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A Regression Approach to Injured Worker Mortality

by

Gary G. Venter
Barbara Schill
Jack Barnett
Workers Compensation Reinsurance Bureau

(REVIEW OF REPORT OF COMMITTEE ON MORTALITY FOR DISABLED LIVES)

Loss reserves for workers compensation cases in the U.S. now are in the area of $50 billion, much of which is tied up in long term cases. Typically standard mortality is used to reserve these cases, but in serious cases a factor (e.g. 10) is applied to the mortality rates on a judgment basis, as in Snader (1987). Some disabled life tables have been calculated from other benefit systems, involving, for example heart disease or cancer cases, but these are probably not appropriate for injured workers.

Faced for the 25 years since the inception of workers compensation insurance with the need for injured worker mortality tables, the CAS decided to take action, and in 1937 appointed a Committee of Three to investigate the feasibility of undertaking a study. Coincidentally, the Committee of Three came up with three conclusions:

1. Very substantial results could not be expected from the data then available.

2. A start should be made in order to get carriers to keep appropriate records.

3. It was as feasible then as it would be at any later time to do a mortality study based on the statistical system in place.

Thus, working with the National Council on Compensation Insurance, a call for disability data was sent out in October 1938. The data used in the study was for accident years or policy years 1930-1935, depending on how carriers reported, and the first year of disability was excluded from each case. Although the first year after the accident was excluded, the data represented fairly new claimants, who might be expected to display higher mortality than more stabilized cases. The results of the study would thus be most applicable to such cases.

This review looks at the data from that study to see if there are any relationships between disabled worker mortality and standard mortality that might endure to the present. A regression methodology is used to explore this question. As the uniform variance assumption of least squares regression is not met, a method for dealing with this heteroscedasticity is developed. The information matrix from the (non-linear) regression is used to test goodness of fit and to develop prediction intervals.

COMMITTEE REPORT

The report of the committee on mortality for disabled lives produced a mortality table for lives disabled by industrial accidents. The table is based on permanent total cases and nondismemberment permanent partial cases involving 50% or more disability. In total there were 8,598 life years of exposure with 285 claim terminations. The 285 claim terminations included deaths and the few cases where the injured person recovered. These claim terminations did not include cases where permanent partial disability followed permanent total, the benefit period ended, or a lump sum settlement was made. Since the mortality table in workers compensation is primarily used to determine expected claim size it is appropriate to include terminations due to either death or recovery. An alternative method is a multiple decrement model in which deaths and recoveries are measured separately. However the committee chose to consider both types of terminations together.

In the original study, mortality rates for each age were calculated based on the reported data. For those ages with sparse data, below age 22 and over age 73, the reported mortality rates were weighted with the mortality rates from the 1930 U.S. life tables for white males. The resulting mortality rates for ages 10 to 105 were graduated using the Whittaker-Henderson technique. Mortality tables were then constructed with these mortality rates.

The authors state that the mortality rate for these disabled lives is 144% of that for white males in the 1930 U.S. Life Tables. This was determined by comparing the expected number of deaths in the next year under the disabled workers table of mortality rates versus the U.S. Life Table mortality rates. The expected number of deaths is determined by multiplying the number of lives exposed for each age group by the respective mortality rate and summing for all ages. It is clear from the data, however that this 144% varies dramatically and systematically by age.

RELATIONSHIP BETWEEN DISABLED WORKER MORTALITY AND STANDARD MORTALITY

We projected the mortality rates for disabled workers based on our hypothesis that the ratio, q_d/q_u, between the mortality rate for disabled workers, q_d; and that of the U.S. population, q_u, is a decreasing function of age. This is an alternate method of graduation to the Whittaker-Henderson formula used by the committee. Initially we set the mortality rate of disabled workers equal to a constant plus a power of the mortality rate of the U.S. multiplied by a function of age;

$$q_d = a + q_u^b \times f(\text{age})$$

We found that the constant, a, was insignificant. In all regressions attempted of q_d on q_u and age our estimate of the power of q_u was approximately one. Together these suggest that the ratio of q_d/q_u can be adequately expressed as a function of age.

Let y_t be the ratio of observed disabled worker mortality to U.S. population standard mortality at age t. A fairly simple model was found to fit quite well:

$$y_t = be^{c/t} + \varepsilon_t \text{ ;} \qquad \text{with } b = 0.32 \text{ and } c = 84$$

The ratio of the parameter to its estimated standard deviation is 3.72 for b and is 10.83 for c.

Graph 1 shows three regressions of y_t on $be^{c/t}$ with the parameter c set equal to 1, 40 and 84. The graph illustrates the importance of c in the model.

In addition, in graph 2 a comparison of the ratio of q_d/q_u to the confidence intervals for the model indicates heteroscedasticity (the variance around the fitted line is not constant over age). The observed q_d/q_u has a much greater variance at younger ages where, on average, q_d/q_u is greater. Therefore rather than assume the constant variance of standard least squares regression it was assumed that errors were normally distributed with mean equal to zero and standard deviation proportional to the mean of the regression. This is referred to as the multiplicative error model and is described further in Appendix 1. The distribution of the error term ε_t is approximated by a normal distribution:

$$\varepsilon_t = y_t - be^{c/t} \sim N(0, b^2e^{2c/t}\sigma^2) \qquad \text{where } \sigma^2 = \text{constant of proportionality}$$

In Appendix 1 it is shown that this model can be fit by a standard regression with the "dependent variable" set equal to one , and $y_t/be^{c/t}$ as the independent variable. Then the parameters b and c are found to be, respectively, 0.35 and 88 which are respectively, 6.86 and 13.08 times the estimated parameter standard deviations. Graph 3 shows the observed data along with the confidence intervals for this multiplicative model. This illustrates the basis for the assumption that the standard deviation of ε_t is proportional to the mean, in that the model confidence intervals more closely approximate the data variations. Table 1 compares the observed y_t and the values from the two fitted models.

To estimate the standard deviations of the parameters for this model we calculated the variance-covariance matrix which is the inverse of the information matrix as described on page 81 of *Loss Distributions* by Robert V. Hogg and Stuart A. Klugman. The calculations of the information matrix and its resulting variance-covariance matrix for both the constant variance and the proportional variance model are described in Appendix 2.

A comparison of mortality rates for 1930 and 1980 from the U.S. Life Tables and the projected mortality rates for disabled workers based on the models is shown in

Table 2. Since the committee used the 1930 U.S. Life Table for white males we used the same 1980 table.

DISCUSSION

The hypothesis that the ratio between the mortality rate for disabled workers versus the population, q_d/q_u, is a decreasing function of age is supported by the data analysis described above.

It is possible that the ratio q_d/q_u is closer to one now than is reflected in the 1930's data. The improvements in mortality of the general population may be heavily influenced by a disproportionately larger improvement in the mortality of disabled people. It will require another study of disabled workers mortality to determine if disabled worker mortality is now closer to standard mortality.

