Some Functional Equations in Actuarial Mathematics

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Abstract

This paper discusses Sinzow's functional equation $f(r, s)f(s, t) = f(r, t)$ and Cauchy's functional equation $f(s)f(t) = f(s + t)$ and gives applications to Life Contingencies, Compound Interest, Risk Theory and Finite Differences.

1. A Generalization of Nesbitt's Formula

In [B-H-N] C.J. Nesbitt gave the formula:

$$\frac{\bar{s}_{\overline{n-t}|}}{\bar{s}_{\overline{n}|}} = \exp\left(-\int_0^t \frac{d\tau}{\bar{a}_{\overline{n-\tau}|}}\right). \tag{1.1}$$

Applying the relation

$$\frac{1}{\bar{a}_{\overline{m}|}} = \frac{1}{\bar{s}_{\overline{m}|}} + \delta$$

to the integrand of (1.1), we obtain

$$\frac{\bar{a}_{\overline{n-t}|}}{\bar{a}_{\overline{n}|}} = \exp\left(-\int_0^t \frac{d\tau}{\bar{s}_{\overline{n-\tau}|}}\right). \tag{1.2}$$

Formula (1.2) can be generalized as

$$\frac{\bar{a}_{x+t:\overline{n-t}|}}{\bar{a}_{x:\overline{n}|}} = \exp\left(-\int_0^t [\bar{P}(\bar{A}_{x+\tau:\overline{n-\tau}|}) - \mu_{x+\tau}]d\tau\right), \tag{1.3}$$

which had been given by P. Vasmoen [Va]. The left-hand side of (1.3) is, of course,

$$1 - {}_t\bar{V}(\bar{A}_{x:\overline{n}|}),$$

the *net amount at risk* at duration t of a continuous n-year endowment insurance issued to (x). The following generalizes (1.1):

$$1 - {}_t\overline{W}(\overline{A}_{x:\overline{n}|}) = \frac{1/\overline{P}(\overline{A}_{x+t:\overline{n-t}|})}{1/\overline{P}(\overline{A}_{x:\overline{n}|})} = \exp\left(-\int_0^t \left(\frac{1}{\overline{a}_{x+\tau:\overline{n-\tau}|}} - \frac{\mu_{x+\tau}}{\overline{A}_{x+\tau:\overline{n-\tau}|}}\right) d\tau\right)$$

$$= \exp\left(-\int_0^t \frac{\overline{P}(\overline{A}_{x+\tau:\overline{n-\tau}|}) - \mu_{x+\tau}}{\overline{A}_{x+\tau:\overline{n-\tau}|}} d\tau\right), \tag{1.4}$$

where

$${}_t\overline{W}(\overline{A}_{x:\overline{n}|}) = \frac{{}_t\overline{V}(\overline{A}_{x:\overline{n}|})}{\overline{A}_{x+t:\overline{n-t}|}}$$

is the amount of *paid-up insurance* that can be provided on an n-year continuous endowment insurance policy at duration t by the full net level premium reserve.

It is easy to check that the following relations hold for all positive numbers s and t, $s + t < n$,

$$[1 - {}_s\overline{V}(\overline{A}_{x:\overline{n}|})][1 - {}_t\overline{V}(\overline{A}_{x+s:\overline{n-s}|})] = 1 - {}_{s+t}\overline{V}(\overline{A}_{x:\overline{n}|}) \tag{1.5}$$

and

$$[1 - {}_s\overline{W}(\overline{A}_{x:\overline{n}|})][1 - {}_t\overline{W}(\overline{A}_{x+s:\overline{n-s}|})] = 1 - {}_{s+t}\overline{W}(\overline{A}_{x:\overline{n}|}). \tag{1.6}$$

In this paper, we shall discuss some equations in actuarial mathematics similar to (1.5) and (1.6) .

