RENEWAL PROCESSES GENERATED BY DISTRIBUTIONS

WITH PERIODIC FAILURE RATES

by

Stefanka   Chukova,   Concordia   University   and   Technical
University of Sofia

Boyan Dimitrov[1], Concordia University and Sofia University

José Garrido[2], Concordia University, Montreal

Abstract

Two renewal processes, known in reliability maintenance as
minimal repair and replacement policy, are introduced. Their
properties are studied in the case where the generating random
sequence has a distribution with periodic failure rate. A
characterization theorem establishes necessary and sufficient
conditions for a non-stationary Poisson process to have a periodic
failure rate. Applications in risk theory are shown.

Key words: Non-stationary Poisson process, characterization
theorem, failure rate function, risk theory.

1. Introduction.

The recent works of Chukova and Dimitrov (1991, 1992) and
Dimitrov et al. (1992) introduced random variables (r.v.'s) having
the so called "almost lack of memory" (ALM) property and studied
their physical interpretations and their analytical and
probabilistic properties. For non-negative r.v.'s X these results
can be summarized as follows:

Theorem 1. These statements are all equivalent:

(i) The distribution function of X has the form

$$F_X(x) = 1 - \alpha^{[x/c]} \{1-(1-\alpha)F_Y(x-[x/c]c)\}, \quad x\geq 0, \qquad (1)$$

where $c>0$ and $\alpha \in (0,1)$ are given constants and F_Y is an arbitrary c.d.f. for a r.v. Y with support on the interval $[0,c)$;

(ii) For a given $c>0$ and all $x\geq 0$ the following relation holds:

$$P(X-c\geq x \mid X\geq c) = P(X\geq x) \qquad (2)$$

We say that X has the lack of memory (LM) property at the point c;

(iii) For a given c, any positive integer n and any $x\geq 0$:

$$P(X-nc\geq x \mid X\geq nc) = P(X\geq x) \qquad (3)$$

This property is called "almost lack of memory", since there is a sequence of infinitely many different points, namely nc, $(n=0,1,2,...)$ at which X has the LM property;

(iv) The r.v. X is representable as the sum

$$X = Y + cZ \qquad (4)$$

of two independent r.v.'s: Y with distribution F_Y on $[0,c)$, and Z with geometric distribution $P(Z=k) = \alpha^k(1-\alpha)$, $k=0,1,...$, $\alpha\in(0,1)$;

(v) The Laplace – Stieltjes transform $\varphi_X(s) = E[e^{-sX}]$ of X satisfies the following equation for any positive integer n:

$$\varphi_X(s) = \int_0^{nc} e^{-sx}dF_Y(x)/(1-e^{-ncs}\{1-F_X(nc)\}), \quad s\geq 0 \quad ;$$

(vi) The failure rate function (when it exists)

$$\lambda_X(t) = \frac{f_X(t)}{1 - F_X(t)} \qquad (5)$$

is periodic in t with period $c>0$; i.e. for any integer $k>0$ we have

$$\lambda_X(kc+t) = \lambda_X(t) \quad \text{for all } t\geq 0. \qquad (6)$$

The proofs of (i)-(iv) can be found in Dimitrov et al. (1992), the proof of (v) is given in Chukova and Dimitrov (1992) while that of (vi) is given in Chukova and Dimitrov (1991). As mentioned

before, this class of distributions appears to be a generalization of the well known exponential and geometric distributions. These have the LM property at every given point $c>0$, and therefore are a part of the class of distributions with the ALM property.

A natural question arises: What is the corresponding generalization of the Poisson process, which would not be generated by a renewal process with exponential distribution, but rather with ALM distributions? If such a generalization exists what physical and probabilistic properties does it have and is it characterized by such properties?

In this article we consider two definitions of generalized Poisson processes generated by a sequence of i.i.d. r.v.'s $\langle X_n \rangle$ with ALM distributions, and establish some of their properties.

