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MOMENTS OF SURVIVAL FUNCTIONS WITH CERTAIN CLASS OF HAZARD RATES

Hung-ping Tsao

ABSTRACT

The proof of the following result will be presented via geometrical realization of moments.

Main theorem. Let $R(x \mid x > c)$ be the reciprocal of the hazard rate function $\lambda(x \mid x > c)$ of a conditional survival function $S(x \mid x > c)$ of the age-at-death random variable X for any $c \geq 0$. $R(x \mid x > c)$ is linear if and only if the n-th conditional moment can be expressed as

$$E[X^n \mid X > c] = \prod_{i=1}^{n} x_i ,$$

where x_i is the solution of the equation

$$x = iR(x \mid x > c) + c^i \left[\prod_{j=1}^{i-1} x_j\right]^{-1}$$

for $i = 1, 2, 3, \ldots, n$, with $\prod_{j=1}^{0} x_j = 1$.

Geometrical realization of monents of some other classes of survival functions will also be discussed in the hope of shedding some light for estimation of moments of more general class of survival functions.

Let us first look at the case where $c = 0$. For any survival function $S(x)$, we note that

$$E[x^n] = n\int_0^w x^{n-1} S(x)\,dx.$$

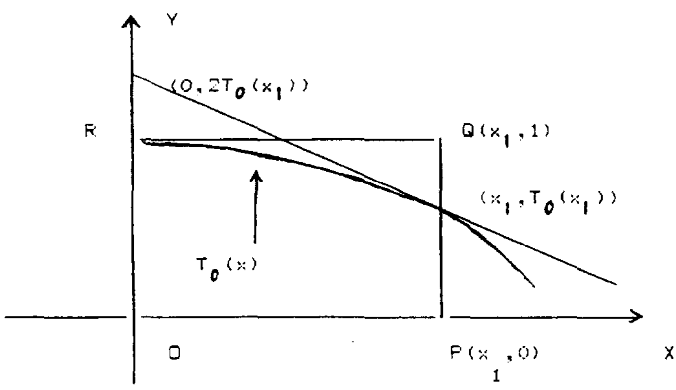
For convenience, we define

$$T_i(x) = x^i S(x)$$

and note that $T_i(w) = 0$ for all $i = 1, 2, 3, \ldots$.

We shall first investigate $\int_0^w T_0(x)\,dx$, where $T_0(x) = S(x)$. To obtain an estimate of the integral, we suggest the following method of approximation.

Find x_1 such that the tangent to the graph of $T_0(x)$ at point $(x_1, T_0(x_1))$ has y-intercept $(0, 2T_0(x_1))$ as shown in the figure.



We shall use the area of rectangle OPQR for approximation of

$$\int_0^w T_0(x)\,dx.$$

Thus

$$E[X] \doteq x_1 . \qquad (1)$$

Since the equation of the tangent in question is

$$y - T_0(x_1) = T_0'(x_1)(x - x_1),$$

we see that x_1 is the solution of the equation

$$T_0(x) = -xT_0'(x),$$

or

$$x = R(x). \qquad (2)$$

Let us look at the uniform distribution and see how good this approximation is. Since

$$\lambda(x) = (w - x)^{-1},$$

the equation (2) is

$$x = w - x$$

and hence $x_1 = \frac{w}{2}$. So the approximation (1) is exact.

For the exponential distribution, we have

$$\lambda(x) = \lambda ,$$

The equation (2) is

$$x = \frac{1}{\lambda}$$

and hence $x_1 = \frac{1}{\lambda}$. The approximation (1) is also exact.

For the survival function

$$S(x) = \sqrt{1 - \frac{x}{w}}$$

we can derive the equation (2) to be

$$x = 2(w - x),$$

which gives $x_1 = \frac{2w}{3}$. Again, the approximation (1) is exact.

Note that R(x) is linear for all the above cases. In fact, we can show the following theorem using the fact that

$$E[X] = R(0) + \int_0^w R'(x)S(x)dx. \qquad (3)$$

Theorem 1. If $R(x) = ax + b$, then E[X] is equal to the solution of the equation

$$x = ax + b,$$

namely

$$E[X] = \frac{b}{1 - a},$$

where $a \leq 0$ and $b > 0$.

Proof. Since S(x) is decreasing, we see that

$$b = R(0) = - \frac{S(0)}{S'(0)} = - \frac{1}{S'(0)} > 0.$$

Since $R(x) > 0$ for all $x < w$, we require that $a \leq 0$.

Since

$$ax + b = R(x) = -\frac{S(x)}{S'(x)},$$

it follows from (3) that

$$E[X] = b + \int_0^w aS(x)\,dx = b + aE[X]$$

and that

$$E[X] = \frac{b}{1 - a}$$

with $a \le 0$.