2. Sinzow's Equations

At the beginning of this century, the Russian mathematician D.M. Sinzow investigated the functional equation

$$f(x, y) + f(y, z) = f(x, z) \qquad \forall\, x, y \text{ and } z. \tag{2.1}$$

Since $f(x, z) - f(y, z)$ is independent of z, the general solution of (2.1) is of the form

$$f(x, y) = g(y) - g(x)$$

[M2, sec. 6.1; A2, p. 223]. (In [A2] Sinzow is spelled as Sincov.) A variant of (2.1) is

$$f(x, y) \cdot f(y, z) = f(x, z), \tag{2.2}$$

which includes equations (1.5) and (1.6) as special cases. Two other examples in *Life Contingencies* are:

$$ {}_{s}p_{x}\, {}_{t}p_{x+s} = {}_{s+t}p_{x} $$

and

$$ {}_{s}E_{x}\, {}_{t}E_{x+s} = {}_{s+t}E_{x}\,. $$

The general solution of (2.2) is of the form

$$ f(x, y) = \frac{h(y)}{h(x)}. \tag{2.3} $$

If h is differentiable, then (2.3) becomes

$$ f(x, y) = \exp\left(\int_{x}^{y} \frac{h'(s)}{h(s)}\, ds\right). \tag{2.4} $$

For the two examples just mentioned, we have

$$ {}_{t}p_{x} = \frac{l_{x+t}}{l_{x}} = \exp\left(-\int_{x}^{x+t} \mu_{s}\, ds\right) $$

and

$$ {}_{t}E_{x} = \frac{D_{x+t}}{D_{x}} = \exp\left(-\int_{x}^{x+t} (\mu_{s} + \delta)\, ds\right). $$

Before we proceed to the next section, we note the similarity between (2.2) and the Chapman-Kolmogorov formula for Markov processes,

$$ \int P(r, v; s, dw) P(s, w; t, A) = P(r, v; t, A)\,. $$

3. Cauchy's Equations

The following definition of *compound interest* has been proposed by Broffitt and Klugman [B-K]: Let $a(t)$ denote the value at time t of an original investment of 1; then interest is said to be compounded at annual rate i if

$$ a(1) = 1 + i \tag{3.1} $$

and

$$ a(s)a(t) = a(s + t) \tag{3.2} $$

for all real s and t.

It has been pointed out by Promislow [P1] that equation (3.2) is a variant of the functional equation

$$f(s) + f(t) = f(s + t), \qquad \forall s, t \in \mathbb{R}. \tag{3.3}$$

Functional equations such as (3.3) and (3.2) had been studied by the eminent French mathematician A.L. Cauchy (1821). Clearly, for each constant c the function

$$f(t) = ct \tag{3.4}$$

is a solution of (3.3). Cauchy showed that, if the function f is continuous, (3.3) has no other solution. In fact, the conclusion that (3.3) has no solutions other than those given by (3.4) holds under any one of the following weaker conditions: (i) f is continuous at a point; (ii) f is bounded above in a neighbourhood of a point; (iii) f is bounded above on a set with positive Lebesgue measure; (iv) there exists a set of positive measure on which f does not take any value between two distinct numbers; (v) f is measurable in a neighbourhood of a point; (vi) f is bounded above on a second category Baire set. On the other hand, it had been shown by G. Hamel and H. Lebesgue by transfinite induction that (3.3) has infinitely many nonmeasurable solutions. For expositions on (3.3), we refer the reader to [A2; A3; Ei; R-V; Saa; Wi]. We note that equations (3.2) and (3.3) have appeared in several of the Part 1 sample examinations released by the Society of Actuaries.

Now consider equation (3.2). Since $a(t) = [a(t/2)]^2$, a(t) is nonnegative for all t. If a(t) vanishes at some point t_0, then $a(t) = a(t - t_0)a(t_0) = 0$ for all t. By Broffitt and Klugman's definition, $a(0) = 1$. Thus, a(t) is strictly positive for all t. (We remark that, if equation (3.2) were assumed to hold just for *nonnegative* t and s, then

$$a(t) = \begin{cases} 1 & t = 0 \\ 0 & t > 0 \end{cases} \tag{3.5}$$

would be a possible solution. However, (3.5) cannot be an accumulation function.)