2. Two types of renewal processes related to the same distribution.

We introduce two renewal processes related with a r.v. $X \geq 0$. These are based on considerations from reliability theory, specially from the theory of the technical maintenance of items. Consider renewal processes generated as follows:

1) The process $N_t^{(1)}$ is a process of minimal repair actions. It means that if an item fails at a given moment t, then the failure is counted and an operating item *of the same age t* is immediately put in operation, in replacement of the failed one. This mechanism of generation of failures leads to the consideration of a non-stationary Poisson process determined by its hazard function

$$\Lambda_X(t) = \int_0^t \lambda_X(u)du = -\ln[1-F_X(t)]. \qquad (7)$$

Many authors, as Beichelt (1981,1991) and Block et al.(1985), have proved this almost evident fact expressed by (7); here we view

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it as an immediate consequence of (5). The main problem is that if X is a bounded r.v., then the process explodes in a finite time horizon. But when X is unbounded, i.e. when for any $t>0$ the probability $P(X>t)$ is positive, then the non-stationary Poisson process defined above (denoted briefly by NPP1) does not explode. We denote by $N_t^{(1)}$ the number of failures until time t of a new item put in operation at time $t_o=0$, i.e. $N_t^{(1)}$ is the number of events of the above NPP1 on the interval $[0,t)$, $t>0$. It is known, see Baxter (1982), that

$$P(N_t^{(1)}= k) = \frac{[\Lambda(t)]^k}{k!} e^{-\Lambda(t)}, \qquad (8)$$

where $\Lambda(t) = \Lambda_X(t)$ is defined by (7);

2) The process $N_t^{(2)}$ is generated by the replacement of failed items by new ones. For any $t\geq0$ the r.v. $N_t^{(2)}$ denotes the number of failures (replacements) of the operating items with life times X_n, $n=1,2,\ldots$ on the interval $[0,t)$, i.e.

$$N_t^{(2)} = max (n: X_1+X_2+\ldots+X_n < t). \qquad (9)$$

Here $\langle X_n \rangle$ is a sequence of i.i.d. r.v.'s, each with the same distribution as X.

The two processes $N_t^{(1)}$ and $N_t^{(2)}$ may be considered as potential candidates to extend the homogeneous Poisson process to the case where the r.v. X, used to generate both processes, has an ALM distribution. In the next section we see how the two processes characterize the distribution of X.

3. Characterization of the exponential distribution.

It is well known from renewal theory that

$$P(N_t^{(2)} = k) = F_X^{(k)}(t) - F_X^{(k+1)}(t) , k = 0,1,2,\ldots \qquad (10)$$

where $F_X^{(0)}(t) = I_{[0,\infty)}(t)$, and $F_X^{(k)}$ is the k-fold convolution of

the c.d.f. F_X with itself $(I_{[0,\infty}(.)$ is the indicator function of the set $[0,\infty))$.

Theorem 2. If the two processes $N_t^{(1)}$ and $N_t^{(2)}$ are generated by the same r.v. X, with c.d.f. F_X, the equality

$$P(N_t^{(1)} = 1) = P(N_t^{(2)} = 1) \qquad (11)$$

is true for any $t>0$ if and only if X is exponential, i.e. when

$$F_X(t) = 1 - exp(-\lambda t). \qquad (12)$$

Proof. We first prove that it is a necessary condition. Let $F_X(t) = 1 - exp(-\lambda t)$. Then the renewal process $< N_t^{(2)}, t>0 >$ is a time-homogeneous Poisson process and with parameter λ, i.e.

$$P(N_t^{(2)} = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad t \geq 0, \text{ for any } k = 0,1,2,\ldots \qquad (13)$$

If at the same time one puts the c.d.f. (12) into the hazard rate equation (7), we then see that $\Lambda(t) = \lambda t$. Thus for exponential F_X, the expression in (8) coincides with (13) for any integer $k \geq 0$, not only for $k=1$.

