A MULTIVARIATE APPROACH TO IMMUNIZATION THEORY

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Abstract

Extending the general nonparallel shift approach to duration analysis developed previously by the author, this paper explores the immunization properties of that model. In particular, results are developed regarding directional immunization, in which the yield curve shift direction vector is specified, as well as for nondirectional immunization. Throughout, the goal of immunization at time k is seen to be intimately linked to the relationship between the durational and convexity attributes of the portfolio and those of a k-period zero coupon bond. Applications to asset/liability management are then explored.

I. Introduction

The concepts of duration and immunization have been the subjects of an increasing amount of interest, both from a theoretical and an applied perspective. Originally discovered over 50 years ago, duration was defined to better reflect the length of a payment stream (Macaulay (1938)). A short time later (Hicks (1939)), it was independently derived in an investigation into the elasticity of the price of a bond with respect to the discount factor $v = (1 + i)^{-1}$.

Soon thereafter (Samuelson (1945), Redington (1952)), duration was rediscovered in the context of the immunization of a firm's or portfolio's net worth. That is, in pursuit of conditions under which assets and liabilities would be equally responsive to changes in an underlying interest rate. Redington's approach was later adapted by Vanderhoof (1972) and became what to many actuaries represented an introduction to this field of thought and its application to insurance company portfolios. Common to the above investigations was the assumption of a single interest rate for all discountings of cash flows; that is, a flat yield curve.

Fischer and Weil (1971) first extended the Redington model to reflect a non-flat term structure, and developed a corresponding duration measure often denoted D_2, to distinguish it from the Macaulay duration, D_1. This measure reflected price sensitivity to parallel shifts in the term structure. That is, shifts for which each yield point moves by the same amount. Other definitions of duration were then developed (Bierwag (1977), Khang

(1979), Brennan and Schwartz(1982)) corresponding to other models of yield curve dynamics, or the manner in which the term structure changed. Surveys of these models and related matters can be found in Bierwag, Kaufman and Toevs(1983d) and Bierwag(1987), the latter reference also providing an excellent introduction to many aspects of this theory and its applications. The importance of the correct choice regarding yield curve dynamics was first noted in Bierwag, Kaufman and Toevs(1983a), which investigated stochastic process risk.

Other extensions of Redington's work include Grove(1974), which immunized a non-zero initial net worth, Kaufman(1984), which investigated the immunization of the net worth asset ratio, and Bierwag, Kaufman and Toevs(1983b,c) which introduced a methodology for developing an immunizing asset portfolio, and investigated the concept of an efficient frontier in this context.

More recent approaches have involved immunizing multiple liabilities (Shiu(1988)), tax adjusting the duration measure (Stock and Simonson(1988)), and utilizing a duration vector approach to immunization (Chambers, Carleton and McEnally(1988)). This latter approach defined a vector in which each component reflected a "moment" of adjusted times-to-receipt of the underlying cash flows. In this context, traditional duration is closely related to their first moment, while the concepts of "convexity" and "inertia" (Bierwag(1987)) are closely related to their second moment. The adjustment made to the times-to-receipt of the cash flows was the reduction by one time unit.

A general nonparallel shift approach to duration analysis was developed in Reitano(1989), and an application to measuring potential yield curve risk exemplified in Reitano(1990). For this analysis, a yield curve is identified with a "vector" of values representing yields at the commonly quoted maturities. The underlying technique employed was a general multivariate analysis. While multivariate models are not in general new (Bierwag(1987)), the particular model utilized was found to provide great insight to the sensitivity to general yield curve shifts. In particular, "partial" durations were defined to reflect yield sensitivities point by point along the yield curve. These measures could then be easily combined to produce "directional" duration measures which reflected portfolio sensitivity to any yield curve shift. The traditional duration measure, for example, reflecting sensitivity in the parallel shift direction, is seen to be the sum of the underlying partial durations.

The current article extends this theory to the question of immunization. The yield curve is again modelled as a vector of quoted maturity yields, with other yields assumed to be functionally dependent, such as via interpolation. Consequently, all yield curve changes are identified with vector shifts, and immunization pursued within this multivariate context.

This immunization model is introduced in Section II, along with the necessary definitions from Reitano(1989). Section III then develops an extension of Redington's approach to general nonparallel yield curve shifts. Here, we define and explore directional immunization and extend and exemplify the general results within the context of spot rate and forward rate models.

In this context, as throughout the paper, the goal of immunization at time k is seen to be intimately connected with the relationship of the portfolio's directional duration and convexity attributes to those of a k period zero coupon bond. Naturally, immunization results for the special case of parallel shifts are seen to be equivalent to well-known results. Also in this section, the concept of an immunization boundary is explored, extending the idea of duration window (Bierwag(1987)), as is the return on investment, generalizing Babcock(1984).

Section IV then develops immunization results in the general nondirectional context. That is, conditions under which portfolio values at time k are preserved under all yield curve shifts. The spot and forward rate models are revisited, as are practical issues related to implementing this approach. General return on investment results are then developed.

Section V investigates the relationship of earlier immunization properties to the yield curve model employed.

Finally, Section VI applies the previous results to the context of asset/liability management. Surplus immunization conditions are developed in both the absolute and asset ratio contexts and the results translated to implications for the immunization boundary.

A technical appendix is included and contains the proofs of the duration theory underlying the immunization results.

II. Multivariate Immunization

A. Multivariate Price Model

Let $P(\mathbf{i})$ denote a positive valued multivariate price function which reflects the dependency of the price of a portfolio of securities on an underlying yield curve vector, $\mathbf{i} = (i_1, \ldots, i_m)$. This portfolio could equally well reflect assets, liabilities, or a net worth or surplus position. The cash flows anticipated by $P(\mathbf{i})$ may be fixed or interest dependent, with $P(\mathbf{i})$ correspondingly representing a simple present value price function, or the price values obtained via a model which incorporates the options or other interest dependencies (for example, Clancy (1985), Ho, Lee (1986), and Jacob, Lord and Tilley (1987)).

The yield curve above is modelled as a discrete vector, representing as previously noted, the quoted maturity points or yield drivers in a given valuation model. This yield curve may reflect any system of units (bond yields, spot or forward rates), and any nominal basis (annual, semi-annual, etc.). In practice, yield points at other maturities are typically derived from these values via interpolation and/or other conversion, so it is appropriate to view the price of the portfolio, $P(\mathbf{i})$, as a function of this yield curve vector. For example, with $\mathbf{i}$ corresponding to bond yields, pivotal yield values for maturities .25, 1, 3, 5, 7, 10, 20 and 30 years are sufficient for many valuations, and $P(\mathbf{i})$ can be modelled as a function of these eight observed values.

As in Reitano (1989), we make the following definitions, which generalize the notions of duration and convexity to this yield vector basis. Accordingly, we assume throughout that $P(\mathbf{i})$ is twice differentiable, with continuous second order partial derivatives.

Definition 1: Given $P(\mathbf{i})$, the jth _partial duration function_, denoted $D_j(\mathbf{i})$, and the jkth _partial convexity function_, denoted $C_{jk}(\mathbf{i})$, are defined for $P(\mathbf{i}) \neq 0$ as follows:

$$(2.1) \qquad D_j(\mathbf{i}) = -d_jP(\mathbf{i})/P(\mathbf{i}), \qquad j = 1,\ldots,m$$

$$(2.2) \qquad C_{jk}(\mathbf{i}) = d_{jk}P(\mathbf{i})/P(\mathbf{i}), \qquad j,k = 1,\ldots,m$$

where $d_jP(\mathbf{i})$ and $d_{jk}P(\mathbf{i})$ denote the corresponding partial derivatives of $P(\mathbf{i})$.

The _total duration vector_, denoted $\mathbf{D}(\mathbf{i})$, and _total convexity matrix_, denoted $\mathbf{C}(\mathbf{i})$, are defined as follows:

$$(2.3) \qquad \mathbf{D}(\mathbf{i}) = (D_1(\mathbf{i}),\ldots,D_m(\mathbf{i})),$$

$$(2.4) \qquad \mathbf{C}(\mathbf{i}) = \begin{vmatrix} C_{11}(\mathbf{i}) & \ldots & C_{1m}(\mathbf{i}) \\ & & \\ & & \\ C_{m1}(\mathbf{i}) & \ldots & C_{mm}(\mathbf{i}) \end{vmatrix}. \quad ||$$

Intuitively, $D_j(\mathbf{i})$ reflects the sensitivity of $P(\mathbf{i})$ to movements in the jth yield point. For example, if $j = 10$ in the above bond yield model, changes in this yield will affect the value of cash flows at time 10 years, as well as those in the range from 7 to 20 years because of the interpolation of yields at these maturities.

Similarly, $C_{jk}(\mathbf{i})$ reflects a "second order" sensitivity of $P(\mathbf{i})$ to movements on the jth and kth yield points.

When appropriate, $\mathbf{D}(\mathbf{i})$ will be interpreted as a row matrix. Also, note that by the above continuity assumption that $C_{jk}(\mathbf{i}) = C_{kj}(\mathbf{i})$, and hence, $\mathbf{C}(\mathbf{i})$ is a symmetric matrix.

Definition 2: Given $P(\mathbf{i})$, and yield curve direction vector $\mathbf{N} = (n_1,\ldots,n_m)$ with $\mathbf{N} \neq \mathbf{0}$, the <u>directional duration function</u> in the direction of $\mathbf{N}$, denoted $D_N(\mathbf{i})$, and the <u>directional convexity function</u> in the direction of $\mathbf{N}$, denoted $C_N(\mathbf{i})$, are defined for $P(\mathbf{i}) \neq 0$ as follows:

$$(2.5) \qquad D_N(\mathbf{i}) = -d_N P(\mathbf{i})/P(\mathbf{i}),$$

$$(2.6) \qquad C_N(\mathbf{i}) = d_{NN} P(\mathbf{i})/P(\mathbf{i}),$$

where $d_N P(\mathbf{i})$ and $d_{NN} P(\mathbf{i})$ denote the first and second directional derivatives of $P(\mathbf{i})$ in the direction of $\mathbf{N}$. !!