The definition $f(t) = \ln a(t)$ transforms equation (3.2) into equation (3.3). Obviously, the accumulation function a(t) has to be bounded above on each finite interval; thus by (3.4) each solution of (3.2) is of the form

$$a(t) = e^{ct}. \tag{3.6}$$

Applying equation (3.1) yields

$$a(t) = (1 + i)^t. \tag{3.7}$$

The constant c is δ, the force of interest. We note that formula (3.7) is derived from (3.2) without any continuity assumption.

Sinzow's equation (2.1) becomes Cauchy's equation (3.3) if $f(x, y)$ is a function of $(y - x)$. Hence, Promislow [P2, sec. 2.3] has shown that an accumulation function $a(s, t)$, which is both *Markov*, i.e.,

$$a(r, s)a(s, t) = a(r, t), \qquad r < s < t,$$

and *stationary*, i.e., $a(s, t)$ is a function of $(t - s)$, must be of the form $a(s, t) = (1 + i)^{t-s}$. Other discussions on functional equations and compound interest can be found in [Lo; Pe; Ei, sec. 1.5].

4. Counting Processes

In collective risk theory, the number-of-claims distribution is usually assumed to be Poisson. We now apply the theory of Cauchy's and Sinzow's equations to determine the distributions of certain stochastic processes which are generalizations of the Poisson process.

A stochastic process $\{N(t)\}_{t\geq 0}$ is called a *counting process* if $N(t)$ represents the number of "events" that have occurred in the time interval $(0, t]$. A Poisson process is a counting process that satisfies the following conditions [B-P-P, p. 19; H-C, sec. 3.2; Wo, p. 27] (cf. [Br, p. 120]):

(i) independence of increments,

(ii) stationarity of increments,

(iii) exclusion of multiple events .

We are interested in determining the distributions of counting processes which do not satisfy all of conditions (i), (ii) and (iii).

Let $\{N(t)\}_{t\geq 0}$ be a counting process that satisfies conditions (i) and (ii), but not necessarily condition (iii). Define

$$p_n(t) = \Pr(N(t) = n) .$$

It follows from (i) and (ii) that the following system of functional equations holds:

$$p_n(s + t) = \sum_{k=0}^{n} p_k(s) \cdot p_{n-k}(t) \qquad n = 0, 1, 2, \ldots \tag{4.1}$$

Consider the generating function

$$f(t, z) = \sum_{n=0}^{\infty} p_n(t) z^n . \tag{4.2}$$

By (4.1)

$$f(s+t, z) = f(s, z) \cdot f(t, z).$$

Since f, as a function in its first variable, satisfies Cauchy's equation (3.2), there exists a function h(z) such that $f(t, z) = e^{t\,h(z)}$. In view of (4.2), put

$$h(z) = \lambda_0 + \lambda_1 z + \lambda_2 z^2 + \ldots$$

As $f(t, 1) = 1$, we have

$$\lambda_0 = -\sum_{i=1}^{\infty} \lambda_i .$$

Consequently,

$$f(t, z) = \exp(-t\sum_{1}^{\infty} \lambda_i) \exp(t\sum_{1}^{\infty} \lambda_i z^i). \tag{4.3}$$

Comparing the expansion of (4.3) in powers of z with (4.2), we obtain the formula

$$p_n(t) = e^{-\sum_1^{\infty} \lambda_k t} \sum_{\substack{\gamma_1+2\gamma_2+\ldots+n\gamma_n=n \\ \gamma \geq 0}} \prod_{i=1}^{n} \frac{(\lambda_i t)^{\gamma_i}}{\gamma_i!} . \tag{4.4}$$

Put

$$\lambda = -\lambda_0 = \sum_{j=1}^{\infty} \lambda_j .$$

Then, $p_0(t) = e^{-\lambda t}$, $p_1(t) = e^{-\lambda t} \lambda_1 t$, $p_2(t) = e^{-\lambda t} [(\lambda_1 t)^2/2 + \lambda_2 t]$, etc. Furthermore, since $p_n(t) \geq 0$ for all n, it can be shown that $\lambda_i \geq 0$, $i = 1, 2, 3, \ldots$.

Jánossy, Rényi and Aczél [J-R-A] proved formula (4.4) by mathematical induction. Their proof is repeated in the books [Sax] and [A2]. Formula (4.4) has been given in [Lū]. In [Th], λ_i is denoted by Q_i and λ is assumed to be 1.

