characteristic function of the discounted value of payments $C_1, C_2, \ldots$ (i.i.d. and independent of (G_k)) is given by (4). In particular if $C_k \sim$ **exp** (m) , then the discounted value has distribution $\Gamma(a, m)$.

This example also has a risk theoretic interpretation. Consider the classical compound Poisson process with claims (C_k) occurring at times (T_k) . The discounted value, at rate g , of claims up to time t is

$$D_t = \sum_{i=1}^{N_t} e^{-g\,T_i} C_i$$

$$= \sum_{i=1}^{N_t} V_1 \ldots V_i C_i$$

where $V_i = \exp[-g(T_i - T_{i-1})]$. Willmot (1989) has studied the determination of the distribution of D_t (a difficult task). It is easily seen that, by suitably redefining the parameters, the limit of D_t as t goes to infinity is identical to Z.

Example 2. (Dufresne, 1991a). As an extension of the first example, suppose the discount factors are the product of two independent random variables each of which has a beta distribution of the first kind:

$$V_k = W_{k,1} \cdot W_{k,2} \quad , \quad W_{k,1} \sim \beta^{(1)}_{a,1} \quad , \quad W_{k,2} \sim \beta^{(1)}_{b,1}$$

(i.e. the density of $W_{k,1}$ is $a\,x^{a-1}\,1_{(0,1)}(x)$) . This is equivalent to assuming geometric rates of return to be the sum of two independent exponential distributions.

If, furthermore, the cash flows have an exponential distribution with mean m, then it can be shown that

$$Z = \sum_{k=1}^{\infty} V_1 \dots V_k C_k \sim \beta^{(1)}_{b,1+a} \otimes \Gamma(a, m)$$

i.e. that Z is distributed as the product of two independent random variables, one with a beta distribution and the other a gamma distribution.

Once again this example is related to the discounted value, at a constant rate g, of claims (C_k) occurring at times (T_k). Since the Vs now have a distribution which is the *product* of the two beta distributions, the interarrival times are distributed as the *sum* of two independent exponential distributions.

Example 3.(Chamayou and Letac, 1991). The beta distribution of the second kind is defined as follows: if $a > 0$, $b > 0$ and

$$f_v(x) = (\text{constant})\, x^{a-1}\, (1 + x)^{-a-b}\ 1_{(0,\infty)}(x)$$

then we say that V has a beta distribution of the second kind with parameters a and b. This is also written $V \sim \beta^{(2)}_{a,b}$. This family of distributions is also known as "generalized F" or "generalized Pareto".

Chamayou and Letac have derived the following remarkable fact:

$$\text{if} \quad V_k \sim \beta^{(2)}_{a,b}\,,\ a < b\,,\ \text{then} \ z = \sum_{k=1}^{\infty} V_1 \dots V_k \sim \beta^{(2)}_{a,b-a}\ .$$

5. When discount factors are not independent

The dependence or independence of the discount factors has a significant effect on the distribution of discounted values. This has been noted by Dufresne (1990 and 1991b) and Frees (1991). This section contains some further comments on this topic.

Three cases will be compared: (*i*) independence, (*ii*) perfect positive correlation, i.e. $\rho = \mathrm{Corr}\,(V_t\,, V_{t+1}) = 1$, and (*iii*) perfect negative correlation, i.e. $\rho = -1$. The basis of comparison will be the expectation of the value of one unit discounted frome time t to time 0 , that is to say the mean value of

$$W_t = V_1 \ldots V_t \,.$$

When discount factors are independent and identically distributed,

$$E\, W_t = (E\, V_1)^t \,.$$

Next, suppose the (V_k) are still identically distributed but that $\rho = \mathrm{Corr}(V_t, V_{t+1}) = 1$ for all t. Then (see the remark at the end of this section)

$$\mathrm{P}(V_1 = \ldots = V_t) \;=\; 1\,, \quad t \geq 1\,.$$

This means

$$E\, W_t = E\left(V_1^t\right).$$

By Jensen's inequality

$$E\left(V_1^t\right) > (E\, V_1)^t$$

unless $\mathrm{Var}\; V_1 = 0$ (let us disregard this possibility). Hence discounted values are larger on average when there is perfect positive correlation than when discount factors are independent.

Now turn to the last case, $\rho \;=\; \mathrm{Corr}\,(V_t\,, V_{t+1}) = -1$ for all t. Then (see the remark at the end of this section)

$$\mathrm{P}(V_t + V_{t+1} = 2E\; V_1) = 1\;, \quad t \geq 1\,.$$

In other words $V_2 = 2E\mathrm{V}_1 - \mathrm{V}_1$, $\mathrm{V}_3 = \mathrm{V}_1$, $\mathrm{V}_4 = \mathrm{V}_2$, etc. (N.B. We need to assume that $\mathrm{P}(V_t \in [0,\; 2E\mathrm{V}_1]\,) = 1$ in this case.) Consequently

$$E\,W_{2t} = E[V_1^t\,(2E\,V_1 - V_1)^t]$$

$$E\,W_{2t+1} = E[V_1^{t+1}\,(2E\,V_1 - V_1)^t]$$

for $t = 0, 1, 2, \ldots$. The functions $x \longmapsto x^t\,(2E\,V_1 - x)^t$ and $x \longmapsto x^{t+1}\,(2E\,V_1 - x)^t$ are both concave, and

$$E\,W_t < (E\,V_1)^t \quad , \quad t \geq 1$$

(unless Var $V_1 = 0$). Finally, discounted values are smaller on average when there is perfect negative correlation than when there is independence.

Among the three cases considered, we see that $\rho = -1$ produces the lowest average discounted values, and $\rho = +1$ the highest (given that the (V_k) have the same distribution in all three cases).

In order to give a concrete meaning to what has just been said, define an *equivalent discount factor*

$$v = \lim_{t \to \infty} (E\,W_t)^{1/t}$$

This constant discount factor produces discounted values approximately equal to expected discounted values obtained using V_1, V_1, ... Similarly,

$$r = v^{-1} - 1$$

is an *equivalent arithmetic rate of return.*

We have the following conclusion: given discount factors which are identically distributed, the equivalent arithmetic rate of return is highest when $\rho = -1$ and lowest when $\rho = +1$, the independent case being in between. Thus the choice of a single valuation rate of interest such as r should depend on the distribution of discount factors *and* on the dependence between them. On the basis of the (admittedly simple) example given, it would appear that r should decrease as ρ increases from -1 to $+1$.

Remark. The following result was used: if

$$a X + b \overset{\mathcal{D}}{=} X ,$$

then

(i) $X = b/(1 - a)$ with probability one,

(ii) $a = +1$ and $b = 0$, or

(iii) $a = -1$.

Proof. Of course the result is immediate if Var X exists. The proof given here does not assume the existence of any moment.

Suppose $|a| \neq 1$. If $|a| > 1$, then (5) implies

$$X \overset{\mathcal{D}}{=} \frac{1}{a} X - \frac{b}{a}$$

where $|\frac{1}{a}| < 1$. We can therefore suppose $|a| > 1$.

Iterating Eq. (5), we find

$$a^n X + b(1 + a + \ldots + a^{n-1}) \overset{\mathcal{D}}{=} X , \ n \geq 1 .$$

Since $|a| < 1$, the left hand side converges in distribution to the constant $b/(1- a)$, and X is degenerate.

If $a = +1$, then b has to be zero, since we would otherwise have

$$P(X \leq x) = P(X \leq x - b) \quad , \quad \forall x \in \mathbb{R} . \qquad \square$$

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