Intuitively, $\mathbf{N}$ equals the "direction" of the yield curve shift in that it reflects the relative magnitude of the individual shift amounts. A typical shift can then be modelled as $t\mathbf{N} = (tn_1,\ldots,tn_m)$, corresponding to each yield point i_j shifting by the amount tn_j. When all $n_j = 1$, the classical parallel shift model results.

As developed in Reitano (1989), the directional measures can be easily obtained from the corresponding partial measures as follows:

(2.7) $$D_N(\mathbf{i}) = \mathbf{D}(\mathbf{i})\cdot\mathbf{N} = \Sigma\, n_j D_j(\mathbf{i}),$$

(2.8) $$C_N(\mathbf{i}) = \mathbf{N}^T\mathbf{C}(\mathbf{i})\mathbf{N} = \Sigma\Sigma\, n_j n_k C_{jk}(\mathbf{i}),$$

where $\mathbf{N}^T$ denotes the transpose of the column vector $\mathbf{N}$.

When $\mathbf{N} = (1,\ldots,1)$, the associated directional measures above reduce to the more traditional modified duration and convexity measures, $D(\mathbf{i})$ and $C(\mathbf{i})$, calculated with respect to parallel yield curve shifts. In addition, we have from (2.7) and (2.8), that these traditional measures equal the sums of the corresponding partial measures:

(2.9) $$D(\mathbf{i}) = \Sigma\, D_j(\mathbf{i}),$$

(2.10) $$C(\mathbf{i}) = \Sigma\Sigma\, C_{jk}(\mathbf{i}).$$

When necessary for clarity, duration and convexity functions will explicitly reflect $P(\mathbf{i})$, such as $D_N(P;\mathbf{i})$ for $D_N(\mathbf{i})$.

B. Immunization Definitions

Let $P_k(\mathbf{i})$ denote the forward value of the portfolio at time $k \geq 0$, on the yield curve vector $\mathbf{i}$, where it is assumed that no securities are either added or removed from the portfolio. In addition, we assume that the yield vector changes from $\mathbf{i}_0$ to $\mathbf{i}$ immediately after time 0, and remains fixed at this level throughout the period. Extending the classical notions of immunization, we have the following:

Definition 3: The price function $P(\mathbf{i})$ is said to be <u>locally immunized at time k on the yield vector $\mathbf{i}_0$</u> if:

$$P_k(\mathbf{i}) \geq P_k(\mathbf{i}_0), \tag{2.11}$$

for $\mathbf{i}$ sufficiently close to $\mathbf{i}_0$. That is, for $\|\mathbf{i} - \mathbf{i}_0\| < r$, where $r > 0$ and $\|\mathbf{i}\|$ denotes the standard Euclidean norm:

$$\|\mathbf{i}\|^2 = \Sigma i_j{}^2. \tag{2.12}$$

Similarly, $P(\mathbf{i})$ is said to be <u>globally immunized at time k on the yield vector $\mathbf{i}_0$</u> if (2.11) is satisfied for all feasible yield vectors $\mathbf{i}$. ||

For the purposes of Definition 3, "feasibility" will not be rigorously defined. Certainly, the restriction:

$$0 < i_j < 1 \tag{2.13}$$

is a minimal requirement for feasibility, though in practice other bounds may be more practical.

We analogously define local and global immunization in the direction of $\mathbf{N}$ by:

$$P_k(\mathbf{i}_0 + t\mathbf{N}) \geq P_k(\mathbf{i}_0) \tag{2.14}$$

for $|t| < r$ (local), and for all feasible t (global).

Note that for the purposes of directional immunization, we restrict our attention to yield curve shifts of a fixed type, **N**, so only the amount of the shift, t, is variable. For example, **N** could reflect the classical parallel shift direction vector, or a shift vector which changes the yield curve level and slope, or more general types of shifts. In the nondirectional immunization model, we consider all possible shift directions from $\mathbf{i}_0$.

For $P(\mathbf{i})$ to be immunized at time k on $\mathbf{i}_0$, it is clear from (2.11) and (2.14) that $\mathbf{i}_0$ needs to be a relative minimum of $P_k(\mathbf{i})$ in the local immunization case, and a global minimum in the global immunization case. For the results below, we utilize the well-known sufficient conditions for a point to be minimum value. For example, a sufficient condition for $\mathbf{x}_0$ to be a local minimum of $f(\mathbf{x})$ in the direction of **N** is that:

(2.15) $$d_N f(\mathbf{x}_0) = 0,$$

(2.16) $$d_{NN} f(\mathbf{x}_0) > 0.$$

A sufficient condition for $\mathbf{x}_0$ to be a global minimum is that (2.16) is satisfied for all $\mathbf{x}$.

Similarly, a sufficient condition for $\mathbf{x}_0$ to be a local minimum of $f(\mathbf{x}_0)$ is that (2.15) and (2.16) hold for all **N**. That is:

(2.17) $$d_j f(\mathbf{x}_0) = 0, \qquad j = 1,\ldots,m$$

(2.18) $\|d_{jk}f(x_0)\|$ is positive definite,

where $\|d_{jk}f(x_0)\|$ denotes the second derivative matrix, or Hessian matrix of $f(x)$. A sufficient condition for x_0 to be a global minimum is that (2.18) hold for all x.

The sufficiency of such conditions follows from the mean value property of a Taylor series. Given $x = x_0 + N$, there exists ξ satisfying $0 < \xi < 1$ so that:

$$f(x) = f(x_0) + \Sigma\, d_j f(x_0) n_j + \tfrac{1}{2}\Sigma\Sigma\, d_{jk} f(x_0 + \xi N) n_j n_k$$

$$= f(x_0) + d_N f(x_0) + \tfrac{1}{2} d_{NN} f(x_0 + \xi N).$$

In other words, a function's value at $x = x_0 + N$ can be expressed in terms of a linear approximation at x_0:

$$f(x_0) + d_N f(x_0),$$

plus a correction term equal to half the second derivative evaluated "somewhere" on the line segment joining x_0 and x.

Consequently, if conditions (2.15) and (2.16) are satisfied, x_0 must be a local minimum of $f(x)$ in the direction of N. This is because (2.16) implies that $d_{NN}(x) > 0$ for x "close" to x_0 by continuity. Similarly, conditions (2.17) and (2.18) imply a local minimum relative to any direction, since:

$$d_{NN}(x_0 + \xi N) = N^T \|d_{jk} f(x_0 + \xi N)\| N,$$

and by continuity, $\|d_{jk}f(x)\|$ is positive definite for x close to

x_0.

C. The Forward Price Function: $P_k(\mathbf{i})$

Given the yield curve vector $\mathbf{i}$, let $Z_k(\mathbf{i})$ denote the price of a k-period zero-coupon bond with maturity value of 1. Clearly, $1/Z_k(\mathbf{i})$ then equals the forward value at time k of 1 invested now, and consequently:

$$P_k(\mathbf{i}) = P(\mathbf{i})/Z_k(\mathbf{i}). \tag{2.19}$$

For example, if $i_j = i$ for all j, then $Z_k(i) = (1 + i)^{-k}$ and $P_k(i) = (1 + i)^k P(i)$.

We next investigate the immunization of $P(\mathbf{i})$. As will be seen, the durational and convexity properties of $Z_k(\mathbf{i})$ provide insight to sufficient conditions for immunization of $P(\mathbf{i})$ at time k. In particular, for local immunization we require that $P(\mathbf{i})$ have the "same duration" as $Z_k(\mathbf{i})$, and to be "more convex," on the yield vector $\mathbf{i}_0$. For global immunization, we also require convexity relationships on other yield curves besides $\mathbf{i}_0$. The concepts of "same duration" and "more convex" will be made precise below, but will be seen to be natural generalizations of the classical notions to this multivariate context.

III. Directional Immunization

A. General Results

In this section, general results are presented on directional immunization. In this context, it is sufficient for $P(\mathbf{i})$ to have the same directional duration as $Z_k(\mathbf{i})$, and greater directional convexity, to be locally immunized at time k.

Proposition 1: Let $P(\mathbf{i})$, $\mathbf{i}_0$ and $\mathbf{N} \neq \mathbf{0}$ be given and assume there exists $k \geq 0$ so that:

$$(3.1) \qquad D_N(P;\mathbf{i}_0) = D_N(Z_k;\mathbf{i}_0),$$

$$(3.2) \qquad C_N(P;\mathbf{i}_0) > C_N(Z_k;\mathbf{i}_0).$$

Then $P(\mathbf{i})$ is locally immunized in the direction of $\mathbf{N}$ at time k on the yield vector $\mathbf{i}_0$.

The proof of Proposition 1 is readily obtained from the following result, proved in the Appendix (Corollary A.4):

Lemma 1: Let $P(\mathbf{i}) = Q_1(\mathbf{i})/Q_2(\mathbf{i})$, where $Q_1(\mathbf{i})$, $Q_2(\mathbf{i}) \neq 0$. Then:

$$D_N(P;\mathbf{i}) = D_N(Q_1;\mathbf{i}) - D_N(Q_2;\mathbf{i}),$$

$$(3.3) \qquad C_N(P;\mathbf{i}) = C_N(Q_1;\mathbf{i}) - C_N(Q_2;\mathbf{i}) + 2D_N(Q_2;\mathbf{i})[D_N(Q_2;\mathbf{i}) - D_N(Q_1;\mathbf{i})].$$

proof of Proposition 1:

Applying Lemma 1 to $P_k(\mathbf{i})$ in (2.19), we have from (3.1) and (3.2) that:

(3.4) $$D_N(P_k;\mathbf{i}_O) = 0,$$

(3.5) $$C_N(P_k;\mathbf{i}_O) > 0.$$

Consequently, the respective directional derivatives of $P_k(\mathbf{i})$ satisfy the conditions in (2.15) and (2.16) and the result follows. ||

For global immunization in the direction of **N**, we require a convexity constraint on all feasible yield vectors $\mathbf{i} = \mathbf{i}_O + t\mathbf{N}$. While this constraint can be expressed directly in terms of (3.3), we instead chose an equivalent, more symmetric representation.