If we write

$$\pi_\gamma^i(t) = e^{-\lambda_i t} \frac{(\lambda_i t)^\gamma}{\gamma!} ,$$

formula (4.4) becomes

$$p_n(t) = \sum_{\substack{\gamma_1+2\gamma_2+\ldots+n\gamma_n=n \\ \gamma_i \geq 0}} \prod_{i=1}^{n} \pi^i_{\gamma_i}(t) \prod_{i=n+1}^{\infty} \pi^i_0(t) . \tag{4.5}$$

Formula (4.5) implies the decomposition

$$N(t) = \sum_{j=1}^{\infty} jN_j(t) , \tag{4.6}$$

where $N_j(t)$ is a Poisson process with parameter λ_j. It can be proved that the Poisson processes $\{N_j(t)\}$ are independent; see [P-R] and [B-G-H-J-N, Theorem 11.2]. This result is attributed to A.N. Kolmogorov [J-R-A, p. 211]. We remark that Feller [Fe, section XII.2] has shown that a distribution concentrated on the nonnegative integers is compound Poisson if and only if it is infinitely divisible; an elegant proof of this fact by means of recursive formulas has been given by Ospina and Gerber [O-G].

If the counting process $\{N(t)\}_{t\geq 0}$ also satisfies condition (iii), i.e., it is a Poisson process, then $\lambda_2 = \lambda_3 = \lambda_4 = \ldots = 0$ and (4.4) reduces to the usual formula

$$p_n(t) = e^{-\lambda t} \frac{(\lambda t)^n}{n!} . \tag{4.7}$$

On the other hand, if a counting process satisfies conditions (i) and (iii) but not the stationarity condition (ii), we can use the technique of *operational time* to obtain a formula similar to (4.7) [B-P-P, p. 353]. However, for a counting process satisfying only condition (i), the method of operational time is not applicable; we now derive formulas which generalize (4.4) and (4.5).

Let $\{N(t)\}_{t\geq 0}$ be a counting process which satisfies condition (i), but not necessarily conditions (ii) and (iii). For $s < t$, $N(t) - N(s)$ equals the number of events that have occurred in the time interval $(s, t]$. Define

$$p_n(s, t) = \Pr(N(t) - N(s) = n) .$$

Then, for $r < s < t$ and $n = 0, 1, 2, \ldots$,

$$p_n(r, t) = \sum_{k=0}^{n} p_k(r, s) p_{n-k}(s, t) . \tag{4.1'}$$

Define the generating function

$$g(s, t, z) = \sum_{n=0}^{\infty} p_n(s, t) z^n. \tag{4.2'}$$

For each fixed z, the function g, as a function in its first two variables, satisfies Sinzow's functional equation,

$$g(r, t, z) = g(r, s, z) g(s, t, z), \qquad r < s < t.$$

Hence, there exists a function k(x, z) such that

$$g(s, t, z) = \exp\left(\int_s^t k(x, z)dx\right).$$

Let

$$k(x, z) = \sum_{j=0}^{\infty} \lambda_j(x) z^j$$

and

$$\int_s^t \lambda_j(x)\,dx = \Lambda_j(t) - \Lambda_j(s).$$

It follows from g(s, t, 1) = 1 that

$$p_n(s, t) = \exp\left[-\sum_1^{\infty} \Lambda_j(t) - \Lambda_j(s)\right] \sum_{\substack{\gamma_1 + 2\gamma_2 + \ldots + n\gamma_n = n \\ \gamma_j \geq 0}} \prod_{k=1}^{n} \frac{[\Lambda_k(t) - \Lambda_k(s)]^{\gamma_k}}{\gamma_k!}. \tag{4.4'}$$

With the definition

$$\pi_\gamma^i(s, t) = \exp[\Lambda_i(s) - \Lambda_i(t)] \frac{[\Lambda_i(t) - \Lambda_i(s)]^\gamma}{\gamma!},$$

formula (4.4') becomes

$$p_n(s, t) = \sum_{\substack{\gamma_1 + 2\gamma_2 + \ldots + n\gamma_n = n \\ \gamma_j \geq 0}} \prod_{i=1}^{n} \pi_{\gamma_i}^i(s, t) \prod_{i=n+1}^{\infty} \pi_0^i(s, t). \tag{4.5'}$$

Rényi [Ré] had derived (4.4') with an additional "rarity" assumption. Aczél [A1; A2, sec. 5.1.2] recognized that Rényi's "rarity" condition is not necessary and proved (4.4') by mathematical induction.

