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An actuarial model for AIDS when transition intensities follow jump-diffusion processes

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ABSTRACT

Improved treatment for AIDS, possible vaccines for HIV, changing behaviour on the one hand, the increasing proportion of females and IV drug users and thus the evolving interface between the at risk and the not at risk population on the other hand suggests that transition intensities are not constant over time but experience gradual drifts and sudden jumps. The paper considers these features in a model for prevalence and survivorship.

1. Ruin problem for geometric Brownian motion

In this section we lay down the necessary mathematical background. Recall that a Brownian motion, also known as a Wiener process, is a stochastic process $\{ W_t : t \in [0,\infty) \}$ with the following properties:

(i) $W_0 = 0$

(ii) W_t is normal with mean 0 and variance t

(iii) $W_{t_2} - W_{t_1}$ and $W_{t_4} - W_{t_3}$ are independent

(iv) $W_{t+s} - W_s$ distributes like W_t

Considers the following stochastic differential equation

$$\frac{dY_t}{Y_t} = \mu\, dt + \sigma\, dW_t \tag{1}$$

with solution

$$Y_t = Y_0 \; e^{\mu t + \sigma W_t - \frac{\sigma^2 t}{2}} \tag{2}$$

which is known as the geometric Brownian motion. See for example, Malliaris (1983, pp. 485-486) or Karlin and Taylor (1975, p.357) for further discussion. The extra term $-\frac{\sigma^2 t}{2}$ in the exponent arises from an application of the

Ito's lemma: [Malliaris (1983, p. 484) or Karlin and Taylor (1981, pp.347-348)]

If

$$dX_t = \mu(X_t,t)\ dt + \sigma(X_t,t)\ dW_t$$

$$Y_t = f\ (X_t,t)$$

then

$$dY_t = \left(\frac{\partial f}{\partial x}\ \mu + \frac{\partial f}{\partial t} + \frac{1}{2}\ \frac{\partial^2 f}{\partial x^2}\ \sigma^2 \right) dt + \frac{\partial f}{\partial x}\ \sigma\ dW_t$$

Proof:

Start with the Taylor series for dY_t to the second derivative

$$dY_t = \frac{\partial f}{\partial x}\ dX_t + \frac{\partial f}{\partial t}\ dt + \frac{\partial^2 f}{\partial x \partial t}\ dX_t\ dt + \frac{1}{2}\ \frac{\partial^2 f}{\partial x^2}\ (dX_t)^2 + \frac{1}{2}\ \frac{\partial f^2}{\partial t^2}\ (dt)^2.$$

Observe that $(dW_t)^2 = dt$, and that all other double differentials lead to dt of power 1.5 or higher. q.e.d.

The behavior of the geometric Brownian motion is well known. For example, Karlin and Taylor (1975, p.357) tell us that

$$E\ (\ Y_t\ ) = Y_0\ \ e^{\mu t},$$

$$V\ (\ Y_t\ ) = Y_0^2\ e^{\ 2\mu t}\ [\ e^{t\sigma^2} - 1\ ] = (E\ Y_t)^2\ [\ t\sigma^2 - 1\ ].$$

To find the probability of Y_t ever going down to a portion of the current value, to be used in section 3 to indicate the diseased population shrinking down to a small enough portion so that the epidemic is considered cured, write (2) as

$$Y_t = Y_0 \ e^{X_t} \quad (3)$$

with

$$X_t = \mu t + \sigma W_t - \frac{\sigma^2 t}{2} \quad (4)$$

and study whether X_t would go down a distance u from the starting level. Thus we have transformed the Y_t problem to a previously solved X_t problem, the ruin problem for Brownian motion with a drift. For such a process,

$$P \{\text{maximum drop} \geq u \} = e^{-\frac{2 \ \text{drift}}{\text{variance}} u} \quad (5)$$

[Karlin and Taylor (1975, p.361)].

To see that (5) is a reasonable result without going to the proof, let us consider what this say in the context of classical risk theory. In the context of classical ruin process, the quantity

$$e^{-\frac{2 \ \text{drift}}{\text{variance}} u} = e^{-\frac{2\theta\lambda p_1}{\lambda p_2} u} = e^{-\frac{2\theta p_1}{p_2} u} < e^{-Ru}$$

is known to be a first approximation of e^{-Ru} [Bowers et al (1986, Exercise 12.4)] and is on the same side of e^{-Ru} as the probability of ruin

$$\psi(u) = P \{ L \geq u \} = \frac{e^{-Ru}}{E [e^{-RU_T} \mid T < \infty]} < e^{-Ru}$$

2. Geometric Brownian motion as model for HIV infections

When we use the geometric Brownian motion to model HIV infections, the parameters are estimated by published growth rate of reported AID cases with the knowledge that exponential rate of HIV infections lead to exponential rate of AID occurances which leads to exponential rate of AID reportings. Consistent with various published data of AIDS doubling in one to seven years in different studies, times, and places, values of μ and σ in (1) are chosen to be .16 and .4 as simple numbers for illustration purposes.

$$\frac{dY_t}{Y_t} = .16\ dt + .4\ dW_t \tag{6}$$

$$Y_t = Y_0\ e^{.08\ t\ +\ .4\ W_t} \tag{7}$$

3. Ruin problem for geometric Brownian motion with jumps

When we use (5) and (7), we found that

$$P\ \{ \min_{t \le s} \frac{Y_s}{Y_t} \le e^{-u} \} = P\ \{\text{maximum drop in } X_t \ge u\ \}$$

$$= e^{-\frac{2\ \text{drift}}{\text{variance}}\ u} = e^{-\frac{2\ (.08)}{.16}\ u} = e^{-u},$$

which says the probability of the diseased population $\{ Y_s : t \leq s \leq \infty \}$ ever go down to p times the current diseased population Y_t is p. [$p = e^{-u}$] The simplicity of the statement comes from our choice of parameters .16 and .4. For general values of the parameters, the probablity of the disease winding down by p is constant times p.

Form the above discussion, we found that the epidemic is not likely to go away by itself. Suppose now there is a new process describing the epidemic

$$\frac{dY_t}{Y_t} = \mu \, dt + \sigma \, dW_t + \text{jumps} \tag{8}$$

where the jumps occurs as a Poisson process with the values of Y_t after the jumps are p times that of before the jumps with $p \in (0,1)$. Incidentally, this model has be used extensicely to descripe investment returns. [Merton (1976)] In agreement with the convention in classical risk theory and in cadlag semimartingales [Metivier (1982)], the diffusion with jumps is consider to be right continuous.

$$p = \frac{Y_t}{Y_{t-}} \tag{9}$$

By (3), we can study the process Y_t by applying what we know about X_t.

$$p = \frac{Y_t}{Y_{t-}} = \frac{e^{X_t}}{e^{X_{t-}}} = e^{X_t - X_{t-}} \quad ,$$

or,

$$X_t = X_{t-} - \ln \frac{1}{p} \qquad (10)$$

The equivalence of (9) and (10) tells us that Y_t drops to a porportion of p is equivalent to X_t drops by a value of $\ln \frac{1}{p}$. Furthermore, p distributes uniformly on [0,1] is equivalent to the drops in X_t is distributed exponentially with parameter 1; p^β distributes uniformly on [0,1] is equivalent to the drops in X_t is distributed exponentially with parameter β.

Geometric Brownian motion with jumps, Y_t, has been studied extensively for stock returns since Merton (1976) and the surplus process with Brownian premium receipts has be recently be studied in Dufresne (1989).

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