Proposition 2: Let $P(\mathbf{i})$, $\mathbf{i}_O$ and $\mathbf{N} \neq \mathbf{0}$ be given and assume that there exists $k \geq 0$ so that:

(3.6) $$D_N(P;\mathbf{i}_O) = D_N(Z_k;\mathbf{i}_O),$$

(3.7) $$d_N D_N(P;\mathbf{i}) < d_N D_N(Z_k;\mathbf{i}) + [D_N(P;\mathbf{i}) - D_N(Z_k;\mathbf{i})]^2,$$

for all feasible yield vectors $\mathbf{i} = \mathbf{i}_O + t\mathbf{N}$. Then $P(\mathbf{i})$ is globally immunized in the direction of **N** at time k on the yield vector $\mathbf{i}_O$.

proof: By Definition 2, $d_N P(\mathbf{i}) = -P(\mathbf{i})D_N(\mathbf{i})$. Taking directional derivatives and dividing by $P(\mathbf{i})$ produces:

(3.8) $$d_N D_N(P;\mathbf{i}) = D_N^2(P;\mathbf{i}) - C_N(P;\mathbf{i}),$$

as well as a similar identity for $d_N D_N(Z_k;\mathbf{i})$. Hence, using Lemma 1, (3.7) assures that (3.5) is satisfied for all feasible $\mathbf{i}$, while (3.6) implies (3.4), and the result follows as before. ||

We note in passing that the convexity constraint (3.2) in Proposition 1 can readily be expressed in terms of directional derivatives as in Proposition 2. Specifically, due to (3.8) and (3.1), we can rewrite this constraint as:

$$d_N D_N(P;\mathbf{i}_0) < d_N D_N(Z_k;\mathbf{i}_0).$$

For fixed $\mathbf{N} \neq \mathbf{0}$, the pair $(k, \mathbf{i}_0)$ of the above Propositions give rise to a "duration window" $(k, P_k(\mathbf{i}_0))$ as defined in Bierwag (1987). Specifically, consider the graph of $y = P_x(\mathbf{i})$ in the xy-plane for each feasible $\mathbf{i} = \mathbf{i}_0 + t\mathbf{N}$. All such graphs will equal or exceed the value $P_k(\mathbf{i}_0)$ when $x = k$ in the case of global immunization, while all graphs with $|t| < r$ will have this property in the local immunization case. That is, each will pass through a "window" at $x = k$ with lower bound equal to $P_k(\mathbf{i}_0)$. Consequently, the value $P_k(\mathbf{i}_0)$ also gives rise to the minimum annual return on investment over the interval $[0, k]$.

It is natural to inquire into the existence of other such duration windows. That is, given $\mathbf{i}_t = \mathbf{i}_0 + t\mathbf{N}$, does there exist $k = k(t)$ so that $P(\mathbf{i})$ is immunized at time $k(t)$ on $\mathbf{i}_t$? We next consider all such pairs, $(k(t), \mathbf{i}_t)$ and the associated duration windows, as forming an immunization boundary.

Definition 4: Given $P(\mathbf{i})$ and $\mathbf{N} \neq \mathbf{0}$, let $\mathbf{i}_t = \mathbf{i}_0 + t\mathbf{N}$ denote the yield vector on which $P(\mathbf{i})$ is locally (globally) immunized in the direction of $\mathbf{N}$ at time $k = k(t)$ if such a k exists. Then the <u>local (global) immunization boundary</u> for $P(\mathbf{i})$, in the direction of

N, denoted $IB_N(P)$, is defined:

$$(3.9) \qquad IB_N(P) = \{(k, P_k(\mathbf{i}_t)) \mid k = k(t)\} \quad !!$$

The immunization boundary then has the same property as does the duration window, yet over a range of forward times k. That is, the collection of graphs $y = P_x(\mathbf{i})$ for $\mathbf{i} = \mathbf{i}_0 + s\mathbf{N}$ will be minimized at each such k(t) on the yield vector $\mathbf{i}_t$ in the global case, and for more limited ranges of yield values in the local immunization case. Therefore, $P_k(\mathbf{i}_t)$ reflects the minimum portfolio value in this sense at each such time k(t), and consequently gives rise to the minimum annual return on investment, i(k), over every such interval [0, k], where:

$$(3.10) \qquad i(k) = [P_k(\mathbf{i}_t)/P(\mathbf{i}_0)]^{\frac{1}{k}} - 1, \qquad k = k(t),$$

and $\mathbf{i}_0$ is the initial yield vector. Note that for t = 0, the minimum return given in (3.10) equals the k period return on the zero coupon bond, $Z_k(\mathbf{i}_0)$, due to (2.19).

We next investigate the above concepts within the framework of two common yield vector models: the spot rate and forward rate models.

B. The Spot Rate Model

Assume that $\mathbf{i}_0 = (i_0, i_1, \ldots, i_m)$ is given and reflects the current spot rate structure. For example, i_0 might be the overnight rate, i_1 the 1 year spot rate, etc. For notational simplicity, we assume that these values are already converted to the nominal basis consistent with the period length.

To have a continuous discounting model, we interpolate spot rates for other periods as follows, where l is an integer:

$$i_{l+s} = (1 - s)i_l + si_{l+1}, \qquad 0 \le s \le 1.$$

Consequently, the zero coupon bond $Z_k(\mathbf{i})$ has value:

$$(3.11) \qquad Z_k(\mathbf{i}) = (1 + (1 - s)i_l + si_{l+1})^{-k},$$

where $k = l + s$. For notational convenience we set:

$$v_k = (1 + i_k)^{-1} = (1 + (1 - s)i_l + si_{l+1})^{-1}.$$

A calculation produces the following partial durations and convexities where $[|k|]$ denotes the greatest integer less than or equal to k:

$$(3.12) \qquad D_j(Z_k;\mathbf{i}_0) = \begin{cases} kv_k(1-s) & j = [|k|] \\ kv_ks & j = [|k|] + 1 \\ 0 & \text{otherwise} \end{cases}$$

$$(3.13) \qquad C_{ij}(Z_k;\mathbf{i}_0) = \begin{cases} k(k+1)v_k^2(1-s)^2 & i = j = [|k|] \\ k(k+1)v_k^2 s(1-s) & i = [|k|],\ j = [|k|] + 1 \\ & i = [|k|] + 1,\ j = [|k|] \\ k(k+1)v_k^2 s^2 & i = j = [|k|] + 1 \\ 0 & \text{otherwise} \end{cases}$$

Hence, condition (3.1) in Proposition 1 that $D_N(P;\mathbf{i}_0) = D_N(Z_k;\mathbf{i}_0)$ can be expressed as follows using (2.7) and (3.12), where

$k = \ell + s$:

(3.14) $$\Sigma n_j D_j(P;i_0) = kv_k((1-s)n_\ell + sn_{\ell+1}).$$

Note that the right hand side of (3.14) is not necessarily monotonic, so multiple solutions may exist. In addition, (3.14) may fail to have a solution.

In the special case where $\mathbf{N} = (1,\ldots,1)$, the parallel shift direction vector, (3.14) reduces to:

(3.15) $$D(P;i_0) = kv_k.$$

When the spot rate vector is flat, $i_j = i$ for all j, (3.15) is readily solved for k, producing the classic result:

(3.16) $$k = (1+i)D(P;i) = D^M(P;i),$$

where D^M denotes the Macaulay duration of P. In this case, the immunization boundary is also easy to describe, subject to convexity constraints. Specifically, for any feasible rate i, the associated point is $(k, P_k(i))$ where k is given by (3.16). Subject to a convexity test, this boundary then defines the minimum value of $P_k(i)$ at each time k so produced. That is, for all k in the range of $D^M(P;i)$ considered as a function of i.

For spot rate vectors which are not flat, (3.15) can only be solved with extra effort. Because kv_k is continuous, it can first be evaluated for integer k, producing bounds for the exact solution(s). For a decreasing spot rate yield curve, the solution will be unique if it exists, since then kv_k will be monotonically increasing. Otherwise, multiple solutions are possible as noted above. In any case, it is interesting to note that the solution k again appears to be a Macaulay-type duration, in that:

$$(3.17) \qquad k = (1 + i_k)D(P, \mathbf{i}_0).$$

To investigate the convexity constraint in (3.2), a calculation using (2.8) and (3.13) produces the following, where $k = \ell + s$:

$$(3.18) \qquad C_N(Z_k; \mathbf{i}_0) = k(k+1)v_k^2\ ((1-s)n_\ell + sn_{\ell+1})^2$$

$$= (1 + {}^1/_k)\ D_N^2(Z_k; \mathbf{i}_0)$$

$$= D_N(Z_k; \mathbf{i}_0)[D_N(Z_k; \mathbf{i}_0) + v_k((1-s)n_\ell + sn_{\ell+1})].$$

Consequently, because $D_N(Z_k; \mathbf{i}_0) = D_N(P; \mathbf{i}_0)$, (3.2) becomes:

$$(3.19) \qquad C_N(P; \mathbf{i}_0) > (1 + {}^1/_k)D_N^2(P; \mathbf{i}_0),$$

or an equivalent inequality using the last expression in (3.18).