We conclude this section with a remark on stochastic processes (not necessarily counting processes) which have independent and stationary increments. Let $\{X(t)\}$ be such a stochastic process with $X(0) = 0$. For $0 < s < t$,

$$\begin{aligned}
\phi(t, \theta) &= E\{\exp[i\theta X(t)]\} \\
&= E\{\exp[i\theta(X(t) - X(s) + X(s))]\} \\
&= E\{\exp[i\theta(X(t) - X(s))]\}\, E\{\exp[i\theta X(s)]\} \\
&= E\{\exp[i\theta(X(t-s))]\}\, E\{\exp[i\theta X(s)]\} \\
&= \phi(t-s, \theta)\, \phi(s, \theta)\,.
\end{aligned}$$

Since ϕ, as a function in its first variable, satisfies Cauchy's equation (3.2), we have

$$\phi(t, \theta) = [\phi(1, \theta)]^t$$

under mild assumptions. The branch of the multivalued function $\log[\phi(1, \theta)]$ with $\log[\phi(1, 0)] = 0$ is called the *cumulant generating function per unit time* and had been shown by Kolmogorov [Cr, chapter 8; Se] to be of the form

$$i\mu\theta - \frac{\sigma^2}{2}\theta^2 + \int_{-\infty}^{\infty} \frac{e^{i\theta x} - 1 - i\theta x}{x^2}\, dK(x)\,, \tag{4.8}$$

where $K(x)$ is a bounded and nondecreasing function continuous at $x = 0$. Formula (4.8) can be expressed in another way, known as the Lévy-Khintchine canonical form. Cf. [Ta, Appendix 3].

5. Interpolation Theory

Recall the translation operator E in *Finite Differences,*

$$E^t f(x) = f(x+t)\,.$$

It is easy to see that the following operator equation holds:

$$E^s E^t = E^{s+t}. \tag{5.1}$$

As equation (5.1) is similar to (3.2), one may wonder if there is an equation analogous to (3.6). If D denotes the differentiation operator, the Taylor expansion formula may be written symbolically as

$$E^t = \sum_{j=0}^{\infty} \frac{t^j}{j!} D^j\,. \tag{5.2}$$

Hence, we have [Fr, p. 126]

$$E^t = e^{tD}. \tag{5.2'}$$

For a rigorous functional analytic treatment of the operator equation (5.2'), see [H-P, chapter 19] or [Yo, sec. IX.5] .

We remark that formula (5.2') can be generalized to higher dimensions. For $f: \mathbf{R}^n \to \mathbf{R}$, define

$$E^{\mathbf{t}} f(\mathbf{x}) = f(\mathbf{x} + \mathbf{t}),$$

where $\mathbf{t} = (t_1, \dots, t_n)^T$ and $\mathbf{x} = (x_1, \dots, x_n)^T$. Then [Fr, p. 141; M1, p. 109]

$$E^{\mathbf{t}} = e^{t_1 \frac{\partial}{\partial x_1} + \dots + t_n \frac{\partial}{\partial x_n}}$$

$$= e^{\mathbf{t} \cdot \frac{\partial}{\partial \mathbf{x}}},$$

where $\frac{\partial}{\partial \mathbf{x}}$ is the gradient operator $(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})^T$.

Recall the Newton forward-difference and backward-difference formulas:

$$E^t = (1 + \Delta)^t = \sum_{j=0}^{\infty} \binom{t}{j} \Delta^j \tag{5.3}$$

and

$$E^t = (1 - \nabla)^{-t} = \sum_{j=0}^{\infty} \binom{-t}{j} (-\nabla)^j = \sum_{j=0}^{\infty} \binom{t+j-1}{j} \nabla^j. \tag{5.4}$$

For $j = 0, 1, 2, \dots$, define [S1, p. 8]

$$t^{[j]} = t(t + \frac{j}{2} - 1)(t + \frac{j}{2} - 2) \dots (t - \frac{j}{2} + 1).$$

Then,

$$E^t = \sum_{j=0}^{\infty} \frac{t^{[j]}}{j!} \delta^j, \tag{5.5}$$

where δ is the central-difference operator. In examining formulas (5.2), (5.3), (5.4) and (5.5), one would feel that there should be a general theory behind these expansions.