For the sufficiency part, let (11) be satisfied for any $t>0$. We see from (10), (7) and (8) that $P(N_t^{(1)} = 0) = P(N_t^{(2)} = 0)$ for any $t>0$. Also from (8), (10) and (11) we obtain that for any $t>0$

$$\Lambda(t)e^{-\Lambda(t)} = F_X(t) - F_X*F_X(t).$$

In view of (7) and from the convolution rule this last equality is equivalent to

$$\int_0^t e^{-\Lambda(t-x)} dF_X(x) = \Lambda(t)e^{-\Lambda(t)}. \qquad (14)$$

Integrating the left-hand side by parts we get

$$\int_0^t e^{-\Lambda(t-x)} dF_X(x) = \int_0^t e^{-\Lambda(t-x)} d[1 - e^{-\Lambda(x)}]$$

$$= \int_0^t e^{-\Lambda(t-x)} e^{-\Lambda(x)} d\Lambda(x) = e^{-\Lambda(t-x)} e^{-\Lambda(x)} \Lambda(x) \Big|_0^t$$

$$+ \int_0^t \Lambda(x) e^{-[\Lambda(t-x)+\Lambda(x)]} [\lambda(x)-\lambda(t-x)] dx.$$

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Since $\Lambda(0) = 0$ the first term above equals $\Lambda(t)e^{-\Lambda(t)}$ and coincides with the right-hand side of (14). Therefore

$$\int_0^t \Lambda(x) \; e^{-[\Lambda(t-x)+\Lambda(t)]} [\lambda(x)-\lambda(t-x)]dx = 0$$

This means that where ever $\Lambda(x) \neq 0$ we have

$$\lambda(x) = \lambda(t-x)$$

for $t > x$. Since there exist at least one positive x_0 for which $\Lambda(x_0) \neq 0$ (and then $\Lambda(x) \neq 0$ for any $x > x_0$) the last equation holds for any $x > x_0$, $t > x$ if and only if $\lambda(x)$ is a constant. Thus from (7) we obtain $F_X(t) = 1 - exp(-\lambda t)$, and $N_t^{(1)}$ as well as $N_t^{(2)}$ are homogeneous Poisson processes. ∎

Remark. An analogous result is shown by Baxter (1982). He suggested the conditions

$$P(N_t^{(1)}= k \;) = P(N_t^{(2)}= k \;)$$

for any $t > 0$ and any $k=0,1,2,...$ and refers to a result of Grosswald et al. (1980) characterizing the exponential distribution in the cases where F_X has a power series tail.

4. The NPP with periodic failure rate and it characterization.

Here we consider the NPP $N_t^{(1)}$ introduced by the help of a r.v. X having an ALM - distribution. According to the previous discussion and Theorem 1-(vi), the failure rate function $\lambda_X(t)$ is periodic with period $c > 0$. Also from the definition of an NPP from relations (7) and (8) we see that the failure rate of the process $N_t^{(1)}$ (defined as the first derivative of the hazard function) coincides with the above $\lambda_X(t)$ and is therefore periodic also. The following is true:

Lemma 1. If the NPP $N_t^{(1)}$ has periodic failure rate function $\lambda_X(t)$ with period $c > 0$, then for all $t \geq 0$ and for any fixed integer $k \geq 0$ its hazard function Λ has the property

$$\Lambda(kc+t) = \Lambda(kc) + \Lambda(t) = k\Lambda(c) + \Lambda(t). \tag{15}$$

Proof. From (7) and (6), and from the periodicity of λ_χ we get

$$\Lambda(kc+t) = \int_0^{kc} \lambda(u)du + \int_{kc}^{kc+t} \lambda(u)du$$

$$= \sum_{\nu=0}^{k-1} \int_{\nu c}^{(\nu+1)c} \lambda(u)du + \int_0^t \lambda(kc+\upsilon)d\upsilon$$

$$= k\Lambda(c) + \int_0^t \lambda(\upsilon)d\upsilon = = k\Lambda(c) + \Lambda(t).$$

The substitution $u=\nu c+\upsilon$ is used when the lower bound of the integral is νc and then (6) is applied. ∎

We call (15) "almost linearity", a property of the hazard function Λ. To introduce the next property we denote by $N_{[\tau,\tau+t)}^{(1)}$ the random number of failures of the principal NPP on the interval $[\tau,\tau+t)$. Alternatively, if the initial item is a new one (at time $t_0 = 0$), we can interpret τ as the age of the initial item at the beginning of the observation period then $N_{[\tau,\tau+t)}^{(1)}$ gives the overall number of failures after t time units.