At an advanced age, there is a crossover point at which the mortality rate of disabled workers becomes less than that of the general population (Table 2). With the committee's method this occurs at age 81. With the multiplicative error model the crossover occurs at age 85. It is reasonable to assume that since these disabled workers had recently been in the work force at an advanced age they were healthier than the general population. The permanent injuries received were not necessarily serious enough to increase the mortality of these exceptionally healthy individuals to the level of the general population at that age.

In fact a fairly minor injury may be "permanent" at an older age in that the person may not return to work. This may contribute to the existence of a crossover point since permanent disability benefits supplement retirement income for older workers and could thus discourage return to work. Since on average today's workers retire earlier than they did in the 1930's the crossover point may be earlier now.

Below are the annuity values for certain ages calculated with the 1979-81 U.S. Life Tables and with estimated disabled workers' mortalities based on the proportional variance model. These annuity values contain an interest rate assumption of 3.5% and escalating benefits are assumed to increase at 7% per year.

Lifetime Annuity Values

	U.S. Life Table		Disabled Mortality	
Age	Nonescalating	Escalating	Nonescalating	Escalating
25	22.756	136.298	20.272	111.229
45	17.776	58.464	16.631	52.366
65	11.009	21.442	10.507	20.364
85	4.606	6.117	4.811	6.486

These disabled worker mortalities are created from the general population of permanent total disabled workers and may not apply to the most severely injured workers. As mentioned earlier since the mortality rates are based on recently injured workers they may not be appropriate for claimants who have been disabled for many years. The disabled worker annuity values do not change drastically from those for the general population but they do decrease. However for advanced ages the annuities under the disabled worker mortalities are actually greater than under the U.S. Life Table mortalities.

CONCLUSIONS

1. A model which declines with age seems appropriate for q_d/q_u, the ratio between the mortality rate for disabled workers and that of the U.S. population.

2. At some age this ratio goes below unity and this may now occur at an earlier age.

3. The impact of the disabled mortality rates on the annuity values was moderate then and would probably be even less now.

4. These results may not be applicable to the first year of injury when higher mortality rates are likely or to longer period after injury where mortality rates closer to standard are expected.

Table 1

Age	Ratio of Observed Mortality Rate to 1930 U.S. Standard Mortality Rate	Fitted Ratio from Constant Variance Model (1)	Fitted Ratio from Proportional Variance Model (2)
24	8.2541	10.6254	13.7001
25	9.6604	9.2373	11.8330
26	14.7013	8.1175	10.3362
27	8.0420	7.2020	9.1196
28	2.6410	6.4446	8.1185
29	2.1841	5.8113	7.2855
30	6.3777	5.2764	6.5853
31	5.2512	4.8207	5.9914
32	4.9615	4.4293	5.4833
33	0.0000	4.0907	5.0453
34	8.4568	3.7956	4.6651
35	3.9529	3.5369	4.3329
36	1.1813	3.3088	4.0409
37	2.0036	3.1066	3.7828
38	4.4908	2.9264	3.5536
39	3.2170	2.7652	3.3489
40	2.1517	2.6202	3.1654
41	1.3040	2.4894	3.0002
42	1.2320	2.3709	2.8509
43	2.1564	2.2631	2.7154
44	2.9405	2.1648	2.5922
45	2.8654	2.0749	2.4796
46	1.7136	1.9924	2.3765
47	2.4772	1.9165	2.2818
48	1.5980	1.8464	2.1946
49	2.3456	1.7816	2.1141
50	1.5227	1.7216	2.0396
51	2.8791	1.6658	1.9705
52	1.2276	1.6139	1.9062
53	1.3889	1.5654	1.8464
54	1.3349	1.5201	1.7905
55	1.5800	1.4778	1.7383
56	1.6526	1.4380	1.6894
57	1.6292	1.4006	1.6435
58	1.8961	1.3655	1.6004
59	0.5384	1.3324	1.5598
60	2.1415	1.3012	1.5215
61	1.6078	1.2716	1.4854
62	1.7536	1.2437	1.4513
63	1.3142	1.2172	1.4190
64	0.7567	1.1921	1.3884
65	1.1449	1.1683	1.3594
66	0.9790	1.1457	1.3318
67	1.2446	1.1241	1.3056
68	0.6668	1.1036	1.2806
69	0.7997	1.0840	1.2569
70	0.2978	1.0653	1.2342
71	0.9891	1.0474	1.2126
72	1.5846	1.0304	1.1919
73	0.8659	1.0140	1.1721
74	0.9447	0.9984	1.1532
75	1.3963	0.9834	1.1351
76	0.8882	0.9690	1.1177
77	1.6805	0.9552	1.1010
78	1.1974	0.9419	1.0850
79	0.6338	0.9292	1.0697
80	0.4526	0.9169	1.0549
81	1.3872	0.9051	1.0407
82	1.1605	0.8937	1.0270
83	0.6815	0.8828	1.0138
84	0.3539	0.8722	1.0011
85	1.2400	0.8620	0.9889
86	0.5859	0.8521	0.9770

(1) Y(t)=0.32086e**(84/t)
(2) Y(t)=0.35155e**(87.9074/t)

Table 2

1930

AGE	U.S. Life Table tQx	Disabled Mortality: Raw Data tQx'	Disabled Mortality: Committee tQx''	Disabled Mortality: Fit(REG) tQx'''	Disabled Mortality: Fit(MAX) tQx''''
18					
19					
20					
21					
22					
23					
24	.0037	.0302	.0259	.0389	.0501
25	.0037	.0358	.0255	.0343	.0439
26	.0037	.0551	.0250	.0304	.0388
27	.0038	.0306	.0243	.0274	.0347
28	.0039	.0103	.0236	.0251	.0317
29	.0040	.0088	.0227	.0234	.0293
30	.0041	.0263	.0218	.0218	.0272
31	.0043	.0224	.0209	.0205	.0255
32	.0044	.0219	.0201	.0196	.0242
33	.0046	.0000	.0192	.0189	.0234
34	.0049	.0411	.0185	.0184	.0227
35	.0051	.0202	.0178	.0180	.0221
36	.0053	.0063	.0173	.0177	.0216
37	.0056	.0113	.0169	.0175	.0213
38	.0060	.0268	.0166	.0175	.0212
39	.0064	.0205	.0165	.0176	.0213
40	.0068	.0146	.0166	.0178	.0215
41	.0073	.0095	.0169	.0181	.0218
42	.0078	.0096	.0174	.0184	.0221
43	.0082	.0178	.0180	.0187	.0224
44	.0087	.0257	.0187	.0189	.0227
45	.0093	.0266	.0195	.0193	.0230
46	.0099	.0169	.0204	.0197	.0235
47	.0105	.0261	.0214	.0202	.0240
48	.0112	.0179	.0224	.0207	.0246
49	.0120	.0281	.0234	.0213	.0253
50	.0128	.0195	.0245	.0220	.0261
51	.0136	.0393	.0256	.0227	.0269
52	.0146	.0179	.0268	.0235	.0278
53	.0157	.0217	.0281	.0245	.0289
54	.0169	.0225	.0294	.0256	.0302
55	.0182	.0287	.0308	.0269	.0316
56	.0197	.0325	.0322	.0283	.0332
57	.0212	.0346	.0335	.0298	.0349
58	.0229	.0434	.0347	.0313	.0366
59	.0246	.0132	.0358	.0328	.0384
60	.0264	.0566	.0367	.0344	.0402
61	.0284	.0456	.0376	.0361	.0422
62	.0305	.0535	.0383	.0380	.0443
63	.0330	.0433	.0391	.0401	.0468