Consider the linear space of polynomials. A linear operator on this space is called *translation invariant* if it commutes with the translation operator E. Each translation-invariant operator can be expressed as a power series in D [Ga, Theorem 1.1]. A translation-invariant operator of the form

$$Q = \sum_{k=1}^{\infty} c_k D^k, \qquad c_1 \neq 0, \tag{5.6}$$

is called a *delta operator* [M-R, p. 180]. Obvious examples of delta operators are D, Δ (= $e^D - I$), ∇ (= $I - e^{-D}$) and δ (= $e^{D/2} - e^{-D/2}$). We note that, since $c_1 \neq 0$, it follows from the Lagrange inversion formula (1768) that we can invert (5.6) and express the differentiation operator D as a power series in Q [Ga, Lemma 1.1; Kn, p. 508].

J.F. Steffensen [S3], the late Professor of Actuarial Science at the University of Copenhagen, observed that, for each delta operator Q, there is a unique sequence of polynomials $q_0(t) \equiv 1, q_1(t), q_2(t), \dots$ such that $q_j(t)$ is a polynomial of degree j,

$$0 = q_1(0) = q_2(0) = \dots$$

and, for j = 1, 2, 3, ... ,

$$Qq_j(t) = jq_{j-1}(t). \tag{5.7}$$

Steffensen called these polynomials *poweroids.* Since the sequence of polynomials $\{q_j(t)\}$ forms a basis for the linear space of all polynomials and

$$Q^k q_j(t)\Big|_{t=0} = \begin{cases} j! & k = j \\ 0 & k \neq j \end{cases},$$

we have the following generalization of the Taylor expansion formula:

$$E^t = \sum_{j=0}^{\infty} \frac{q_j(t)}{j!} Q^j. \tag{5.8}$$

It follows from (5.1) and (5.8) that, for j = 0, 1, 2, ... and s, t $\in$ **R**,

$$\frac{q_j(s+t)}{j!} = \sum_{k=0}^{j} \frac{q_k(s)}{k!} \frac{q_{j-k}(t)}{(j-k)!}$$

or

$$q_j(s+t) = \sum_{k=0}^{j} \binom{j}{k} q_k(s) q_{j-k}(t). \tag{5.9}$$

Note the similarity between the systems of functional equations (5.9) and (4.1). In the special case where

$$q_j(t) = t(t-1)(t-2) \dots (t-j+1),$$

(5.9) is known as Vandermonde's Theorem.

Formula (5.9) generalizes the binomial theorem and was first observed by Steffensen [S3, sec. 13]. Three decades later, R. Mullin and G.-C. Rota [M-R, Theorem 1b] showed that the converse is also true: A sequence of polynomials, one for each *degree and satisfying the binomial identity (5.9) for all j, s and t, must be the poweroids of* a delta operator.

Remarks. (1) Recall the *Z-method* in Life Contingencies [Jo, sec. 10.3], where subscripts are treated as exponents. Similarly, equation (5.9) can be written *umbrally* as

$$q(s+t)^j = [q(s)+q(t)]^j .$$

For a recent exposition on the umbral calculus, see the book [Ro].

(2) Consider the delta operator

$$Q = E^a \frac{E^b - I}{b} = \frac{1}{b} E^a \Delta_b ;$$

it can easily be shown [S2; S3, p. 346] that its poweroids are $q_0(t) \equiv 1$ and, for $j = 1, 2, \ldots$,

$$\begin{aligned} q_j(t) &= t\,E^{-ja}(t-b)(t-2b)\cdots(t-(j-1)b) \\ &= t(t-ja-b)(t-ja-2b)\cdots(t-ja-(j-1)b). \end{aligned}$$

This formulation includes as special cases the delta operators Δ, ∇, δ and D ($b \to 0$).