For example if $\mathbf{N} = (1, \ldots, 1)$, *this* can be equivalently rewritten:

$$(3.20) \qquad C(P; \mathbf{i}_0) > D(P; \mathbf{i}_0)[D(P; \mathbf{i}_0) + v_k].$$

When the yield curve is flat, a calculation below shows that (3.20) is always satisfied when cash flows are fixed and positive.

That is, we always have:

$$(3.21) \qquad C(P,i_0) > D^M(P,i_0)[D^M(P,i_0) + 1]v^2.$$

Hence, in this well-known case, the local Macaulay immunization boundary is as noted above:

$$(3.22) \qquad IB_M(P) = \{(k, (1+i_k)^k P(i_k)) \mid k = D^M(P,i_k),\ i_k \text{ feasible}\}.$$

In other words, for price functions with fixed positive cash flows, such as those for noncallable bonds, the immunization boundary exists for all k in the range of the Macaulay duration function. That is, for each feasible yield rate i, we calculate $k = D^M(P;i)$ and obtain the associated minimum value of the price function at k, namely $(1+i)^k P(i)$. In addition, the associated minimum returns on investment, i(k), are given by:

$$(3.23) \qquad i(k) = (1 + i)[P(i)/P(i_0)]^{\frac{1}{k}} - 1,$$

where i_0 is the initial yield to maturity.

In the more general directional immunization case with fixed cash flows, not necessarily all positive, we have the following Proposition, in which the convexity constraint is expressed in terms of the last expression in (3.18).

Proposition 3: Let $P(i) = \Sigma a_j v^j$ be a price function with fixed cash flows and $P(i) \neq 0$. A necessary and sufficient condition for P(i) to satisfy:

(3.24) $$C_N(P;\mathbf{i}) > D_N(P;\mathbf{i})[D_N(P;\mathbf{i}) + v],$$

where $v = v_K((1-s)n_\ell + sn_{\ell+1})$, $K=\ell+s$, is that:

(3.25) $$Var(X_J) > E[X_J] - E[X_J^2/J],$$

where $X_J = Jn_Jv_J/v$, and the implied "probabilities" p_J are defined by $p_J = a_Jv_J^J/P(\mathbf{i})$.

proof: A calculation using (2.7) and (2.8) shows that given $P(\mathbf{i})$ above:

$$D_N(\mathbf{i}) = \Sigma\, Jn_Ja_Jv_J^{J+1}/P(\mathbf{i}) = v\Sigma\, p_JX_J,$$

$$C_N(\mathbf{i}) = \Sigma\, J(J+1)n_J^2a_Jv_J^{J+2}/P(\mathbf{i}) = v^2\Sigma\, p_J(X_J^2 + X_J^2/J).$$

Hence,

$$C_N(\mathbf{i}) - D_N(\mathbf{i})(D_N(\mathbf{i}) + v)$$
$$= v^2[E(X_J^2 + X_J^2/J) - E^2(X_J) - E(X_J)],$$

and the result follows. ||

It should be noted that in the classical model with $\mathbf{N} = (1,\ldots,1)$, $i_J = i$ and $a_J \geq 0$, (3.24) is always satisfied since then $X_J = J$, $p_J \geq 0$, and the right hand side of (3.2) is 0. In addition, $Var(X_J)$ is equal to the portfolio "inertia" in this case, as defined in Bierwag(1987).

In the most general case, a simple necessary condition for (3.24) can be cited. Here, however, we express this convexity constraint in terms of (3.19).

Proposition 4: Let $P(\mathbf{i})$ be given. A necessary condition for

(3.19) to be satisfied is that:

$$(3.26) \qquad d_N D_N(\mathbf{i}_0) < 0.$$

That is, $D_N(\mathbf{i})$ is a decreasing function at $\mathbf{i}_0$ in the direction of **N**.

proof: By (3.8), the above condition implies that $C_N(\mathbf{i}_0) > D_N^2(\mathbf{i}_0)$, which is clearly a necessary condition for (3.19). ||

The above proposition provides an intuitive necessary condition for immunization when the directional duration identity in (3.1) is satisfied. Clearly, this condition can be sharpened to be sufficient as well. Specifically, (3.19) is equivalent to:

$$(3.27) \qquad d_N D_N(\mathbf{i}_0) < -\frac{1}{k} D_N^2(\mathbf{i}_0).$$

Consequently, (3.26) is also a necessary condition if and only if $D_N(\mathbf{i}_0) = 0$.

C. The Forward Rate Model

We assume here that $\mathbf{i}_0 = (i_1,\ldots,i_m)$ is given and reflects the current forward rate structure. Corresponding to (3.11), we have, for $k = \ell + s$, $0 \le s \le 1$:

$$(3.28)\qquad Z_k(\mathbf{i}) = \prod_{j=1}^{\ell} (1 + i_j)^{-1}(1 + i_{\ell+1})^{-s}.$$

Taking partial derivatives in (3.28) and applying (2.7), the condition in (3.1) that $D_N(P;\mathbf{i}_0) = D_N(Z_k;\mathbf{i}_0)$ becomes:

$$(3.29)\qquad \Sigma\, n_j D_j(P;\mathbf{i}_0) = \sum_{1}^{\ell} n_j v_j + s n_{\ell+1} v_{\ell+1},$$

where $v_j = (1 + i_j)^{-1}$. Clearly, the right hand side of (3.29) is monotonic in k if and only if all n_j have the same sign. In this case, the solution for k is unique when it exists. Otherwise, multiple solutions are possible.

For the special case $\mathbf{N} = (1,\ldots,1)$, (3.29) becomes:

$$(3.30)\qquad D(P;\mathbf{i}_0) = k\bar{v}_k,$$

where $\bar{v}_k$ is an "average" discount factor:

$$(3.31)\qquad \bar{v}_k = \frac{1}{k}\left(\sum_{0}^{\ell} v_j + s v_{\ell+1}\right).$$

Consequently, in this case k is again seen to be a Macaulay-type duration as in (3.17):

(3.32) $$k = (1 + \bar{i}_k)D(P;\mathbf{i}_0),$$

where $\bar{i}_k$ is the interest rate associated with $\bar{v}_k$ in (3.31). Note that $\bar{v}_k$ reduces to $v = (1 + i)^{-1}$ when $i_j = i$ for all j and (3.32) is identical to (3.16).

Taking second partial derivatives in (3.28) and applying (2.8), we have:

(3.33) $$C_N(Z_k;\mathbf{i}_0) = D_N^2(Z_k;\mathbf{i}_0) + \sum_1^{\ell} n_j^2 v_j^2 + s n_{\ell+1}^2 v_{\ell+1}^2.$$

Hence, given that $D_N(Z_k;\mathbf{i}_0) = D_N(P;\mathbf{i}_0)$, the convexity constraint in (3.2) becomes:

(3.34) $$C_N(P;\mathbf{i}_0) > D_N^2(P;\mathbf{i}_0) + \sum_1^{\ell} n_j^2 v_j^2 + s n_{\ell+1}^2 v_{\ell+1}^2.$$

Clearly, in order for (3.34) to be satisfied, it is necessary to have $C_N - D_N^2 > 0$. Consequently, Proposition 4 applies in this context as well, as does the obvious counterpart to (3.27).

D. Returns on Investment: I_k

As noted above, the immunization boundary gives rise to the minimum return on investment, $i(k)$, over every period $[0,k]$ for which $P(\mathbf{i})$ can be immunized at time k. The return on investment over $[0,k]$ is in fact a random variable, I_k, the value of which depends on the yield vector $\mathbf{i}$. Here as before, we assume the initial yield vector to be $\mathbf{i}_0$, and that this value changes to $\mathbf{i}$ immediately after time 0, and remains fixed at this level throughout the period.

Similar to (3.10), which provided the minimum value of $I_k(\mathbf{i})$, we have:

$$(3.35) \qquad I_k(\mathbf{i}) = [P_k(\mathbf{i})/P(\mathbf{i}_0)]^{\frac{1}{k}} - 1,$$

where $\mathbf{i} = \mathbf{i}_0 + t\mathbf{N}$. Following Babcock(1984), we seek an approximation for $I_k(\mathbf{i})$, where the approximation reflects the dependency on t. To this end, let $\psi(t)$ denote the right hand side of (3.35), expressed as a function of t, where $\mathbf{i} = \mathbf{i}_0 + t\mathbf{N}$. The first order Taylor series approximation is then $\psi(0) + \psi'(0)t$.

By substitution, we have that $\psi(0) = j(k)$, where $j(k)$ is the k period return on the zero coupon bond, $Z_k(\mathbf{i}_0)$, due to (3.35). To evaluate $\psi'(t)$, note that:

$$d_t P_k(\mathbf{i})|_{t=0} = d_N P_k(\mathbf{i}_0) = -P_k(\mathbf{i}_0) D_N(P_k;\mathbf{i}_0).$$

Consequently, we obtain the approximation:

$$(3.36) \qquad I_k(\mathbf{i}) \approx j(k) + (1 + j(k))[D_N(Z_k;\mathbf{i}_0) - D_N(P;\mathbf{i}_0)]t/k.$$

Note that if $P(\mathbf{i})$ is immunized at time k, then the above linear approximation reduces to: $I_k(\mathbf{i}) \approx j(k)$. In this context, however, $j(k) = i(k)$ as defined in (3.10). Since $i(k)$ is the minimum value of $I_k(\mathbf{i})$, by definition, it is clear that the above formula is somewhat crude in this case.