1980

AGE	U.S. Life Table tQx	Disabled Mortality tQx'''	Disabled Mortality tQx'''
18	.0015	.0515	.0701
19	.0016	.0435	.0585
20	.0017	.0374	.0499
21	.0019	.0326	.0430
22	.0019	.0282	.0369
23	.0019	.0239	.0310
24	.0019	.0201	.0259
25	.0018	.0169	.0217
26	.0018	.0144	.0183
27	.0017	.0124	.0157
28	.0017	.0108	.0136
29	.0017	.0097	.0122
30	.0017	.0088	.0109
31	.0016	.0080	.0099
32	.0017	.0074	.0091
33	.0017	.0069	.0085
34	.0017	.0066	.0082
35	.0018	.0065	.0080
36	.0020	.0065	.0079
37	.0021	.0065	.0079
38	.0022	.0066	.0080
39	.0024	.0066	.0080
40	.0026	.0068	.0083
41	.0029	.0071	.0086
42	.0032	.0075	.0090
43	.0035	.0079	.0094
44	.0038	.0083	.0099
45	.0042	.0087	.0104
46	.0046	.0092	.0110
47	.0051	.0099	.0117
48	.0057	.0106	.0126
49	.0064	.0114	.0135
50	.0071	.0122	.0144
51	.0077	.0129	.0153
52	.0085	.0137	.0162
53	.0093	.0146	.0172
54	.0103	.0156	.0184
55	.0112	.0166	.0196
56	.0123	.0176	.0207
57	.0134	.0187	.0220
58	.0146	.0200	.0234
59	.0160	.0214	.0250
60	.0176	.0229	.0268
61	.0193	.0246	.0287
62	.0212	.0264	.0308
63	.0232	.0282	.0329

