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INTEREST, AMORTIZATION AND SIMPLICITY

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Interest, Amortization and Simplicity

Consider simple interest for a moment. Suppose you have an effective interest rate of 12% per year, and you want to bring a quantity forward with interest for three months. Obviously, you multiply by 1.03. Simple interest seems so easy, but what if you have to go backward with interest for three months instead? Do you multiply by 0.97 or divide by 1.03?

Suppose you have a mid-year adjustment. Do you multiply by 1.06, make the adjustment, and then multiply by 1.06 again to get to the end of the year? You will end up with 0.36% too much interest for the year. Perhaps you divide by 1.06 and multiply by 1.12 after the adjustment, in order to end up with exactly 12% annually. Does that mean that the quantity in the first example should have been divided by 1.09 and multiplied by 1.12, instead of just multiplied by 1.03?

Simple interest is not so simple after all. In fact, simple interest is very complicated. It is much more confusing to compute than compound interest, because with compound interest, every step is defined precisely, and no confusion ever arises. Moreover, compound interest describes how interest behaves in the real world much better than simple interest does.

Amortization schedules are analogous to interest schedules in many ways, and a linear amortization schedule has all the problems of simple interest and then some. On the one hand, bases with the same amortization period but different starting dates cannot be combined, so detailed records of prior bases must be maintained from year to year. On the other hand, bases with differing signs can produce peculiar and undesirable effects when aggregated.

Imagine that you have a $50,000 gain one year and a $50,000 loss the next. Suppose you neglect interest, as accountants are wont to do, and amortize each base over ten years. After one year, $5,000 of the gain has been realized, but the unrealized portion of the gain, coupled with the loss, produces an unrealized net loss of $5,000. Because the annual amounts of amortization of the two bases cancel one another, the unrealized net loss of $5,000 remains on the books undisturbed for nine more years, then vanishes in the course of the single final year.

Why should an accountant with an unrealized net loss of $5,000 sit around twiddling his thumbs for nine full years and suddenly realize all of it in the eleventh year? To do so is nothing less than silly. The starting date of an unrealized gain or loss being amortized is of no more relevance than the year printed on a dollar bill that is earning interest in a bank. Each dollar of unrealized gain or loss should be treated equal. The only way to aggregate several gains and losses, and always amortize them smoothly, is to amortize a fixed proportion of the unrealized balance every period, which means using an exponential amortization schedule instead of a linear amortization schedule.

Recall that the force of mortality is the negative of the derivative (with respect to time) of the logarithm of an expected population. By analogy, let the <u>force of amortization</u> $\alpha(t)$ at time t be equal to the negative of the derivative (with respect to time) of the logarithm of the balance $b(t)$ of a base.

The outstanding balance $b(t)$ of a base amortized linearly from time $t = 0$ to time $t = n$ is given by

$$b(t) = \frac{\bar{a}_{\overline{n-t}|}}{\bar{a}_{\overline{n}|}}\, b(0),$$

so the force of amortization is given by

$$\begin{aligned}
\alpha(t) &= -\frac{d}{dt} \log b(t) \\
&= -\frac{d}{dt}\left[\log \bar{a}_{\overline{n-t}|} + \log b(0) - \log \bar{a}_{\overline{n}|}\right] \\
&= -\frac{d}{dt} \log \bar{a}_{\overline{n-t}|} = \frac{v^{n-t}}{\bar{a}_{\overline{n-t}|}} = \frac{1}{\bar{s}_{\overline{n-t}|}} .
\end{aligned}$$

With simple interest, the force of interest varies periodically, but with linear amortization, the force of amortization diverges to infinity at time $t = n$, inasmuch as the denominator in the last expression vanishes at time $t = n$.