Lemma 2. Under the conditions of Lemma 1, for all $t \geq 0$ and any fixed integer $k \geq 0$ we have

(i) $P \ (N_{[kc,kc+t)}^{(1)} = m \ ) = P \ (N_{[0,t)}^{(1)} = m \ )$, $m = 0,1,2,\ldots$

(ii) The r.v.'s $N_{[kc,kc+t)}^{(1)}$ and $N_{[0,kc)}^{(1)}$ are mutually independent.

Proof. (i) From the general properties of a NPP, (see Çinlar, 1974), it is known that for any $m = 0,1,2,\ldots$

$$P(N_{[\tau,\tau+t)}^{(1)}=m) = \frac{[\Lambda(\tau+t)-\Lambda(\tau)]^m}{m!} e^{-[\Lambda(\tau+t)-\Lambda(\tau)]}.$$

Substitute here $\tau=kc$ and apply the result of Lemma 1 to obtain

$$P(N_{[kc,kc+t)}^{(1)} = m) = \frac{[\Lambda(t)]^m}{m!} e^{-\Lambda(t)} = P(N_{[0,t)}^{(1)}=m), \quad m=0,1,2,\ldots$$

(ii) Consider the probability generating function

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$$P^{(1)}_{[o,\,kc+t)}(z) = \sum_{m=0}^{\infty} P(N^{(1)}_{[o,\,kc+t)}=m)z^m = exp[\Lambda(kc+t)(z-1)]$$

Apply here (15) and the result of (i) to see that

$$P^{(1)}_{[o,\,kc+t)}(z) = exp[\Lambda(kc)(z-1)] \; exp[\Lambda(t)(z-1)]$$

$$= P^{(1)}_{[o,\,kc)}(z) \; P^{(1)}_{[kc,\,kc+t)}(z)$$

Since $N^{(1)}_{[o,\,kc+t)} = N^{(1)}_{[o,\,kc)} + N^{(1)}_{[kc,\,kc+t)}$ and the generating function on the left-hand side is equal to the product of the p.g.f. of the summands on the right-hand side, these summands are independent. ∎

Theorem 3. A NPP $\langle N^{(1)}_t, \; t \geq 0 \rangle$ has a periodic failure rate function if and only if the next two conditions are fulfilled:

(i) There exists some $c > 0$ such that for any $t \geq 0$ the random number of failures $N^{(1)}_{[c,\,c+t)}$ and $N^{(1)}_{[t,\,t)}$ coincide in distribution;

(ii) For the same $c > 0$ as in (i) and for any $t > 0$, the random number of failures $N^{(1)}_{[o,\,c)}$ and $N^{(1)}_{[c,\,c+t)}$ are independent.

Proof. The necessity part of the theorem follows from Lemma 2 when $k=1$. To prove the sufficiency we observe that from assumptions (i) and (ii) the following chain of implications holds:

$$P(N^{(1)}_{[o,\,c+t)} = 0) = P(N^{(1)}_{[t,\,c)} + N^{(1)}_{[c,\,c+t)} = 0) \qquad (16)$$

$$= P(N^{(1)}_{[t,\,c)} = 0) \; P(N^{(1)}_{[c,\,c+t)} = 0)$$

$$= P(N^{(1)}_{[t,\,c)} = 0) \; P(N^{(1)}_{[o,\,t)} = 0).$$

Further we remark that on the basis of the relation

$$P(N_{[o,\,y)} = 0) = P(X_1 \geq y) \quad for \; any \; y \geq 0,$$

equation (16) means that for the given $c > 0$ and any $t \geq 0$ we have

$$P(X_1 \geq c+t) = P(X_1 \geq c) \; P(X_1 \geq t), \qquad (17)$$

where X_1 is the random life time of the first component in the NPP model, which started the renewal process and determines it. But

(17) is exactly the LM property at point c (see Theorem 1-(ii)). Referring to Theorem 1-(vi) we verify the assertion and see that it is equivalent to say that X_1 has the ALM property. ∎

Corollary. If $N_t^{(1)}$ is an NPP with periodic failure rate function of period $c>0$, then for any $t>0$ the r.v. $N_{[0,t)}^{(1)}$ is representable in the form

$$N_{[0,t)}^{(1)} = M_c^{(1)} + M_c^{(2)} + \ldots + M_c^{[t/c]} + M_{[0,t-[t/c]c)} \,, \qquad (18)$$

where $M_c^{(\nu)}$ and $M_{(0,y)}$ are independent Poisson r.v.'s, distributed as $N_{[0,c)}^{(1)}$ and $N_{[0,y)}^{(1)}$, respectively, for $y \in [0,c)$.