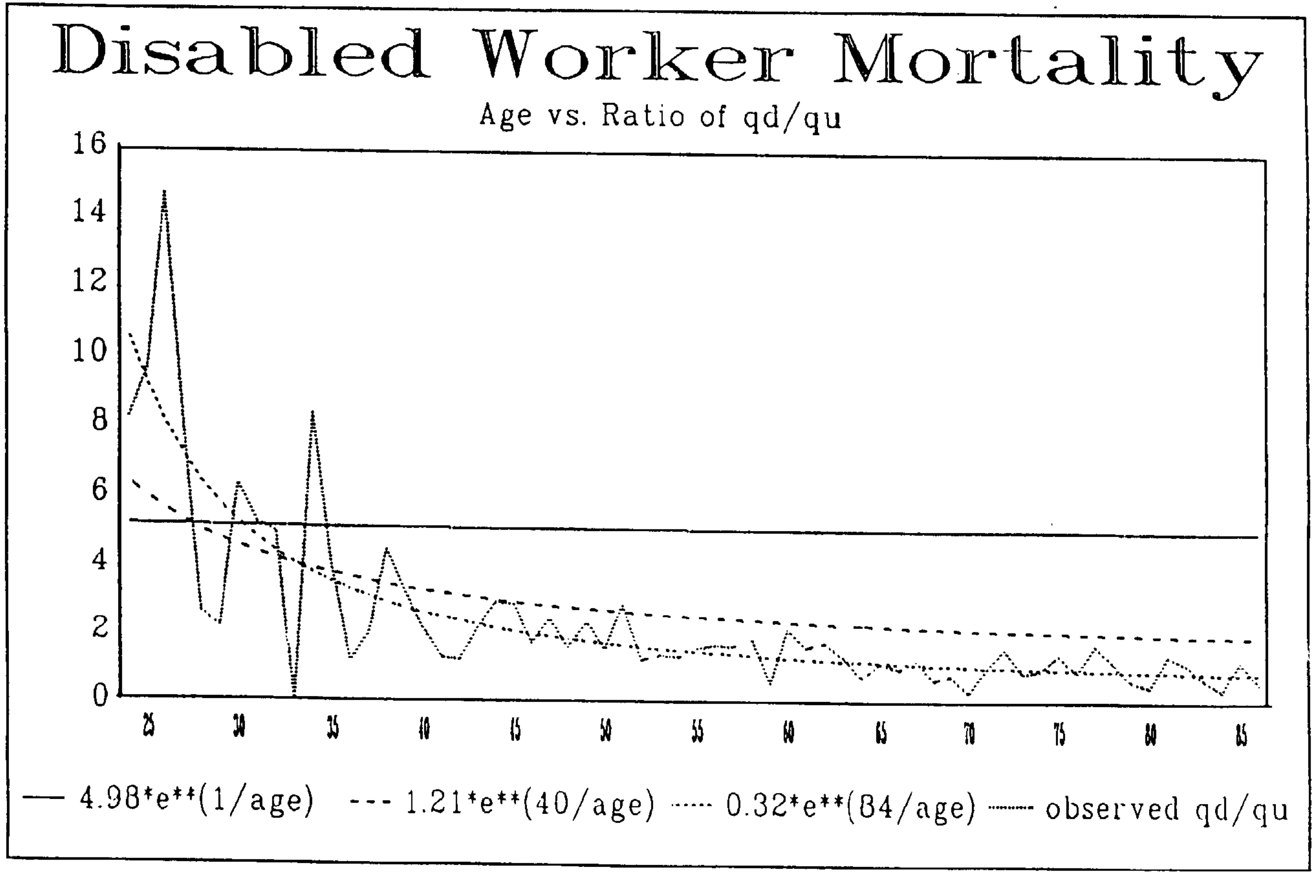
1930

AGE	U.S. Life Table	Disabled Mortality Raw Data	Committee	Fit(REG)	Fit(MAX)
	tQx	tQx'	tQx''	tQx'''	tQx''''
64	.0357	.0270	.0400	.0425	.0495
65	.0386	.0442	.0412	.0452	.0525
66	.0420	.0411	.0428	.0481	.0559
67	.0456	.0567	.0451	.0512	.0595
68	.0495	.0330	.0481	.0546	.0634
69	.0536	.0429	.0519	.0581	.0674
70	.0580	.0173	.0566	.0617	.0715
71	.0625	.0618	.0621	.0655	.0758
72	.0674	.1068	.0682	.0694	.0803
73	.0727	.0630	.0750	.0737	.0852
74	.0786	.0743	.0822	.0785	.0907
75	.0853	.1190	.0898	.0838	.0968
76	.0927	.0824	.0976	.0899	.1037
77	.1010	.1698	.1056	.0965	.1113
78	.1101	.1319	.1137	.1037	.1195
79	.1199	.0759	.1220	.1114	.1282
80	.1300	.0588	.1305	.1192	.1371
81	.1404	.1948	.1393	.1271	.1461
82	.1512	.1754	.1485	.1361	.1553
83	.1621	.1105	.1581	.1459	.1644
84	.1733	.0613	.1681	.1560	.1735
85	.1847	.2290	.1787	.1662	.1826
86	.1962	.1149	.1899	.1766	.1917
87	.2078		.2019	.1870	.2007
88	.2197		.2146	.1977	.2097
89	.2321		.2283	.2089	.2191
90	.2455		.2429	.2209	.2292
91	.2602		.2587	.2342	.2403
92	.2763		.2757	.2487	.2525
93	.2940		.2941	.2646	.2660
94	.3133		.3140	.2820	.2820
95	.3344		.3356	.3010	.3010
96	.3574		.3589	.3217	.3217
97	.3824		.3841	.3442	.3442
98	.4095		.4113	.3686	.3686
99	.4388		.4406	.3949	.3949
100	.4704		.4720	.4233	.4233
101	.5044		.5057	.4539	.4539
102	.5409		.5417	.4868	.4868
103	.5800		.5799	.5220	.5220
104	.6219		.6204	.5597	.5597
105	.6666		.6666	.5999	.5999
106					
107					
108					
109					

1980

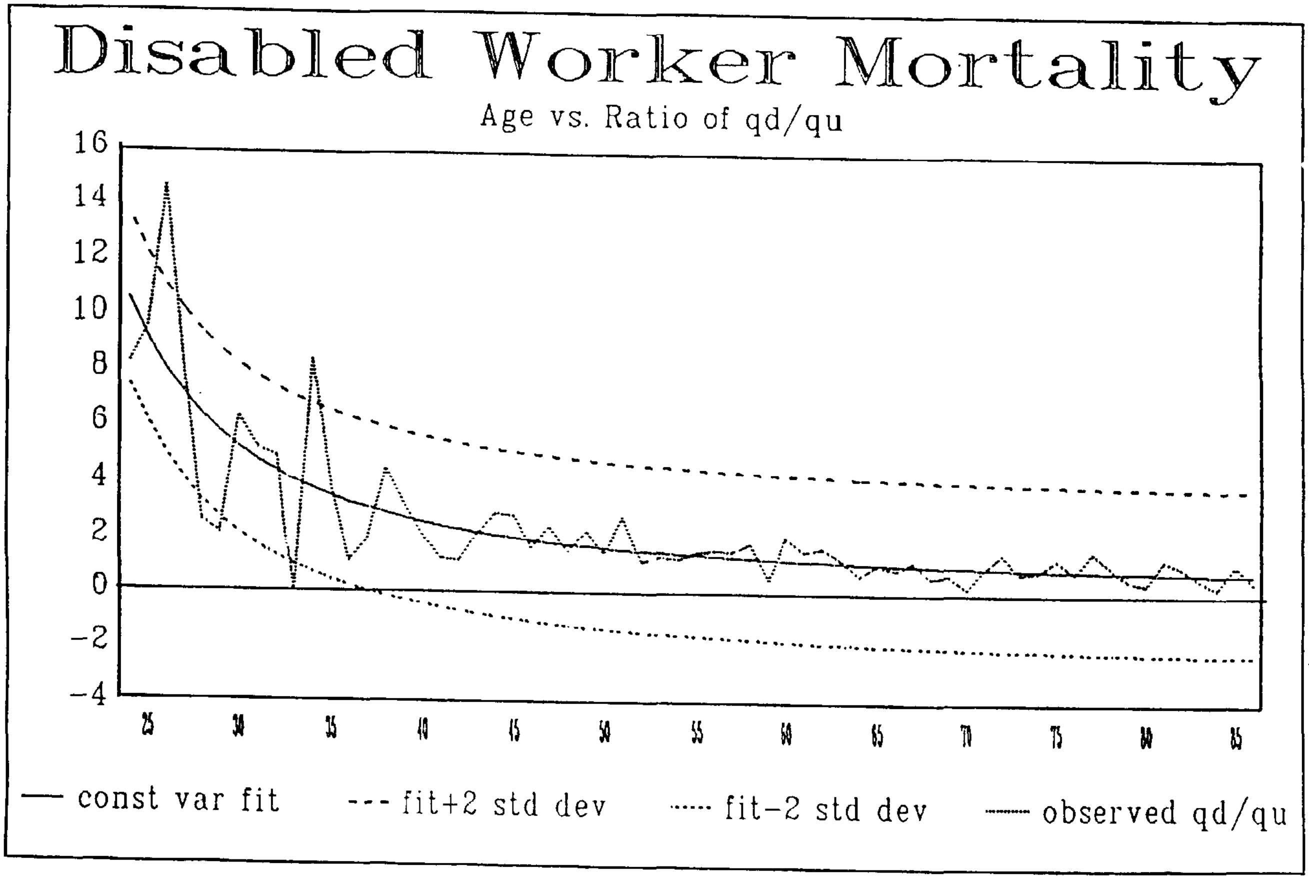
AGE	U.S. Life Table	Disabled Mortality	
	tQx	tQx'''	tQx''''
64	.0252	.0301	.0350
65	.0274	.0320	.0372
66	.0297	.0340	.0395
67	.0322	.0362	.0420
68	.0349	.0386	.0448
69	.0380	.0412	.0478
70	.0415	.0442	.0512
71	.0452	.0473	.0548
72	.0490	.0505	.0584
73	.0529	.0537	.0621
74	.0570	.0569	.0658
75	.0615	.0604	.0698
76	.0664	.0644	.0742
77	.0718	.0686	.0791
78	.0776	.0731	.0842
79	.0839	.0780	.0898
80	.0910	.0834	.0960
81	.0989	.0895	.1029
82	.1073	.0966	.1102
83	.1161	.1045	.1177
84	.1252	.1127	.1254
85	.1351	.1216	.1336
86	.1459	.1313	.1426
87	.1569	.1412	.1515
88	.1677	.1510	.1601
89	.1787	.1609	.1687
90	.1906	.1715	.1779
91	.2039	.1835	.1883
92	.2186	.1968	.1998
93	.2345	.2111	.2122
94	.2506	.2255	.2255
95	.2662	.2396	.2396
96	.2800	.2520	.2520
97	.2931	.2638	.2638
98	.3054	.2749	.2749
99	.3170	.2853	.2853
100	.3278	.2951	.2951
101	.3379	.3041	.3041
102	.3472	.3125	.3125
103	.3559	.3203	.3203
104	.3638	.3275	.3275
105	.3712	.3341	.3341
106	.3779	.3401	.3401
107	.3841	.3457	.3457
108	.3897	.3507	.3507
109	.3949	.3554	.3554