Not only does the force of amortization diverge to infinity, but the unweighted mean force of amortization between time $t = 0$ and time $t = n$ is infinite as well and is given by

$$\begin{aligned}
\frac{\int_0^n \alpha(t)\, dt}{n} &= \frac{\int_0^n -\frac{d}{dt} \log \bar{a}_{\overline{n-t}|}\, dt}{n} \\
&= \frac{\log \bar{a}_{\overline{n}|} - \log \bar{a}_{\overline{0}|}}{n} .
\end{aligned}$$

Fortunately, the mean force of amortization weighted by the outstanding balance is finite and is given by

$$\frac{\int_0^n \alpha(t)\, b(t)\, dt}{\int_0^n b(t)\, dt} = \frac{\int_0^n \frac{v^{n-t}}{\bar{a}_{\overline{n}|}}\, b(0)\, dt}{\int_0^n \frac{\bar{a}_{\overline{n-t}|}}{\bar{a}_{\overline{n}|}}\, b(0)\, dt} = \frac{\int_0^n v^{n-t}\, dt}{\int_0^n \bar{a}_{\overline{n-t}|}\, dt}$$

$$= \frac{\bar{a}_{\overline{n}|}}{\bar{D}\bar{a}_{\overline{n}|}} = \frac{\bar{a}_{\overline{n}|}}{\frac{n}{\delta} - \frac{\bar{a}_{\overline{n}|}}{\delta}} = \frac{\left[\frac{1 - \exp(-\delta n)}{\delta}\right]}{\frac{n}{\delta} - \frac{1 - \exp(-\delta n)}{\delta^2}}$$

$$\sim \frac{\left[\frac{1 - (1 - \delta n)}{\delta}\right]}{\frac{n}{\delta} - \frac{1 - \left[1 - \delta n + \frac{(\delta n)^2}{2}\right]}{\delta^2}} = \frac{n}{\left[\frac{n^2}{2}\right]} = \frac{2}{n} .$$

By contrast, if the force of amortization α is constant, then

$$b(t) = \exp(-\alpha t)\, b(0).$$

Thus, a constant force of amortization produces an exponential amortization schedule. Since the effective rate of interest i is given by $1+i = \exp(\delta)$, define the <u>effective rate of amortization</u> to be m such that $1-m = \exp(-\alpha)$. Therefore, the outstanding balance is given by

$$b(t) = (1-m)^t\, b(0)$$

$$= (1-m)\, b(t-1),$$

and the payment made at the end of each year to pay interest and amortize the principal is given by

$$(m+i)\, b(t-1).$$

By holding m fixed, a change of interest rate changes only the yearly payment but not the schedule of unamortized balances, thereby making a change in interest rate easy to implement.

Furthermore, the effective rate of amortization is approximated by

$$m \sim \alpha \sim \frac{2}{n} .$$

Using $m \sim 2/n = 0.2$ to approximate the effective rate of amortization in the $50,000 example above, examine the unrealized net loss under the traditional linear amortization schedule as compared to an exponential amortization schedule, as follows:

Year	Unrealized Net Loss Linear Amortization	Unrealized Net Loss Exponential Amortization
0	$ (50,000)	$ (50,000)
1	5,000	10,000
2	5,000	8,000
3	5,000	6,400
4	5,000	5,120
5	5,000	4,096
6	5,000	3,277
7	5,000	2,621
8	5,000	2,097
9	5,000	1,678
10	5,000	1,342
11	0	1,074
12	0	859
13	0	687
14	0	550
15	0	440

Besides producing a smoother amortization schedule, the exponential amortization schedule allows all the bases, which have the same force of amortization, to be combined and multiplied each year by the simple factor of (1-m). Consequently, like bases occurring at different dates can be combined into a single base, and the record keeping becomes simpler.

Beware that the exponentially amortized bases are never amortized fully. Because like bases can be combined, however, the number of bases is limited. In other words, the exponentially amortized bases do not proliferate like rabbits, despite being immortal.

Exponential amortization is not very useful for mortgages, but it has practical applications for the pension actuary, especially with regard to the actuarial value of assets, funding standard account, and FASB expense and disclosure. It may apply to the insurance business, but I am not familiar with how insurance actuaries use amortization. Of course, there are legal obstacles to overcome.

Compound interest is an improvement upon simple interest that makes the calculation of interest simpler. Likewise, exponential amortization is an improvement upon linear amortization that makes the calculation of amortization simpler. Most important, however, an exponential amortization schedule avoids the peculiar effects produced by the interaction of several bases amortized linearly.