Taking the second derivative of $\psi(t)$, we obtain the following generalization of (3.36), where all durations are evaluated on $\mathbf{i}_0$:

$$(3.37)\quad I_k(\mathbf{i}) \approx j(k) + (1 + j(k))[D_N(Z_k) - D_N(P)]t/k$$
$$+ (1 + j(k))\left\{d_N D_N(Z_k) - d_N D_N(P) + \frac{1}{k}(D_N(Z_k) - D_N(P))^2\right\}t^2/2k.$$

If $P(\mathbf{i})$ is immunized at time k, we see from (3.7) that the second order term in (3.37) is positive, and hence $I_k(\mathbf{i}) > j(k) = i(k)$ as expected.

For other values of k, the linear term in (3.36) will in general be nonzero. Specifically, if $P(\mathbf{i})$ is "longer" than Z_k on $\mathbf{i}_0$ in the direction of **N**, then $I_k(\mathbf{i})$ will decrease with increases in the yield structure in this direction. That is, the capital loss due to the increase in yields cannot be made up by reinvestment gains over the period [0, k]. Similarly, $I_k(\mathbf{i})$ will increase with decreases in the direction of **N**. On the other hand, if $P(\mathbf{i})$ is "shorter" than Z_k on $\mathbf{i}_0$ in this direction, then $I_k(\mathbf{i})$ will increase with yield increases in the direction of **N**, since then reinvestment gains will overcome initial capital losses. In all cases, the second order adjustment in (3.37) will be independent of the "sign" of the yield curve movement, reflecting only the magnitude of t. In general, however, the "sign" of this adjustment will depend on k.

Naturally, either of the above approximations can be used to estimate the mean and variance of I_k, given an assumption as to the probability density of t, or $(\mathbf{i} - \mathbf{i}_0)$ measured in units of the shift vector **N**. For example, from (3.36), we obtain:

(3.38) $$E[I_k(\mathbf{i})] = j(k) + (1 + j(k))[D_N(Z_k) - D_N(P)]E(t)/k,$$

(3.39) $$Var[I_k(\mathbf{i})] = (1 + j(k))^2 [D_N(Z_k) - D_N(P)]^2 Var(t)/k^2.$$

IV. Non-Directional Immunization

A. General Results

In this section, general results on non-directional immunization will be developed and seen to be natural generalizations of the above results. For local immunization, for example, we again require $P(\mathbf{i})$ to have the "same duration" as $Z_k(\mathbf{i})$ on $\mathbf{i}_0$, and be "more convex." Here, however, the constraints **are stated in terms of the total duration vectors and total convexity matrices.** We begin with a definition:

Definition 5: Let $\mathbf{A}$ and $\mathbf{B}$ be square matrices. We say that <u>$\mathbf{A}$ is greater than $\mathbf{B}$</u>, denoted $\mathbf{A} > \mathbf{B}$, if $\mathbf{A} - \mathbf{B}$ is positive definite. That is, $\mathbf{x}^T(\mathbf{A} - \mathbf{B})\mathbf{x} > 0$ for all $\mathbf{x} \neq 0$. !!

For convenience, we will sometimes write $\mathbf{A} > \mathbf{0}$, which by Definition 5 means that $\mathbf{A}$ is positive definite.

The generalization of Proposition 1 is then:

Proposition 5: Let $P(\mathbf{i})$ and $\mathbf{i}_0$ be given and assume that there exists a $k \geq 0$ so that:

(4.1) $$\mathbf{D}(P;\mathbf{i}_0) = \mathbf{D}(Z_k;\mathbf{i}_0),$$

(4.2) $$\mathbf{C}(P;\mathbf{i}_0) > \mathbf{C}(Z_k;\mathbf{i}_0).$$

Then $P(\mathbf{i})$ is locally immunized at time k on the yield vector $\mathbf{i}_0$.

proof: As for the proof of Proposition 1, we require the result of Proposition A.4 relating **D** and **C** for $P_k(\mathbf{i})$ to the respective values for $P(\mathbf{i})$ and $Z_k(\mathbf{i})$. In particular, from (A.13) we see that (4.1) assures that:

$$(4.3) \qquad \mathbf{D}(P_k;\mathbf{i}_0) = \mathbf{0},$$

while (4.2) and (4.1) together imply that:

$$(4.4) \qquad \mathbf{C}(P_k;\mathbf{i}_0) > \mathbf{0}.$$

Recalling the comments following (2.17) and (2.18), we see that the above conclusions regarding $P_k(\mathbf{i})$ assure that $\mathbf{i}_0$ is a local minimum, and the result follows. !!

Clearly, the conditions of Proposition 5 are equivalent to assuming that conditions (3.1) and (3.2) of Proposition 1 are satisfied for a fixed k, for all direction vectors N. A similar statement holds for the generalization of Proposition 2. However, regarding condition (3.7), we utilize an alternative yet equivalent representation.

Proposition 6: Let $P(\mathbf{i})$ and $\mathbf{i}_0$ be given and assume that there exists a $k \geq 0$ so that:

$$(4.5) \qquad \mathbf{D}(P;\mathbf{i}_0) = \mathbf{D}(Z_k;\mathbf{i}_0),$$

$$(4.6) \qquad \mathbf{C}(P;\mathbf{i}) - \mathbf{C}(Z_k;\mathbf{i}) > 2\mathbf{D}(Z_k;\mathbf{i})^T [\mathbf{D}(P;\mathbf{i}) - \mathbf{D}(Z_k;\mathbf{i})],$$

for all feasible $\mathbf{i}$. Then $P(\mathbf{i})$ is globally immunized at time k on the yield vector $\mathbf{i}_0$.

proof: Only (4.6) need be investigated, as the implication of

(4.5) follows as before. Using (A.14) of Proposition (A.4), it is clear that $\mathbf{C}(P_k;\mathbf{i})$ is positive definite if and only if:

$$(4.7) \qquad \mathbf{C}(P) - \mathbf{C}(Z_k) > \mathbf{D}(Z_k)^T[\mathbf{D}(Z_k) - \mathbf{D}(P)] + [\mathbf{D}(Z_k) - \mathbf{D}(P)]^T\mathbf{D}(Z_k).$$

However, as a quadratic form, the right hand side of (4.7) is equivalent to the right hand side of (4.6) as a calculation shows. !!

B. Spot and Forward Rate Models

It is clear from Proposition 5, that even ignoring the convexity condition in (4.2), the restriction on the total duration vector $\mathbf{D}(P;\mathbf{i}_0)$ for local immunization is quite strong. For example, using the spot rate model in (3.12), we see that condition (4.1) requires that there is a $k \geq 0$ so that:

$$(4.8) \qquad \mathbf{D}_j(P;\mathbf{i}_0) = \begin{cases} kv_k(1-s) & j = [|k|] \\ kv_k s & j = [|k|] + 1 \\ 0 & \text{otherwise}. \end{cases}$$

Similarly, using a forward rate model, the total duration vector must satisfy the following, where $k = [|k|] + s$:

$$(4.9) \qquad \mathbf{D}_j(P;\mathbf{i}_0) = \begin{cases} v_j & j \leq [|k|] \\ sv_j & j = [|k|] + 1 \\ 0 & j > [|k|] + 1 \end{cases}$$

Consequently, given a portfolio for which local immunization is required at time k on the current yield vector, $\mathbf{i}_0$, a significant amount of trading may be necessary just to satisfy the above conditions. In theory, such a trade may be impossible. If it is possible, it may be highly impractical to implement.

For example, assume that we have m bonds $\{B_j\}$, where m equals the number of yield points on which partial durations are calculated. If we trade a_j bonds, with $a_j > 0$ representing a buy, $a_j < 0$ a sale, the new portfolio value will be:

$$(4.10) \qquad \tilde{P}(\mathbf{i}_0) = P(\mathbf{i}_0) + \Sigma\, a_j B_j(\mathbf{i}_0).$$

Similary, the new partial durations will satisfy:

$$(4.11) \qquad \mathbf{D}_k(\tilde{P};\mathbf{i}_0) = [P(\mathbf{i}_0)\mathbf{D}_k(P) + \Sigma a_j D_k(B_j)]/[P(\mathbf{i}_0) + \Sigma a_j B_j(\mathbf{i}_0)].$$

Setting the formulas obtained in (4.11) equal to the targeted partial durations, denoted c_k, the following system emerges:

$$(4.12) \qquad \Sigma\, a_j[D_k(B_j) - c_k B_j(\mathbf{i}_0)] = P(\mathbf{i}_0)[c_k - D_k(P)], \quad k = 1,\ldots,m.$$

Naturally, the above system may be unsolvable, or produce impractically large trade values. If solvable, there is no assurance that $\Sigma a_j B_j(\mathbf{i}) = 0$, so new funds may need to be invested or divested. Alternatively, a uniform percentage of the new portfolio $\tilde{P}(\mathbf{i}_0)$ could be bought or sold to achieve the original portfolio value $P(\mathbf{i}_0)$. Clearly, the entire trade could be difficult or impossible in practice, even if possible in theory. The convexity condition in (4.2) would then need to be investigated.

An alternative approach would be one in which the durational characteristics of $P(\mathbf{i})$ are chosen sufficiently "close" to those of $Z_k(\mathbf{i}_0)$, with the additional constraints that the resulting trade satisfy $\Sigma a_j B_j(\mathbf{i}_0) = 0$ and $\Sigma|a_j| < K$ or $\Sigma a_i^2 < K$ for some K. By assuming a probability structure for $\mathbf{i}$, a linear programming or Lagrange multiplier problem emerges, with an objective function equal to the range, volatility or some other measure of the resulting $P_k(\mathbf{i})$.