Graph 1

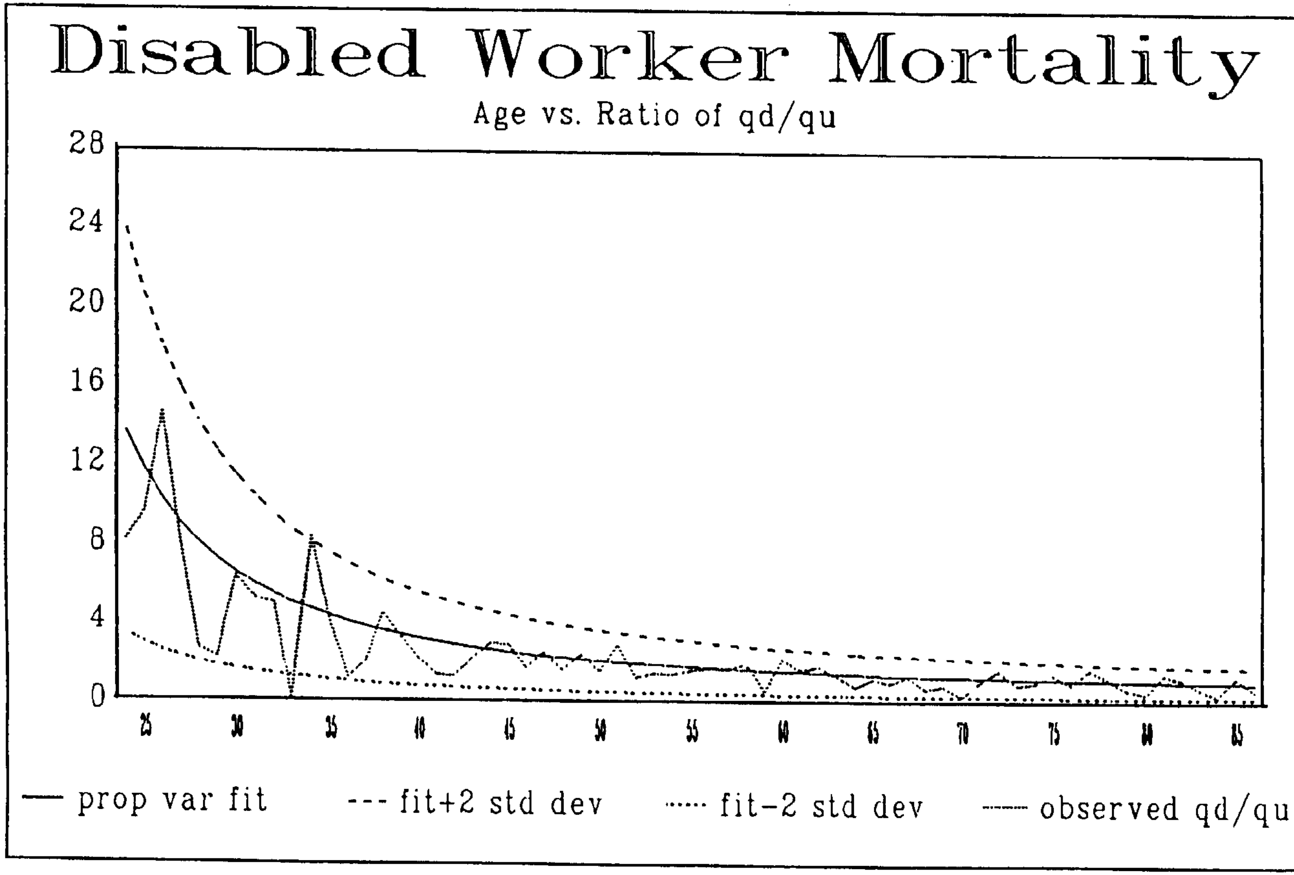


Disabled Worker Mortality
Age vs. Ratio of qd/qu
16
14
12
10
8
6
4
2
0
25
30
35
40
45
50
55
60
65
70
75
80
85
4.98*e**(1/age)
1.21*e**(40/age)
0.32*e**(84/age)
observed qd/qu

Graph 2


Disabled Worker Mortality
Age vs. Ratio of qd/qu
16
14
12
10
8
6
4
2
0
-2
-4
25
30
35
40
45
50
55
60
65
70
75
80
85
const var fit
fit+2 std dev
fit-2 std dev
observed qd/qu

Graph 3


Disabled Worker Mortality
Age vs. Ratio of qd/qu
28
24
20
16
12
8
4
0
25
30
35
40
45
50
55
60
65
70
75
80
85
prop var fit
fit+2 std dev
fit-2 std dev
observed qd/qu

Appendix 1 Regression formulas

<u>Regression with additive error structure</u>

This is the standard least squares regression method.

Model is : $y_t = g(x_{1t} \ldots x_{kt}) + \epsilon_t$

where: y_t is the dependent variable

$x_1 \ldots x_k$ are the independent variables

g is the function with parameters to be estimated

ϵ_t is $\sim \mathcal{N}(0,\sigma^2)$

The additive error structure is appropriate when it can be assumed that the conditional variance $= \mathrm{var}(y_t \mid g(x_{1t} \ldots x_{kt})) =$ constant $= \sigma^2$. In other words the variance σ^2 is independent of t. This is an assumption of least square regression referred to as homoscedasticity.

Assuming a normal distribution of the disturbance term ϵ_t the maximum likelihood estimates for the parameters of g minimize:

$$\sum_t \epsilon_t^2 = \sum_t \left[y_t - g(x_{1t} \ldots x_{kt})\right]^2$$

The regression function used is: $g(x_{1t}) = be^{c/t}$

where $x_{1t} = t =$ age

Our model becomes : $y_t = be^{c/t} + \epsilon_t$

where y_t is the observed ratio of injured worker mortality to standard mortality at age t.

The regression finds b and c which minimize: $\sum_t \left[y_t - be^{c/t}\right]^2$

Regression with multiplicative error structure.

Model is : $y_t = g(x_{1t}\ldots x_{kt})(1+\epsilon_t) = g(x_{1t}\ldots x_{kt}) + \epsilon_t \cdot g(x_{1t}\ldots x_{kt})$

where ϵ_t is $\sim \mathcal{N}(0,\sigma^2)$

Thus the disturbance term increases in size with the function.

This multiplicative error structure is appropriate when it can be assumed that the $\mathrm{var}(\, y_t \mid g(x_{1t}\ldots x_{kt})) = g(x_{1t}\ldots x_{kt})^2\sigma^2$ i.e, the variance increases with the square of the function (the conditional mean).