For the survival function

$$S(x) = (1 + \lambda x)e^{-\lambda x}$$

with $R(x) = \dfrac{1 + \lambda x}{\lambda^2 x}$, the approximation (1) is not exact, since

$$x_1 = \frac{1 + \sqrt{5}}{2\lambda} \text{ while } E[X] = \frac{2}{\lambda}.$$

Now, let us look at $T_1(x)$. Since

$$T_1'(x) = \frac{d[xS(x)]}{dx} = xS'(x) + S(x) = [x - R(x)]S'(x),$$

we see that $x = x_1$ is a critical point. Furthermore,

$$T_1''(x) = xS''(x) + 2S'(x)$$

gives the following

Lemma 1. $T_1(x)$ is concave down wherever $S(x)$ is.

Since

$$T''_1(x) = x \frac{d\left[-\frac{S(x)}{R(x)}\right]}{dx} - \frac{2S(x)}{R(x)} = \frac{x[R'(x)+1]-2R(x)}{[R(x)]^2} S(x),$$

we have

Lemma 2. $T_1(x)$ is concave down if and only if

$$x < \frac{2R(x)}{1+R'(x)} . \qquad (4)$$

Let x_2 denote the solution of the equation

$$x = 2R(x).$$

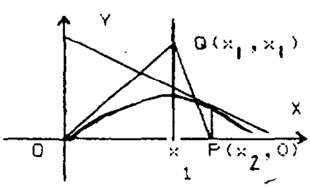
Then $T_1(x)$ attains the global maximum at $x = x_1$ for the interval $[0, x_2]$ provided that $R'(x) < 0$, which is true when $R(x)$ is linear.

Note that the tangent

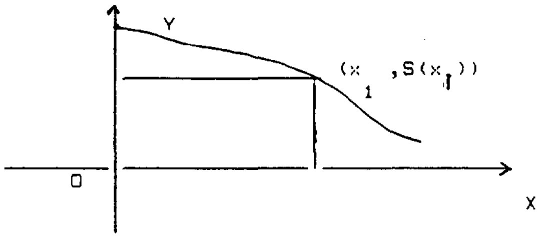
$$y - T_1(x_2) = T'_1(x_2)(x - x_2)$$

to the graph of $T_1(x)$ at point $(x_2, T_1(x_2))$ has y-intercept $(0, 2T_1(x_2))$, since $x_2 = 2R(x_2)$ implies $T_1(x_2) = -x_2 T'_1(x_2)$.

As shown in the figure, we shall use the area of the triangle OPQ for approximation of $\int_0^w T_1(x)dx$. Thus $E[X^2] = x_1 x_2$.



Note that x_1 can also be obtained by finding the maximum rectangle inscribed under the graph of $S(x)$ as shown.



We shall show the following

Theorem 2. If $R(x) = ax + b$, then $E[X^2]$ is equal to the product of the solutions of equations $x = R(x)$ and $x = 2R(x)$, namely

$$E[X^2] = \frac{b}{1-a} \cdot \frac{2b}{1-2b} .$$

Proof. Since

$$E[X^2] = 2\int_0^w T_1(x)\,dx = 2\int_0^w xR(x)S'(x)\,dx$$

$$= 2\int_0^w \frac{d[xR(x)]}{dx} S(x)\,dx$$

$$= 2aE[X^2] + 2bE[X] ,$$

our result follows from Theorem 1.

Now, let us look at the case where $c > 0$. For simplicity, we shall still use $R(x) = ax + b$ in reference to $R(x \mid x > c)$. Since

$$E[X \mid X > c] = \int_c^w xf(x \mid x > c)dx$$

$$= c + \int_c^w S(x \mid x > c)dx$$

$$= c - \int_c^w R(x)S'(x \mid x > c)dx$$

$$= c + R(c) + \int_c^w R'(x)S(x \mid x > c)dx$$

$$= c + ac + b + a\int_c^w S(x \mid x > c)dx$$

$$= c + ac + b + a\{E[X \mid X > c) - c\},$$

we have

$$E[X \mid X > c] = \frac{b + c}{1 - a} .$$

As for geometrical realization, we seek x such that the tangent to the graph of $S(x \mid x > c)$ at point $(x_1, S(x_1 \mid x > c))$ intersects the line $x = c$ at point $(c, 2S(x_1 \mid x > c))$. Since the equation of the tangent in question is

$$y - S(x_1 \mid x > c) = S'(x_1 \mid x > c)(x - x_1),$$

we have $S(x_1 \mid x > c) = S'(x_1 \mid x > c)(c - x_1)$, namely $x_1 = R(x_1) + c$. Hence $x = \frac{b + c}{1 - a}$ is the solution of $x = R(x) + c$, as stated in the main theorem.

Similarly, we can obtain

$$E[X^2 \mid X > c] = \frac{2b(b + c) + (1 - a)c^2}{(1 - 2a)(1 - a)} = x_1 x_2 ,$$

where

$$x_2 = \frac{2b + c^2/x_1}{1 - 2a} = \frac{2b(b + c) + (1 - a)c^2}{(1 - 2a)(b + c)} ,$$

which turns out to be the solution of the equation

$$x = 2R(x) + \frac{c^2}{x_1} ,$$

as stated in the main theorem.