C. Returns on Investment: I_k

The results in Section III.D. above readily generalize to this setting. Again defining $I_k(\mathbf{i})$ as in (3.35), we have the following counterpart to (3.36), which follows from (2.7) and the observation that $\mathbf{i} - \mathbf{i}_0 = t\mathbf{N}$:

$$(4.13)\quad I_k(\mathbf{i}) \approx j(k) + (1 + j(k))[\mathbf{D}(Z_k;\mathbf{i}_0) - \mathbf{D}(P;\mathbf{i}_0)]\cdot(\mathbf{i} - \mathbf{i}_0)/k.$$

Similarly, the second order term in (3.37) can be expressed as follows, using (3.8) and (2.8):

$$(\mathbf{i} - \mathbf{i}_0)^T\{\mathbf{C}(P) - \mathbf{C}(Z_k) + 2\mathbf{D}(Z_k)^T[\mathbf{D}(Z_k) - \mathbf{D}(P)]\}(\mathbf{i} - \mathbf{i}_0)/2k,$$

where all terms are evaluated on $\mathbf{i}_0$.

The comments made above in the directional immunization case regarding the competition between capital gains and losses and reinvestment losses and gains apply here as well. Here, however, the concept of $P(\mathbf{i})$ being "longer" or "shorter" than Z_k refers to the sign of the inner product in (4.13) being negative or positive, respectively.

Corresponding to the moments of $I_k(\mathbf{i})$ in (3.38) and (3.39), we require the following notation. Let $\mathbf{E}(\mathbf{i} - \mathbf{i}_0)$ correspond to the vector mean, and $\mathbf{V}(\mathbf{i} - \mathbf{i}_0)$ correspond to the covariance matrix of $\mathbf{i} - \mathbf{i}_0$, reflecting the underlying density function of $\mathbf{i}$. Then:

(4.14) $$E[I_k(\mathbf{i})] = j(k) + (1 + j(k))[\mathbf{D}(Z_k) - \mathbf{D}(P)]\cdot\mathbf{E}(\mathbf{i} - \mathbf{i}_0),$$

(4.15) $$\mathrm{Var}[I_k(\mathbf{i})] = (1 + j(k))^2[\mathbf{D}(Z_k) - \mathbf{D}(P)]\mathbf{V}(\mathbf{i} - \mathbf{i}_0)[\mathbf{D}(Z_k) - \mathbf{D}(P)]^T/k^2,$$

where all total duration vectors are evaluated on $\mathbf{i}_0$.

V. Yield Vector Transformations

It is natural to inquire to what extent immunization, as developed above, depends on the underlying yield vector basis used. For example, if a portfolio is locally immunized at time k on the yield vector $\mathbf{i}_0$, what can be said if the analysis was to be done using yield basis $\mathbf{j}_0$? A similar question arises for directional immunization. The next proposition shows that the property of local immunization is independent of the yield basis.

Proposition 7: Let $P(\mathbf{i})$ be a price function which satisfies conditions (4.1) and (4.2) of Proposition 5, and hence is locally immunized at time k on the yield vector $\mathbf{i}_0$. Let $\mathbf{A}:\mathbf{i}\rightarrow\mathbf{j}$ be a yield curve transformation, with a nonsingular Jacobian matrix, $\mathbf{J}[\mathbf{A}(\mathbf{i})]$

at $\mathbf{i}_0$. Then $P(\mathbf{j})$ also satisfies those conditions at $\mathbf{j}_0 = \mathbf{A}(\mathbf{i}_0)$, and hence is also locally immunized at time k on the yield vector $\mathbf{j}_0$.

proof: By Proposition A.5, we have:

$$\mathbf{D}(P_k;\mathbf{i}_0) = \mathbf{D}(P_k;\mathbf{j}_0)\mathbf{J}[\mathbf{A}(\mathbf{i}_0)],$$

and hence:

$$(5.1) \qquad [\mathbf{D}(P;\mathbf{i}_0) - \mathbf{D}(Z_k;\mathbf{i}_0)] = [\mathbf{D}(P;\mathbf{j}_0) - \mathbf{D}(Z_k;\mathbf{j}_0)]\ \mathbf{J}[\mathbf{A}(\mathbf{i}_0)].$$

Consequently, since $\mathbf{J}[\mathbf{A}(\mathbf{i}_0]$ is nonsingular, $P(\mathbf{i})$ satisfies (4.1) at $\mathbf{i}_0$ if and only if it satisfies this constraint at $\mathbf{j}_0$.

Similarly, we have:

$$\mathbf{C}(P_k;\mathbf{i}_0) = \mathbf{J}[\mathbf{A}(\mathbf{i}_0)]^T\mathbf{C}(P_k;\mathbf{j}_0)\mathbf{J}[\mathbf{A}(\mathbf{i}_0)] - \mathbf{D}(P_k;\mathbf{j}_0)\mathbf{H}[\mathbf{A}(\mathbf{i}_0)],$$

where $\mathbf{H}[\mathbf{A}(\mathbf{i}_0)]$ is the Hessian matrix of $\mathbf{A}$ at $\mathbf{i}_0$. Substituting for the total convexity matrices using (A.14), and using the fact $\mathbf{D}(P_k;\mathbf{i}_0) = \mathbf{D}(P_k;\mathbf{j}_0) = \mathbf{0}$ by (5.1), we obtain:

$$(5.2) \quad \mathbf{C}(P;\mathbf{i}_0) - \mathbf{C}(Z_k;\mathbf{i}_0) = \mathbf{J}[\mathbf{A}(\mathbf{i}_0)]^T[\mathbf{C}(P;\mathbf{j}_0) - \mathbf{C}(Z_k;\mathbf{j}_0)]\mathbf{J}[\mathbf{A}(\mathbf{i}_0)].$$

Consequently, since $\mathbf{J}[\mathbf{A}(\mathbf{i}_0)]$ is nonsingular, $\mathbf{C}(P)$ satisfies (4.2) at $\mathbf{i}_0$ if and only if it satisfies this constraint at $\mathbf{j}_0$. !!

The implication of Proposition 7 is clear. Namely, that k, the time to which $P(\mathbf{i})$ is immunized, is a coordinate invariate and intrinsic property of the portfolio. It does not depend on the yield curve basis one chooses. As for directional immunization, the situation is of necessity more yield curve dependent, since the direction vector $\mathbf{N}$ clearly reflects the yield curve basis. Transforming $\mathbf{N}$ by the Jacobian of the transformation provides a direction vector $\mathbf{M}$ for which immunization is possible, yet unfortunately not assured without additional constraints as the following result demonstrates.

Propostion 8: Let $P(\mathbf{i})$ be a price function and $\mathbf{N} \neq \mathbf{0}$ a direction vector such that conditions (3.1) and (3.2) of Proposition 1 are satisfied and hence, $P(\mathbf{i})$ is locally immunized at time k in the direction of $\mathbf{N}$ on the yield vector $\mathbf{i}_0$. Let $\mathbf{A}$ be given as above. Then $P(\mathbf{j})$ satisfies condition (3.1) with $\mathbf{M} = \mathbf{J}[\mathbf{A}(\mathbf{i}_0)]\mathbf{N}$ and $\mathbf{j}_0 = \mathbf{A}(\mathbf{i}_0)$. In addition, if $D_{M'}(P;\mathbf{j}_0) \geq D_{M'}(Z_k;\mathbf{j}_0)$, where $\mathbf{M'} = \mathbf{N}^T\mathbf{H}[\mathbf{A}(\mathbf{i}_0)]\mathbf{N}$, then $P(\mathbf{j})$ also satisfies condition (3.2) and hence is also locally immunized at time k in the direction of $\mathbf{M}$ on the yield vector $\mathbf{j}_0$.

proof: Using Corollary A.5, we have:

$$(5.3) \qquad D_N(P_k;\mathbf{i}_0) = D_M(P_k;\mathbf{j}_0),$$

and hence $P(\mathbf{j})$ satisfies condition (3.1) with $\mathbf{M}$ and $\mathbf{j}_0$ if and only if $P(\mathbf{i})$ satisfies this condition with $\mathbf{N}$ and $\mathbf{i}_0$.

Using the corresponding result for directional convexities, and simplifying, we obtain:

$$(5.4) \quad C_N(P\ ;\mathbf{i}_0) - C_N(Z_k;\mathbf{i}_0) = C_M(P\ ;\mathbf{j}_0) - C_M(Z_k;\mathbf{j}_0) - D_{M'}(P_k;\mathbf{j}_0).$$

Consequently, if $P(\mathbf{i})$ satisfies (3.2) with $\mathbf{N}$ and $\mathbf{i}_0$, it does not necessarily follow that $P(\mathbf{j})$ satisfies this condition with $\mathbf{M}$ and

j_0 due to the last term on the right of (5.4). However, if $D_{M'}(P_k;j_0) = D_{M'}(P;j_0) - D_{M'}(Z_k;j_0) \geq 0$, local immunization in the direction of **M** is assured. ‖

Results on global immunization can be treated similarly. Unfortunately, as in Proposition 8, while the duration results carry forward well, the convexity conditions are not preserved without additional constraints. For example, for global immunization, we require $\mathbf{J}[\mathbf{A}(\mathbf{i})]$ to be nonsingular everywhere, and $\mathbf{D}(P_k;\mathbf{j})\mathbf{H}[\mathbf{A}(\mathbf{i})]$ positive definite for all $\mathbf{i}$. Details are left to the interested reader.

VI. Asset/Liability Management

In this section, we translate the above immunization results to an asset/liability management setting. To this end, we consider two objective functions:

(6.1) $$P(\mathbf{i}) = A(\mathbf{i}) - L(\mathbf{i}),$$

(6.2) $$R(\mathbf{i}) = [A(\mathbf{i}) - L(\mathbf{i})]/A(\mathbf{i}),$$

where $A(\mathbf{i})$ and $L(\mathbf{i})$ denote the market values of assets and liabilities, respectively. Immunization in the context of (6.1) then provides a floor for the value of surplus at time k, while use of the objective function in (6.2) provides a floor for the ratio of surplus to assets, or net worth asset ratio.