We now conclude this paper with a derivation of some classical formulas in interpolation theory. Since $E^t = e^{tD}$, differentiating (5.8) with respect to D yields [S3, (32)]

$$E^t = \sum_{j=0}^{\infty} \frac{q_{j+1}(t)}{t\,j!} Q^j \frac{dQ}{dD}. \tag{5.10}$$

Now, consider $Q = \delta$ and $q_j(t) = t^{[j]}$. Writing $t^{[j]}/t$ as $t^{[j]-1}$ and noting that

$$\frac{d\delta}{dD} = \frac{d}{dD}(E^{1/2} - E^{-1/2}) = \frac{E^{1/2} + E^{-1/2}}{2} = \mu$$

(which is the *averaging operator*), we have [S3, (147)]

$$E^t = \sum_{j=0}^{\infty} \frac{t^{[j+1]-1}}{j!} \mu\delta^j. \tag{5.11}$$

Comparing (5.8) and (5.10), one may conjecture the formula [S3, (15); M-R, Theorem 4.4]:

$$\left(\frac{dQ}{dD}\right)^{-1} q_j(t) = \frac{q_{j+1}(t)}{t}.$$

Also, putting $T = dQ/dD$, we can rewrite (5.10) as

$$E^t = \sum_{j=0}^{\infty} \frac{T^{-1} q_j(t)}{j!} TQ^j, \tag{5.12}$$

which is in fact valid for all invertible translation-invariant operators T [R-K-O, section 5].

Formulas (5.5) and (5.11) are not useful for interpolation as they require values at −0.5, 0.5, 1.5, ... etc., in addition to values at integral points. However, by splitting these two formulas into their odd and even components, we can easily derive the classical central-difference interpolation formulas. Indeed, the Gauss forward and backward, Everett and Steffensen formulas are immediate consequence of the odd-even decomposition of (5.11).

For $j = 1, 2, 3, \dots$,

$$t^{[2j]} = \prod_{k=0}^{j-1} (t^2 - k^2)$$

and

$$t^{[2j+1]} = t \prod_{k=0}^{j-1} [t^2 - (k + \tfrac{1}{2})^2] ;$$

hence $\{t^{[2j]}\}$ and $\{t^{[2j+1]-1}\}$ are even functions and $\{t^{[2j-1]}\}$ and $\{t^{[2j]-1}\}$ are odd functions. Therefore,

$$\frac{E^t + E^{-t}}{2} = \sum_{j=0}^{\infty} \frac{t^{[2j]}}{2j!} \delta^{2j} = \sum_{j=0}^{\infty} \frac{t^{[2j+1]-1}}{2j!} \mu\delta^{2j} \tag{5.13}$$

and

$$\frac{E^t - E^{-t}}{2} = \sum_{j=1}^{\infty} \frac{t^{[2j-1]}}{(2j-1)!} \delta^{2j-1} = \sum_{j=1}^{\infty} \frac{t^{[2j]-1}}{(2j-1)!} \mu\delta^{2j-1}. \tag{5.14}$$

(Observing that $\delta = e^{D/2} - e^{-D/2}$ is an odd function in D, we see that formulas (5.13) and (5.14) can be generalized. Consider $Q = g(D)$; then

$$e^{tg^{-1}(Q)} = e^{tD} = E^t = \sum_{j=0}^{\infty} \frac{q_j(t)}{j!} Q^j.$$

If g is an odd function, its inverse g^{-1} is also an odd function; hence $\{q_{2j}(t)\}$ and $\{q_{2j+1}(t)/t\}$ are even polynomials, and $\{q_{2j-1}(t)\}$ and $\{q_{2j}(t)/t\}$ are odd polynomials.)