For the third moment, we have

$$E[X^3 \mid X > c] = \frac{3b[2b\cdot(b + c) + (1 - a)c^2] + (1 - 2a)(1 - a)c^3}{(1 - 3a)(1 - 2a)(1 - a)}$$

$$= x_1 x_2 x_3 ,$$

where

$$x_3 = \frac{3b + c^3/x_1 x_2}{1 - 3a} = \frac{3b[2b(b + c) + (1 - a)c^2] + (1 - 2a)(1 - a)c^3}{(1 - 3a)[2b(b + c) + (1 - a)c^2]}$$

is the solution of the equation

$$x = 3R(x) + \frac{c^3}{x_1 x_2} .$$

The main theorem can be proved by Mathematical Induction.

Conversely, we shall only consider the case where $c = 0$.

Since $S(x) = -R(x)S'(x)$, it follows from Maclaurin's expansion of $R(x)$ that

$$\int_0^w S(x)\,dx = R(0) + \frac{R'(0)}{1!}\int_0^w S(x)\,dx + \frac{R''(0)}{2!}\int_0^w xS(x)\,dx + \ldots$$

Then by the expressions of moments given in the main theorem, we have

$$x_1 = R(0) + R'(0)x_1 + \frac{R''(0)}{2}\cdot\frac{x_1x_2}{2} + \frac{R'''(0)}{6}\cdot\frac{x_1x_2x_3}{3} + \ldots \qquad (5)$$

Since $x_1 = R(x_1)$, we have

$$x_1 = R(0) + R'(0)x_1 + \frac{R''(0)}{2}x_1^2 + \frac{R'''(0)}{6}x_1^3 + \ldots \qquad (6)$$

By comparing (5) and (6), we can conclude that

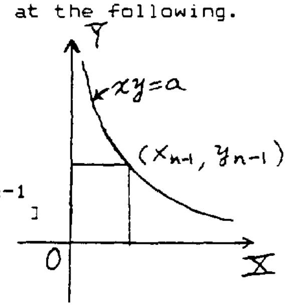
$$x_2 = 2x_1 . \qquad (7)$$

Since $x_2 = 2R(x_2)$, we have

$$x_2 = 2R(0) + 2R'(0)x_2 + 2\cdot\frac{R''(0)}{2}x_2^2 + 2\cdot\frac{R'''(0)}{6}x_2^3 + \ldots \qquad (8)$$

By comparing (6) and (8), we see that $R''(0) = 0$. Otherwise, we would have $x_1^2 = x_2^2$, contradicting (7). Similarly, we can show that all the higher derivatives of $R(x)$ at $x = 0$ are zero. hence $R(x)$ is linear.

Geometrically, the n-th moment of an age-at-death random variable can be approximated by finding the volume of an n-dimensionalrectangular solid with dimensions obtainable iteratively via appropriate tangents. This approximation is not exact if $F(x)$ is non-linear. Let us look at the following.



Case 1. $R(x) = \frac{a}{x} + b$.

We can show, for $n \geq 2$ with $E[X^0] = 1$, that

$$E[X^n] = naE[X^{n-2}] + nbE[X^{n-1}]$$

and $x_n = nR(x_{n-1})$, namely

$$x_n = n\left(\frac{a}{x_{n-1}} + b\right) . \qquad (9)$$

Instead of using a tangent, (9) suggests that we use the hyperbola $xy = a$ to first obtain y_{n-1} from x_{n-1} and then make appropriate shifting and enlargement to arrive at x_n .

Case 2. $R(x) = \frac{1}{ax + b}$.

We can show , for $n \geq 2$, that

$$E[X^{n-2}] = \frac{a}{n} E[X^n] + \frac{b}{n-1} E[X^{n-1}]$$

and

$$x_n = n\left(\frac{1/a}{x_{n-1}} - \frac{1}{n-1} \cdot \frac{b}{a}\right) .$$

We use the hyperbola $xy = 1 / a$ to first obtain y_{n-1} from x_{n-1} and then make appropriate shifting and enlargement to get x_n .

Case 3. $R(x) = ax^2 + bx + c$.

We can show, for $n \geq 2$, that

$$E[X^n] = \frac{1 - (n-1)b}{(n-1)a} E[X^{n-1}] - \frac{c}{a} E[X^{n-2}]$$

and

$$x_n = \frac{-c/a}{x_{n-1}} - [\frac{b}{a} - \frac{1}{(n-1)a}].$$

We use the hyperbola $xy = -c/a$ to first obtain y_{n-1} from x_{n-1} and then make an appropriate shifting to get x_n.

We hope that the above observation may shed some light for more general cases.

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REFERENCES.

London, Dick: Survival Models and Their Estimation, ACTEX Publications, 1988.