Also, $$\epsilon_t = \frac{y_t - g(x_{1t}\ldots x_{kt})}{g(x_{1t}\ldots x_{kt})} = \frac{y_t}{g(x_{1t}\ldots x_{kt})} - 1$$

This ϵ_t satisfies the assumptions of standard least squares regression, that is : $\epsilon_t \sim \mathcal{N}(0,\sigma^2)$, so the maximum likelihood estimates of the parameters of g minimize:

$$\sum_t \left[\frac{y_t}{g(x_{1t}\ldots x_{kt})} - 1\right]^2$$

An alternative model (which we did not use) is : $y_t = g(x_{1t}\ldots x_{kt}) + \epsilon_t\sqrt{g(x_{1t}\ldots x_{kt})}$

Which requires minimization of : $$\sum_t \left[\frac{y_t}{\sqrt{g(x_{1t}\ldots x_{kt})}} - \sqrt{g(x_{1t}\ldots x_{kt})}\right]^2$$

$\mathrm{var}(\, y_t \mid g(x_{1t}\ldots x_{kt})) = g(x_{1t}\ldots x_{kt})\sigma^2$

Here the variance increases linearly with the conditional mean.

Appendix 1 Regression formulas

Both of these error structures are examples of heteroscedasticity, a common violation of the assumptions of least squares regression.

A multiplicative model was used and eventually chosen as the model that best "fit" our data .

The regression function used is: $g(x_{1t}) = be^{c/t}$

where $x_{1t} = t = \text{age}$

Our model becomes : $y_t = be^{c/t}(1 + \epsilon_t)$

For this model , the regression minimizes: $\sum_t \left[\frac{y_t}{be^{c/t}} - 1\right]^2$

This is equivalent to minimizing the sum of the squares of the <u>proportional</u> errors.

Appendix 2 Significance of Parameters

Regression can be regarded as fitting a distribution (often a normal distribution) to the error terms ϵ_t by the method of maximum likelihood.

Variances and covariances of the regression parameters can thus be estimated by the inverse of the information matrix as described in *LOSS DISTRIBUTIONS* by Robert V. Hogg - Stuart A. Klugman (Page 81).

If $f(\epsilon;\theta)$ is the density function for the error terms, and θ is a vector listing the parameters to be estimated, the *ijth element* of the <u>information matrix</u> is:

$$a_{ij}(\theta) = -nE\left[\frac{\partial^2 \ln f(\epsilon;\theta)}{\partial\theta_i\partial\theta_j}\right],$$ Here n is the number of observations.

This is typically estimated by:

$$a_{ij} \approx -\sum_{t=1}^{n}\frac{\partial^2 \ln f(\epsilon_t;\hat\theta)}{\partial\theta_i\partial\theta_j} = -\frac{\partial^2}{\partial\theta_i\partial\theta_j}\ln\prod_{t=1}^{n} f(\epsilon_t;\hat\theta)$$

Where $\hat\theta$ is the vector of parameter estimates and
$\epsilon_t =$ observed deviation from the model for observation t.
Thus the information matrix is estimated by the second partials of the negative loglikelihood.

<u>Additive error structure</u>

For our model: $y_t = be^{c/t} + \epsilon_t$ $\quad\hat\theta = \langle b,c,\sigma^2\rangle$ and $f(\epsilon_t;\theta) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\epsilon_t^2/2\sigma^2}$

so that : $\epsilon_t = y_t - be^{c/t}$ $\quad$ Since $\epsilon_t \sim \mathcal{N}(0,\sigma^2)$

$$\text{Thus}\quad \ln f(\epsilon_t;\theta) = -\tfrac{1}{2}\ln 2\pi - \ln\sigma - \frac{\epsilon_t^2}{2\sigma^2}$$

$$= -\tfrac{1}{2}\ln 2\pi - \ln\sigma - \left[y_t - be^{c/t}\right]^2\frac{1}{2\sigma^2} \qquad \text{Since } \epsilon_t = \left[y_t - be^{c/t}\right]$$

Taking the partial derivatives of $\ln f(\epsilon_t;\theta)$ with respect to b,c and σ^2 (after some algebra) yields the following estimates of the a_{ij} :

$$a_{11} = \frac{1}{\sigma^2}\sum_t c^{2c/t}$$

$$a_{12} = a_{21} = \frac{1}{\sigma^2}\sum_t \frac{e^{c/t}}{t}\left[2be^{c/t} - y_t\right]$$

$$a_{22} = \frac{b}{\sigma^2}\sum_t \frac{e^{c/t}}{t^2}\left[2be^{c/t} - y_t\right]$$

$$a_{13} = a_{31} = \frac{1}{\sigma^4}\sum_t e^{c/t}\left[y_t - be^{c/t}\right] = \frac{1}{\sigma^4}\sum_t e^{c/t}\,\epsilon_t$$

$$a_{23} = a_{32} = \frac{b}{\sigma^4}\sum_t \frac{e^{c/t}}{t}\left[y_t - be^{c/t}\right] = \frac{b}{\sigma^4}\sum_t \frac{e^{c/t}}{t}\,\epsilon_t$$

$$a_{33} = -\frac{n}{2\sigma^4} + \frac{1}{\sigma^6}\sum_t\left[y_t - be^{c/t}\right]^2 = -\frac{n}{2\sigma^4} + \frac{1}{\sigma^6}\sum_t \epsilon_t^2$$

For the data used the sum is from t=24 to t=86.

Appendix 2 Significance of Parameters

For our example the maximum likelihood estimates of the parameters are:

$\hat{b} = .32$, $\hat{c} = 84$ and $\hat{\sigma}^2 = 2.34$ yielding the

<u>Information Matrix:</u>

$$\begin{bmatrix} 2664.4519 & 28.7613 & .9412 \\ 28.7613 & .3271 & .0104 \\ .9412 & .0104 & 5.0397 \end{bmatrix}$$

Taking the matrix inverse gives us the <u>Variance-Covariance Matrix:</u>

$$\begin{bmatrix} .0074 & -.6493 & 0 \\ -.6493 & 60.1556 & -.0028 \\ 0 & -.0028 & .1984 \end{bmatrix}$$

Our final step is to check the significance of our parameters. We do this by observing the ratio of the estimated parameter values to their standard deviations.

Standard error of parameter b : $\sqrt{.0074} = .086$ $.32/.086 = 3.72$

Standard error of parameter c: $\sqrt{60.16} = 7.76$ $84/7.76 = 10.83$

Parameters b and c appear to be significant.