Combining (5.13) and (5.14), we obtain the Stirling and Bessel interpolation formulas:

$$E^t = \sum_{j=0}^{\infty} \frac{t^{[2j]}}{2j!}\delta^{2j} + \frac{t^{[2j+2]-1}}{(2j+1)!}\mu\delta^{2j+1},$$

$$E^t = \sum_{j=0}^{\infty} \frac{t^{[2j+1]-1}}{2j!}\mu\delta^{2j} + \frac{t^{[2j+1]}}{(2j+1)!}\delta^{2j+1}.$$

To derive the Everett formula, note that

$$E^t = \frac{E - E^{-1}}{E - E^{-1}}E^t = \frac{E^t - E^{-t}}{E - E^{-1}}E + \frac{E^{1-t} - E^{-(1-t)}}{E - E^{-1}}$$

and

$$\frac{E^x - E^{-x}}{E - E^{-1}} = \frac{E^x - E^{-x}}{2\mu\delta} = \sum_{j=1}^{\infty} \frac{x^{[2j]-1}}{(2j-1)!}\delta^{2j-2} = \sum_{k=0}^{\infty} \binom{x+k}{2k+1}\delta^{2k}.$$

To obtain the Steffensen formula, consider

$$E^t = \frac{(E^{t+1/2} + E^{-(t+1/2)})E^{1/2} - (E^{t-1/2} + E^{-(t-1/2)})E^{-1/2}}{E - E^{-1}}.$$

Using the formula

$$\frac{E^x + E^{-x}}{2} = \mu + \sum_{j=1}^{\infty} \frac{x^{[2j+1]-1}}{2j!}\mu\delta^{2j},$$

we get

$$E^t = I + \sum_{j=1}^{\infty}\left\{\binom{j+t}{2j}E^{1/2} - \binom{j-t}{2j}E^{-1/2}\right\}\delta^{2j-1}.$$

To derive the Gauss forward and backward formulas we employ the following odd-even decompositions:

$$E^t = \frac{E^t - E^{-t}}{E^{1/2} + E^{-1/2}}E^{1/2} + \frac{E^{t-1/2} + E^{-(t-1/2)}}{E^{1/2} + E^{-1/2}}$$

and

$$E^t = \frac{E^t - E^{-t}}{E^{1/2} + E^{-1/2}}E^{-1/2} + \frac{E^{t+1/2} + E^{-(t+1/2)}}{E^{1/2} + E^{-1/2}}.$$

The odd component of (5.5),

$$\frac{E^{t}-E^{-t}}{2} = \sum_{j=0}^{\infty}\frac{t^{[2j+1]}}{(2j+1)!}\delta^{2j+1},$$

can be used to give a simple derivation of the "summation n" formula and King's pivotal-value formula in Graduation Theory. The operator "summation n" is defined by

$$[n] = E^{(n-1)/2} + E^{(n-3)/2} + \cdots + E^{-(n-1)/2} = \frac{E^{n/2}-E^{-n/2}}{E^{1/2}-E^{-1/2}}.$$

Thus,

$$[n] = \frac{E^{n/2}-E^{-n/2}}{\delta} = 2\sum_{j=0}^{\infty}\frac{(n/2)^{[2j+1]}}{(2j+1)!}\delta^{2j} = n + \frac{n(n^2-1)}{24}\delta^2 + \frac{n(n^2-1)(n^2-3^2)}{1920}\delta^4 + \ldots \qquad (5.15)$$

King's formula is obtained by inverting the "summation n" operator:

$$\begin{aligned} [n]^{-1} &= \frac{E^{1/2}-E^{-1/2}}{E^{n/2}-E^{-n/2}} = \frac{E_n^{1/2n}-E_n^{-1/2n}}{\delta_n} \\ &= 2\sum_{j=0}^{\infty}\frac{(1/2n)^{[2j+1]}}{(2j+1)!}\delta_n^{2j} = \frac{1}{n}\sum_{j=0}^{\infty}\frac{(1/2n)^{[2j+1]-1}}{(2j+1)!}\delta_n^{2j} \\ &= \frac{1}{n} - \frac{n^2-1}{24n^3}\delta_n^2 + \frac{(n^2-1)[(3n)^2-1]}{1920n^5}\delta_n^4 - \cdots \end{aligned} \qquad (5.16)$$

It had been observed by G.J. Lidstone [Li] that the coefficients in (5.16) can be obtained from those in (5.15) by substituting $1/n$ for n. Hence, we have the "formula"

$$[n]^{-1} = [n^{-1}].$$

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