Proof. We apply the result of Theorem 3 to each one of the terms in the decomposition

$$N_{[0,t)}^{(1)} = \sum_{k=0}^{[t/c]-1} N_{[kc,(k+1)c)}^{(1)} + N_{[[t/c]c,t)}^{(1)},$$

and replace them by the variables $M_c^{(\nu)}$ and $M_{(0,y)}^{(1)}$ to obtain (18).

Moreover we know the distributions of each summand;

$$P(\, M_c^{(\nu)} = m\,) = \frac{[\Lambda(c)]^m}{m!}\ e^{-\Lambda(c)}, \ m = 0,1,\ldots;$$

$$P(\, M_{(0,y)} = m\,) = \frac{[\int_0^y \lambda(u)du\,]^m}{m!}\ \exp(-\int_0^y \lambda(u)du), \ m=0,1,2,\ldots \quad ∎$$

Corollary 1 gives a clear and practical structure for the use of NPP with periodic failure rates. For time instants t that are integer multiples of c, i.e. when $t=nc$, the process $\langle N_t^{(1)}\rangle$ is equivalent to the independent sum of n identically distributed Poisson processes, (see (18)). This property is not equivalent to the infinite divisibility of the time-homogeneous Poisson process but is analogous to it.

5. Applications to risk theory.

The properties of the NPP1 established above, i.e. an NPP with periodic failure rate, is expected to find wide applications in

many problems related with risk theory. For illustration purposes we briefly describe here some applications to insurance modelling.

A general insurance contract is usually issued for a limited time period, say one year. During this period the insurance company will pay any reasonable claims of the policyholder; when no claim is made (i.e. no events occurred or they were not reported), no payment is issued. At the end of the contract period, the policyholder may renew the policy, thus buying insurance for the next year. The process so extends in time. We assume permanent renewal of the insurance contracts on the part of the policyholder and a constant number of policyholders within the portfolio.

Let $\lambda_x(t)$ be the intensity of occurrence of insurance events at time $t \geq 0$. When seasonal conditions affect the insured risk (e.g. automobile insurance) it is natural to assume that λ_x is a periodic function with period of 1 year. The policy issue date is irrelevant but must be fixed in advance; e.g. our periods could start on January 1 and end on December 31. Alternatively, we could use fiscal years going from April 1 to March 31, but then $\lambda_x(t)$ would have to be modified appropriately to account for weather patterns. For instance,

$$\lambda_x(t) = rt^p(1-t)^q, \quad \text{for} \quad r>0; \ p>-1; \ q>-1; \ t \in [0,1) \tag{19}$$

which mimics the shapes of the beta distribution. This could be flexible enough to include many expected patterns of claim intensities during the year.

If we assume that $\lambda_x(t)$ is the claim intensity of one policyholder and that there are N similar contracts in the portfolio at the beginning of the year, then from the theory of NPP's (see Çinlar, 1974), the composition of N such processes is

also NPP with intensity

$$\lambda(t) = \lambda_X^{(1)}(t) + \lambda_X^{(2)}(t) + \ldots + \lambda_X^{(N)}(t) = N\lambda_X(t). \qquad (20)$$

Further, if λ_X is a periodic function, so will λ. Therefore if the claim intensity function of a single insured has the form (19) , then the portfolio claim intensity will have the same form, namely

$$\lambda(t) = Nrt^P(1-t)^q. \qquad (21)$$

which is convenient for statistical inference. We assume here that λ_X has a period of 1, as when claim occurrences depend on seasons but are preserved from year to year. Moreover, we assume that if no claim is recorded during the year, it does not alter the probability of an event occurring during the next year.