A. Surplus Immunization

We first investigate local immunization in the direction of $\mathbf{N} \neq \mathbf{0}$. Below, we use the notation r^s to represent the surplus ratio on the current yield vector $\mathbf{i}_0$. That is,

$$r^s = [A(\mathbf{i}_0) - L(\mathbf{i}_0)]/A(\mathbf{i}_0).$$

Proposition 9: Let $P(\mathbf{i}) = A(\mathbf{i}) - L(\mathbf{i})$, $\mathbf{i}_0$ and $\mathbf{N} \neq \mathbf{0}$ be given. Assume that there exists $k \geq 0$ so that:

(6.3) $$D_N(A;\mathbf{i}_0) = (1 - r^s)D_N(L;\mathbf{i}_0) + r^s D_N(Z_k;\mathbf{i}_0),$$

(6.4) $$C_N(A;\mathbf{i}_0) > (1 - r^s)C_N(L;\mathbf{i}_0) + r^s C_N(Z_k;\mathbf{i}_0).$$

Then $P(\mathbf{i})$ is locally immunized at time k in the direction of $\mathbf{N}$ on the yield vector $\mathbf{i}_0$.

proof: Consider first the case where $r^s > 0$. By Proposition 1, we require:

(6.5) $$D_N(P;\mathbf{i}_0) = D_N(Z_k;\mathbf{i}_0).$$

However, by Corollary A.1,

$$D_N(P;\mathbf{i}_0) = D_N(A;\mathbf{i}_0)/r^s - D_N(L;\mathbf{i}_0)(1 - r^s)/r^s,$$

and (6.5) follows from (6.3). A virtually identical argument demonstrates that (6.4) is equivalent to $C_N(P;\mathbf{i}_0) > C_N(Z_k;\mathbf{i}_0)$.

For the case $r^s = 0$, we work directly with the directional derivatives of $P_k(\mathbf{i})$, with the goal that (2.15) and (2.16) be satisfied. The resulting conditions on the directional derivatives of $A(\mathbf{i})$ and $L(\mathbf{i})$ can then be translated to the conditions in (6.3) and (6.14) with $r^s = 0$. !!

Note that for $r^s = 0$, if conditions (6.3) and (6.4) are satisfied, $P(\mathbf{i})$ is locally immunized in the direction of **N** on the yield vector $\mathbf{i}_0$ at all times $k \geq 0$. Consequently, the local immunization boundary given by (3.9) with $\mathbf{i}_t = \mathbf{i}_0$ for all $k \geq 0$. However, since $r_s = 0$, we have that $P_k(\mathbf{i}_0) = 0$ for all k, and hence:

$$IB_N = \{(k,0) \mid k \geq 0\}.$$

For $r^s > 0$, we see that the directional duration of assets required for immunization reflects both the directional durations of liabilities and the discount bond, $Z_k(\mathbf{i})$, corresponding to the immunization horizon k. In some applications, k may be chosen small or equal to zero, providing short term immunization as part of an active management strategy. For $k = 0$, the above inequalities become:

(6.6) $$D_N(A;\mathbf{i}_0) = (1 - r^s)D_N(L;\mathbf{i}_0),$$

(6.7) $$C_N(A;\mathbf{i}_0) > (1 - r^s)C_N(L;\mathbf{i}_0).$$

For **N** = (1,...,1), the parallel shift direction vector, and a flat yield curve, the above conditions are equivalent to those in

Bierwag (1987) which are stated in terms of Macaulay durations and the portfolio "inertias" I_A. This is because in this case:

$$(1 + i)^2 C(A) = I_A + D^M(D^M - 1),$$

and similarly for liabilities. In this special case, we see from (6.3) and (6.4) that immunization at time $k > 0$ requires more asset duration and convexity as k increases, since then $D_N(Z_k;i_0) = kv$ is an increasing function of k, as is $C_N(Z_k;i_0) = k(k + 1)v^2$.

Alternatively, k can be chosen to be consistent with the planning cycle of the organization. For example, $k = 1$ would be an immunization target consistent with stabilizing income over a one period interval, where income is defined as the change in net worth. Similarly, larger values of k can be chosen to reflect a multi-year business plan, or the maturity period of the last liability flow. This last assignment would then be consistent with immunizing pricing margins over the life of a block of liabilities.

For non-directional immunization, the above proposition generalizes in the natural way. We state the result without proof.

Proposition 10: Let $P(\mathbf{i}) = A(\mathbf{i}) - L(\mathbf{i})$ and $\mathbf{i}_0$ be given. Assume that there exists $k \geq 0$ so that:

$$(6.8) \qquad \mathbf{D}(A;\mathbf{i}_0) = (1 - r^S)\mathbf{D}(L;\mathbf{i}_0) + r^S\mathbf{D}(Z_k;\mathbf{i}_0),$$

(6.9) $\mathbf{C}(A;\mathbf{i}_0) > (1 - r^s)\mathbf{C}(L;\mathbf{i}_0) + r^s\mathbf{C}(Z_k;\mathbf{i}_0)$.

Then $P(\mathbf{i})$ is locally immunized at time k on the yield vector $\mathbf{i}_0$. ||

As for Proposition 9, the conclusion of Proposition 10 remains valid when $r^s = 0$. The conditions (6.8) and (6.9) then imply local immunization for all $k \geq 0$.

B. Relative Surplus Immunization

Next, we investigate the immunization of the net worth asset ratio, $R(\mathbf{i}) = [A(\mathbf{i}) - L(\mathbf{i})]/A(\mathbf{i})$. Since $R(\mathbf{i})$ is not really a price function, its forward value at time k, $R_k(\mathbf{i})$, is not given by (2.19). However, we have:

$$R_k(\mathbf{i}) = [A_k(\mathbf{i}) - L_k(\mathbf{i})]/A_k(\mathbf{i})$$

$$= R(\mathbf{i}),$$

since the forward values of $A(\mathbf{i})$ and $L(\mathbf{i})$ satisfy (2.19). Consequently, immunizing $R(\mathbf{i})$ at time 0 ensures its immunization at all times $k \geq 0$.

Proposition 11: Let $R(\mathbf{i})$ be defined as above, and let $\mathbf{i}_0$ and $\mathbf{N} \neq \mathbf{0}$ be given. Assume that:

(6.10) $D_N(A;\mathbf{i}_0) = D_N(L;\mathbf{i}_0)$, (6.11) $$C_N(A;\mathbf{i}_O) > C_N(L;\mathbf{i}_O).$$

Then $R(\mathbf{i})$ is locally immunized at all times $k \geq 0$ in the direction of $\mathbf{N}$ on the yield vector $\mathbf{i}_O$.

proof: Assuming that $R(\mathbf{i}_O) = r^s > 0$, we have from Corollaries A.4 and A.1:

$$D_N(R;\mathbf{i}_O) = D_N(A - L;\mathbf{i}_O) - D_N(A;\mathbf{i}_O)$$

$$= c[D_N(A;\mathbf{i}_O) - D_N(L;\mathbf{i}_O)],$$

where $c = L(\mathbf{i}_O)/S(\mathbf{i}_O)$. Consequently, (2.15) is satisfied due to (6.10). Similarly:

$$C_N(R;\mathbf{i}_O) = c[C_N(A;\mathbf{i}_O) - C_N(L;\mathbf{i}_O)] - 2cD_N(A;\mathbf{i}_O)[D_N(A;\mathbf{i}_O) - D_N(L;\mathbf{i}_O)],$$

and (2.16) is satisfied due to (6.10) and (6.11).

For $r^s = 0$, we proceed as in Proposition 9, working directly with the directional derivatives of $R(\mathbf{i})$. ||

For $\mathbf{N} = (1,\ldots,1)$ and a flat yield vector $\mathbf{i}_O$, the above conditions reduce to those in Bierwag (1987) expressed in terms of Macaulay durations and inertias. Also, for general $\mathbf{N}$, the local immunization boundary in (3.9) is given with $\mathbf{i}_t = \mathbf{i}_O$ for all $k \geq 0$, and hence, $R_k(\mathbf{i}_t) = r^s$. That is,

$$IB_N = \{(k, r^s) \mid k \geq 0\}.$$

Finally, we state without proof the nondirectional immunization result.

Proposition 12: Let $R(\mathbf{i})$ be defined as above, and $\mathbf{i}_0$ be given. Assume that:

$$(6.12) \qquad \mathbf{D}(A;\mathbf{i}_0) = \mathbf{D}(L;\mathbf{i}_0),$$

$$(6.13) \qquad \mathbf{C}(A;\mathbf{i}_0) > \mathbf{C}(L;\mathbf{i}_0).$$

Then $R(\mathbf{i})$ is locally immunized at all times $k \geq 0$ on the yield vector $\mathbf{i}_0$. ||

Proposition A.1: Let $P(\mathbf{i}) = P_1(\mathbf{i}) + P_2(\mathbf{i})$. Then for $P_1(\mathbf{i}), P_2(\mathbf{i}), P(\mathbf{i}) \neq 0$:

$$\mathbf{D}(P) = a_1\mathbf{D}(P_1) + a_2\mathbf{D}(P_2), \quad (A.1)$$

$$\mathbf{C}(P) = a_1\mathbf{C}(P_1) + a_2\mathbf{C}(P_2), \quad (A.2)$$

where $a_j = P_j(\mathbf{i})/P(\mathbf{i})$.

proof: Let d_j denote differentiation with respect to i_j. Then:

$$d_jP = d_jP_1 + d_jP_2,$$

$$d_{jk}P = d_{jk}P_1 + d_{jk}P_2.$$

Dividing by $P(\mathbf{i})$ completes the proof. ||

Corollary A.1: Let $P(\mathbf{i}) = P_1(\mathbf{i}) + P_2(\mathbf{i})$ and $\mathbf{N} \neq \mathbf{0}$ be given. Then for $P_1(\mathbf{i}), P_2(\mathbf{i}), P(\mathbf{i}) \neq 0$:

$$\mathbf{D}_N(P) = a_1\mathbf{D}_N(P_1) + a_2\mathbf{D}_N(P_2), \quad (A.3)$$

$$\mathbf{C}_N(P) = a_1\mathbf{C}_N(P_1) + a_2\mathbf{C}_N(P_2), \quad (A.4)$$

where $a_j = P_j(\mathbf{i})/P(\mathbf{i})$.