Appendix 2 Significance of Parameters

Multiplicative error structure

$\theta = <b,c,\sigma^2>$

$\epsilon_t =$ observed deviation from the model for observation t

Again: $$f(\epsilon_t;\theta) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\epsilon_t^2/2\sigma^2} \quad \text{and}$$

$$\ln f(\epsilon_t;\theta) = -\frac{1}{2}\ln 2\pi - \ln\sigma - \frac{\epsilon_t^2}{2\sigma^2}$$

$$= -\frac{1}{2}\ln 2\pi - \ln\sigma - \left[\frac{y_t}{be^{c/t}} - 1\right]^2 \frac{1}{2\sigma^2} \qquad \text{,since } \epsilon_t = \left[\frac{y_t}{be^{c/t}} - 1\right]$$

Taking the partial derivatives of $\ln f(\epsilon_t;\theta)$ with respect to b,c and σ^2 yields the following estimates of the a_{ij} :

$$a_{11} = \frac{1}{b^2\sigma^2}\sum_t (\epsilon_t + 1)(3\epsilon_t + 1)$$

$$a_{12} = a_{21} = \frac{1}{b\sigma^2}\sum_t \frac{1}{t}(\epsilon_t + 1)(2\epsilon_t + 1)$$

$$a_{13} = a_{31} = \frac{1}{b\sigma^4}\sum (\epsilon_t + 1)\epsilon_t$$

$$a_{22} = \frac{1}{\sigma^2}\sum_t \frac{1}{t^2}(\epsilon_t + 1)(2\epsilon_t + 1)$$

$$a_{23} = a_{32} = \frac{1}{\sigma^4}\sum \frac{1}{t}(\epsilon_t + 1)\epsilon_t$$

$$a_{33} = \frac{-n}{2\sigma^4} + \frac{1}{\sigma^6}\sum_t \epsilon_t^2$$

Appendix 2 Significance of Parameters

For our example: $\hat{b} = .35$, $\hat{c} = 88$ and $\hat{\sigma}^2 = .15$ yielding the

<u>Information Matrix:</u>

$$\begin{bmatrix} 2953.559 & 20.9673 & 17.3812 \\ 20.9674 & .1709 & .1104 \\ 17.3812 & .1104 & 1348.404 \end{bmatrix}$$

Taking the inverse of this matrix gives us the <u>Variance-Covariance Matrix:</u>

$$\begin{bmatrix} .0026 & -.3218 & 0 \\ -.3218 & 45.3341 & .0004 \\ 0 & .0004 & .0007 \end{bmatrix}$$

Standard error of parameter b : $\sqrt{.0026} = .051$ $.35/.051 = 6.86$

Standard error of parameter c: $\sqrt{45.33} = 6.73$ $88/6.73 = 13.08$

Parameters appear to be significant.