4.1. The number of claims within a policy year.

According to the previous sections this r.v. $N_{[\tau,\tau+t)}^{(1)}$ has a Poisson distribution:

$$P(N_{[\tau,\tau+t)}^{(1)} = m) = \frac{[\int_\tau^{\tau+t} \lambda(u)du]^m}{m!} exp(-\int_\tau^{\tau+t} \lambda(u)du), \quad m=0,1,2,\ldots \qquad (22)$$

If $\lambda_X(t)$ is as in (19) then the probabilities in (22) can be calculated by the help of the incomplete beta function.

When in (22) $\tau=0$ and $t=1$ we obtain a total number $N_{(0,1)}^{(1)}$ of claims during the year. Using the beta function

$$\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \int_0^1 t^{p-1}(1-t)^{q-1}dt, \quad p,q \geq 0,$$

where $\Gamma(\omega) = \int_0^1 t^{\alpha-1}e^{-t}dt$, for $\omega > 0$, is the gamma function, we can write for $m=0,1,2,\ldots$

$$P(N_{[0,1]}^{(1)}=m) = \frac{\left[Nr \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}\right]^m}{m!} exp\left[-Nr \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}\right]. \qquad (23)$$

The expected number of claims during the year coincides with its variance and is expressed by the formula

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$$\mathbb{E}[N^{(1)}_{[0,1)}] = Nr \, \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)} \, . \tag{24}$$

Here r, p and q are parameters and can be estimated from the annual data records. The constant number of policies in force, N, is assumed known. Models with random N could be developed.

The expected number of claims during any period $[\tau, \tau+t)$ within the year could be found from (22) in a similar way:

$$\mathbb{E}[N^{(1)}_{[\tau,\tau+t)}] = Nr \int_{\tau}^{\tau+t} u^p (1-u)^q du \tag{25}$$

$$= Nr \, \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)} \left[\beta(p+1, q+1; \tau+t) - \beta(p+1, q+1; \tau) \right].$$

Here $\beta(p,q;t) = \dfrac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \displaystyle\int_0^t u^{p-1}(1-u)^{q-1} du$ is the incomplete beta function. Hence (25) can be calculated numerically.

4.2. The total claims in a given time period.

Let Z_i be the amount of the i-th claim. We assume Z_i does not depend on the time when the claim occurs and let $(Z_i)_{i=1}^{\infty}$ be i.i.d. r.v.'s with c.d.f. $F_Z(x) = P(Z_i < x)$. Then the total claims for the time period $[0,t)$ is represented by

$$S_t = \sum_{i=0}^{N^{(1)}_{[0,t)}} Z_i, \tag{26}$$

where $N^{(1)}_{[0,t)}$ is the NPP defined above. Therefore from (18) we can write (26) in the following form

$$S_t = S_c^{(1)} + S_c^{(2)} + \ldots + S_c^{([t/c])} + S_c^{(t-[t/c]c)}. \tag{27}$$

where the $S_c^{(k)}$ are i.i.d. r.v.'s all distributed as

$$S_c = \sum_{i=0}^{N^{(1)}_{[0,c)}} Z_i. \tag{28}$$

and $N^{(1)}_{[0,c)}$ is a Poisson r.v. with parameter $\Lambda(c) = \int_0^c \lambda(u) du$.

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Therefore the Laplace-Stieltjes transform (LST) of S_c is

$$\varphi_{S_c}(s) = E[exp(-sS_c)] = exp(\Lambda(c)[\varphi_Z(s)-1]), \qquad (29)$$

where $\varphi_Z(s)$ is the LST of the claim amounts Z_i. The last term $S_c^{(t-[t/c]c)}$ in (27) is equivalent to the random sum

$$S_c^{(t-[t/c]c)} = Z_1 + Z_2 + \ldots + Z_{N_{[c,t-[t/c]c)}}^{(i)}. \qquad (30)$$

The representation in (27) generalizes the classical compound renewal sum used in the literature for the ruin problem and related questions. These will be considered in detail in a future paper.

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