proof: Applying (2.7) and (2.8) to Proposition A.1, the result follows. ||

Proposition A.2: Let $P(\mathbf{i}) = P_1(\mathbf{i})P_2(\mathbf{i})$. Then for $P(\mathbf{i}) \neq 0$:

$$\mathbf{D}(P) = \mathbf{D}(P_1) + \mathbf{D}(P_2), \quad (A.5)$$

$$\mathbf{C}(P) = \mathbf{C}(P_1) + \mathbf{C}(P_2) + \mathbf{D}(P_1)^T\mathbf{D}(P_2) + \mathbf{D}(P_2)^T\mathbf{D}(P_1), \quad (A.6)$$

where $\mathbf{D}^T$ is the column matrix transoose of the row matrix **D**.

proof: Let d_j be defined as above, then:

$$d_jP = P_1(d_jP_2) + (d_jP_1)P_2,$$

$$d_{jk}P = (d_{jk}P_1)P_2 + P_1(d_{jk}P_2) + (d_jP_1)(d_kP_2) + (d_jP_2)(d_kP_1).$$

Hence,

$$D_j(P) = D_j(P_1) + D_j(P_2),$$

$$C_{jk}(P) = C_{jk}(P_1) + C_{jk}(P_2) + D_j(P_1)D_k(P_2) + D_j(P_2)D_k(P_1). \quad !!$$

Corollary A.2: Let $P(\mathbf{i}) = P_1(\mathbf{i})P_2(\mathbf{i})$ and $\mathbf{N} \neq \mathbf{0}$ be given. Then for $P(\mathbf{i}) \neq 0$:

(A.7) $$\mathbf{D}_N(P) = \mathbf{D}_N(P_1) + \mathbf{D}_N(P_2),$$

(A.8) $$\mathbf{C}_N(P) = \mathbf{C}_N(P_1) + \mathbf{C}_N(P_2) + 2\mathbf{D}_N(P_1)\mathbf{D}_N(P_2).$$

proof: Applying (2.7) and (2.8) to Proposition A.2, the result follows. !!

Proposition A.3: Let $P(\mathbf{i}) = 1/Q(\mathbf{i})$, $Q(\mathbf{i}) \neq 0$. Then:

(A.9) $$\mathbf{D}(P) = -\mathbf{D}(Q),$$

(A.10) $$\mathbf{C}(P) = -\mathbf{C}(Q) + 2\mathbf{D}(Q)^T\mathbf{D}(Q).$$

proof: As above,

$$d_jP = -d_jQ/Q^2,$$

from which (A.9) follows. Similarly,

$$d_{jk}P = -d_{jk}Q/Q^2 + 2(d_jQ)(d_kQ)/Q^3,$$

from which (A.10) follows. !!

Corollary A.3: Let $P(\mathbf{i}) = 1/Q(\mathbf{i})$, $Q(\mathbf{i}) \neq 0$ and $\mathbf{N} \neq \mathbf{0}$ be given. Then:

$$\text{(A.11)} \qquad D_N(P) = -D_N(Q),$$

$$\text{(A.12)} \qquad C_N(P) = -C_N(Q) + 2D_N^2(Q).$$

proof: Immediate. !!

Proposition A.4: Let $P(\mathbf{i}) = P_1(\mathbf{i})/P_2(\mathbf{i})$, $P_2(\mathbf{i}) \neq 0$. Then for $P(\mathbf{i}) \neq 0$:

$$\text{(A.13)} \qquad \mathbf{D}(P) = \mathbf{D}(P_1) - \mathbf{D}(P_2),$$

$$\text{(A.14)} \qquad \mathbf{C}(P) = \mathbf{C}(P_1) - \mathbf{C}(P_2) + \mathbf{D}(P_2)^T[\mathbf{D}(P_2) - \mathbf{D}(P_1)] + [\mathbf{D}(P_2) - \mathbf{D}(P_1)]^T\mathbf{D}(P_2).$$

proof: Combining Propositions A.2 and A.3;

$$\mathbf{D}(P) = \mathbf{D}(P_1) + \mathbf{D}(1/P_2) = \mathbf{D}(P_1) - \mathbf{D}(P_2),$$

$$\begin{aligned}\mathbf{C}(P) &= \mathbf{C}(P_1) + \mathbf{C}(1/P_2) + \mathbf{D}(P_1)^T\mathbf{D}(1/P_2) + \mathbf{D}(1/P_2)^T\mathbf{D}(P_1) \\ &= \mathbf{C}(P_1) - \mathbf{C}(P_2) + 2\mathbf{D}(P_2)^T\mathbf{D}(P_2) - \mathbf{D}(P_1)^T\mathbf{D}(P_2) - \mathbf{D}(P_2)^T\mathbf{D}(P_1).\end{aligned}$$!!

Corollary A.4: Let $P(\mathbf{i}) = P_1(\mathbf{i})/P_2(\mathbf{i})$, $P_2(\mathbf{i}) \neq 0$ and $\mathbf{N} \neq \mathbf{0}$ be given. Then for $P(\mathbf{i}) \neq 0$:

$$\text{(A.15)} \qquad D_N(P) = D_N(P_1) - D_N(P_2),$$

$$\text{(A.16)} \qquad C_N(P) = C_N(P_1) - C_N(P_2) + 2D_N(P_2)[D_N(P_2) - D_N(P_1)].$$

proof: Immediate. !!

Proposition A.5: Let $\mathbf{A}$: $\mathbf{i} \rightarrow \mathbf{j}$ be a smooth transformation from $\mathbf{R}^m$ to $\mathbf{R}^n$. Let $Q(\mathbf{j})$ be a price function and define $P(\mathbf{i}) = Q(\mathbf{A}\mathbf{i})$. Then:

$$(A.17) \qquad \mathbf{D}(P;\mathbf{i}) = \mathbf{D}(Q;\mathbf{A}\mathbf{i})\mathbf{J}[\mathbf{A}(\mathbf{i})],$$

$$(A.18) \qquad \mathbf{C}(P;\mathbf{i}) = \mathbf{J}[\mathbf{A}(\mathbf{i})]^T\mathbf{C}(Q;\mathbf{A}\mathbf{i})\mathbf{J}[\mathbf{A}(\mathbf{i})] - \mathbf{D}(Q;\mathbf{A}\mathbf{i})\cdot\mathbf{H}[\mathbf{A}(\mathbf{i})],$$

where $\mathbf{J}[\mathbf{A}(\mathbf{i})]_{jk} = \partial A_j/\partial i_k$ is the n x m Jacobian matrix of $\mathbf{A}$, and $\mathbf{H}[\mathbf{A}(\mathbf{i})]_{jk\ell} = \partial^2 A_j/\partial i_k \partial i_\ell$ is the n x m x m Hessian matrix of $\mathbf{A}$.

proof: Applying the chain rule:

$$d_k P(\mathbf{i}) = \sum_j d_j Q(\mathbf{A}\mathbf{i}) d_k A_j(\mathbf{i}) = dQ \cdot d_k \mathbf{A},$$

from which (A.17) follows. Taking second derivatives:

$$d_{\ell k} P(\mathbf{i}) = \sum_j \sum_i d_{ij} Q(\mathbf{A}\mathbf{i}) d_\ell A_i(\mathbf{i}) d_k A_j(\mathbf{i}) + \sum_j d_j Q(\mathbf{A}\mathbf{i}) d_{\ell k} A_j(\mathbf{i})$$

$$= (d_\ell \mathbf{A})^T [d^2 Q] d_k \mathbf{A} + dQ \cdot d_{\ell k} \mathbf{A},$$

from which we obtain (A.18). ||

Corollary A.5: Let $\mathbf{A}$, $Q(\mathbf{j})$ and $P(\mathbf{i})$ be as in Proposition A.5, and $\mathbf{N} \neq \mathbf{0}$ be a given direction vector in $\mathbf{R}^m$. Also, let $\mathbf{M}$ and $\mathbf{M}'$ be defined in $\mathbf{R}^n$ by:

$$(A.19) \qquad \mathbf{M} = \mathbf{J}[\mathbf{A}(\mathbf{i})]\mathbf{N},$$

$$(A.20) \qquad \mathbf{M}' = \mathbf{N}^T\mathbf{H}[\mathbf{A}(\mathbf{i})]\mathbf{N}.$$

Then:

$$(A.21) \qquad D_N(P) = D_M(Q),$$

(A.22) $$C_N(P) = C_M(Q) - D_{M'}(Q).$$

proof: Using (2.7), (A.21) follows immediately from (A.17). Similarly, (2.8) makes the first term on the right of (A.22) clear. For the second term, we have:

$$-\mathbf{N}^T\mathbf{D}(Q;\mathbf{Ai})\mathbf{H}[\mathbf{A}(\mathbf{i})]\mathbf{N}$$

$$= \sum_j d_j Q(\mathbf{Ai}) \sum_{lk} d_{lk}\mathbf{A}_j(\mathbf{i}) n_l n_k \; / \; Q(\mathbf{Ai})$$

$$= dQ \cdot [\mathbf{N}^T\mathbf{H}[\mathbf{A}(\mathbf{i})]\mathbf{N}] \; / \; Q(\mathbf{Ai})$$

$$= -\mathbf{D}(Q) \cdot \mathbf{M'}$$

$$= -D_{M'}(Q). \quad ||$$

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