$$q_d = f(q_u)$$

$$q_d = a + f(age)q_u^{\ b}$$

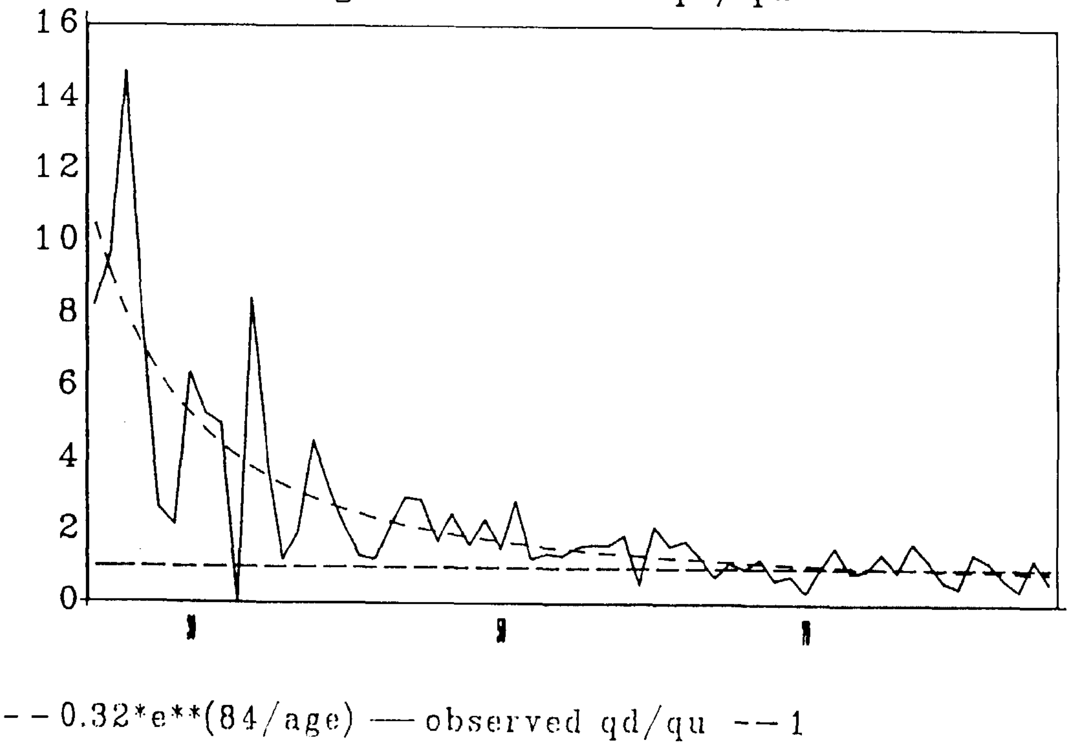
$$a \approx 0, \quad b \approx 1$$

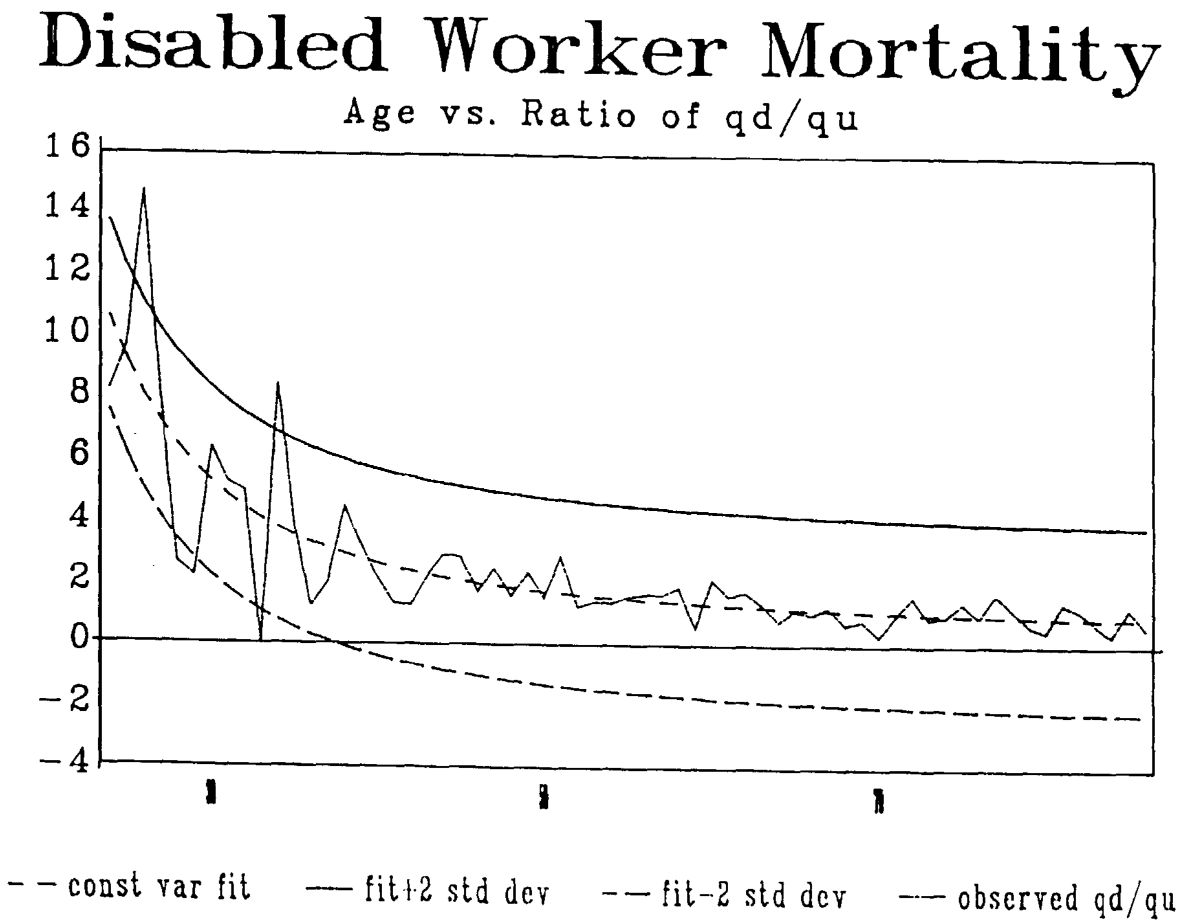
$$\therefore q_d/q_u = f(age)$$

$$f(t) = be^{c/t}$$

Disabled Worker Mortality

Age vs. Ratio of qd/qu




Disabled Worker Mortality
Age vs. Ratio of qd/qu
16
14
12
10
8
6
4
2
0
-2
-4
- - const var fit
— fit+2 std dev
- - fit-2 std dev
— observed qd/qu

Error Structure

Constant Variance

$$y_t = f(t) + \varepsilon_t$$

Proportional Variance

$$y_t = f(t)(1 + \varepsilon_t)$$

Proportional Standard Deviation

$$y_t = f(t) + f(t)^{1/2}\varepsilon_t$$

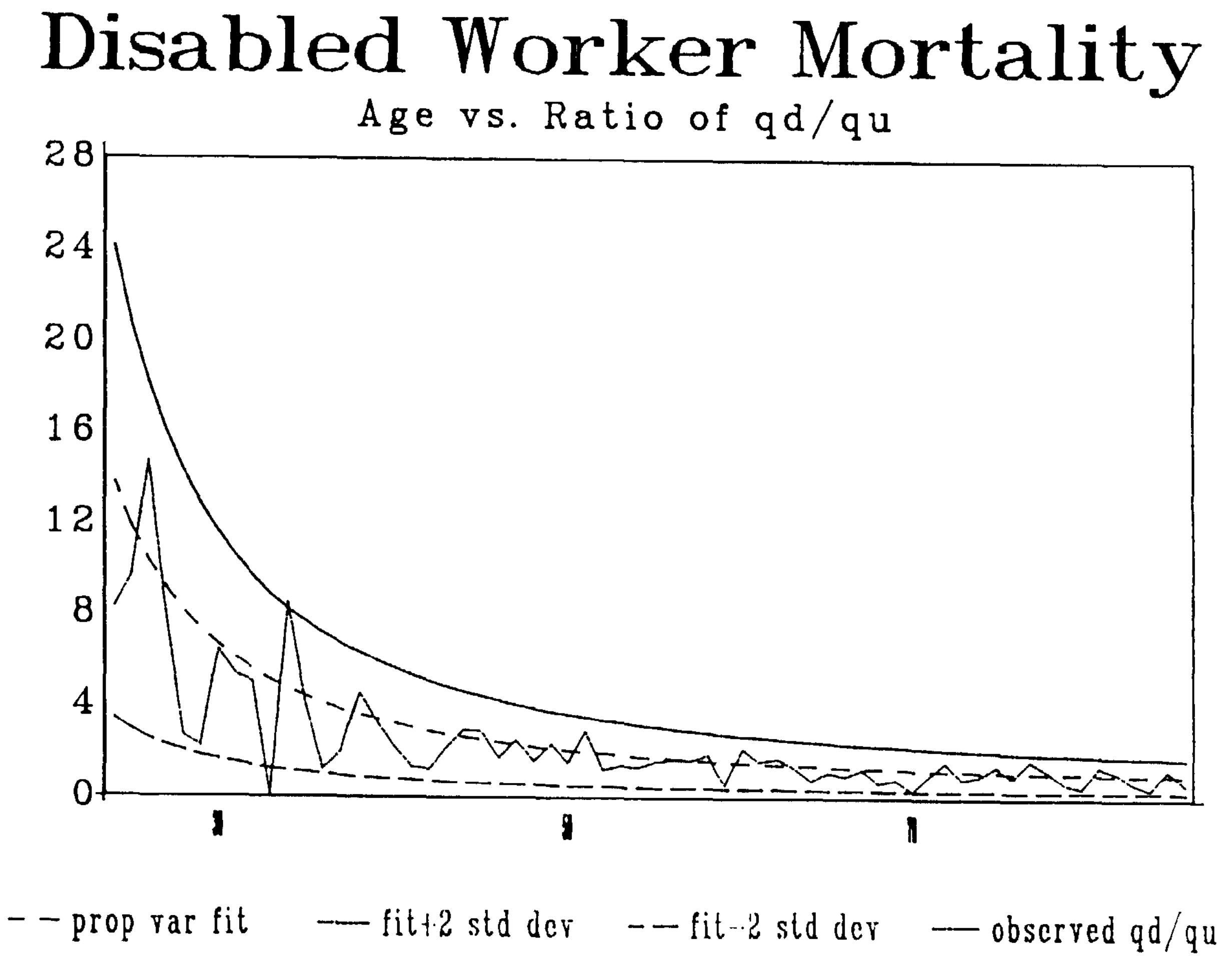
Proportional Variance Model

$$\varepsilon_t = y_t/f(t) - 1$$

Thus minimize:

$$\sum (y_t/f(t) - 1)^2$$

Minimize sum of proportional errors


Disabled Worker Mortality
Age vs. Ratio of qd/qu
28
24
20
16
12
8
4
0
- - prop var fit
— fit+2 std dev
- - fit-2 std dev
— observed qd/qu

Parameter Estimation Error

Information Matrix (parameters θ)

$$a_{ij} = -n\mathrm{E}\left[\partial^2 \ln f(\varepsilon_t;\theta)/\partial\theta_i\partial\theta_j\right]$$

Estimated by:

$$a_{ij} = -\sum\left[\partial^2 \ln f(\varepsilon_t;\theta)/\partial\theta_i\partial\theta_j\right]$$

(Second partials of negative loglikelihood)

f is normal density for ε
θ is b, c, σ^2

E.g., $a_{12} = b^{-1}\sigma^{-2}\sum(\varepsilon_t+1)(2\varepsilon_t+1)/t$