ACTUARIAL RESEARCH CLEARING HOUSE
1990 VOL. 2

Binomial Lattice for the Cox, Ingersoll and Ross Spot Rate Process

Zhaobi Ha and Elias S.W. Shiu
Department of Actuarial & Management Sciences
University of Manitoba
Winnipeg, Manitoba R3T 2N2, Canada

This note is motivated by the paper [MT], which appeared in a recent issue of this journal. Among the many interesting ideas discussed in [MT] is a proposal of a discrete analogue of the Cox-Ingersoll-Ross model [CIR1, CIR2]. The purpose of this note is to comment on this binomial lattice model.

Cox, Ingersoll and Ross postulate that the instantaneous spot rate of interest r(t) is a stochastic process governed by the stochastic differential equation

$$dr(t) = \alpha[\gamma - r(t)]dt + \rho\sqrt{r(t)}dW(t), \tag{1}$$

where α, γ and ρ are positive constants and W(t) is the standardized Gauss-Wiener process. In the context of an intertemporal general equilibrium asset pricing model, Cox, Ingersoll and Ross [CIR2, (23)] derive a closed-form formula for valuing default-free and noncallable zero coupon bonds. Several authors ([Ba], [HW], [NR]) have presented methods for the discretization of the square-root spot rate process. With the closed-form formula for zero coupon bonds, it is possible to check the validity of a discretization method. A derivation of the closed-form formula can be found in [BeS].

It is suggested in [MT] that the stochastic equation (1) be discretized as

$$r(n+1) - r(n) = \alpha[\gamma - r(n)] \pm \rho\sqrt{r(n)}, \quad n = 0, 1, 2, 3, \ldots . \tag{2}$$

Consider the up and down functions:

$$u(x) = \alpha\gamma + (1-\alpha)x + \rho\sqrt{x}, \quad x \geq 0, \tag{3}$$

and

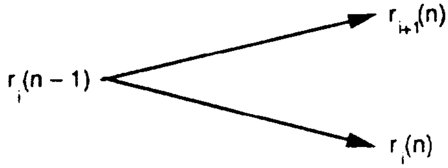
$$d(x) = \alpha\gamma + (1-\alpha)x - \rho\sqrt{x}, \quad x \geq 0. \tag{4}$$

Formula (2) means that $r(n+1) = u(r(n))$ or $r(n+1) = d(r(n))$. (For simplicity, we assume that the constants α, γ and ρ are such that, for each $x \geq 0$, $d(x)$ is nonnegative.) Since

$$u(d(x)) \neq d(u(x)),$$

(2) cannot be "implemented" on a binomial lattice. To overcome this difficulty, the authors of [MT] modify (2). We now show that the volatility of the spot rate in the modified model will **almost always** go to zero as time passes.

In a binomial lattice, there are n + 1 states of nature at time n. Let the one-period rates at time n be denoted by $\{r_0(n), r_1(n), r_2(n), \ldots, r_n(n)\}$.



Write $f^2(x) = f(f(x))$, $f^3(x) = f(f(f(x)))$, ... In the binomial lattice in [MT], $r_0(n) = d^n(r(0))$, $r_n(n) = u^n(r(0))$ and the inequalities

$$r_0(n) \leq r_1(n) \leq r_2(n) \leq \ldots \leq r_n(n)$$

hold. The sequence $\{u^n(r(0))\}$ converges to the unique fixed point of u, which is the larger root of the quadratic equation

$$[\alpha(\gamma - x)]^2 = \rho^2 x.$$

(The smaller root of the quadratic equation is the fixed point of the function d and is the limit of the sequence $\{d^n(r(0))\}$.) Thus there is a positive constant M such that, for all n,

$$0 \leq r_0(n) \leq r_1(n) \leq r_2(n) \leq \ldots \leq r_n(n) \leq M.$$

Consequently, for large n, most of the differences $\{r_{i+1}(n) - r_i(n)\}$ have to be small. Hence the volatility of the one-period interest rate must almost always go to zero as time tends to infinity.

Acknowledgment

We acknowledge with thanks the support from the Great-West Life Assurance Company and the Natural Sciences and Engineering Research Council of Canada.

References

[Ba] Ball, C.A. "A Branching Model for Bond Price Dynamics and Contingent Claim Pricing." Working Paper, University of Michigan, November 1989.

[BeS] Beekman, J.A., and Shiu, E.S.W. "Stochastic Models for Bond Prices, Function Space Integrals and Immunization Theory," *Insurance: Mathematics and Economics* 7 (1988), 163-173.

[CIR1] Cox, J.C., Ingersoll, J.E., Jr., and Ross, S.A. "Duration and the Measurement of Basis Risk," *Journal of Business* 52 (1979), 51-61.

[CIR2] Cox, J.C., Ingersoll, J.E., Jr., and Ross, S.A. "A Theory of the Term Structure of Interest Rates," *Econometrica* 53 (1985): 385-407.

[HW] Hull, J., and White, A. "Valuing Derivative Securities Using the Explicit Finite Difference Method," *Journal of Financial and Quantitative Analysis* 25 (1990): 87-100.

[MT] Macleod, N.J., and Thomison, J.D. "A Discrete Equilibrium Model of the Term Structure," *Actuarial Research Clearing House* (1988.1): 69-109.

[NR] Nelson, D.B., and Ramaswamy, K. "Simple Binomial Processes as Diffusion Approximations in Financial Models." Working Paper, University of Chicago, June 1989.

[PST] Pedersen, H.W., Shiu, E.S.W., and Thorlacius, A.E. "Arbitrage-Free Pricing of Interest-Rate Contingent Claims," *Transactions of the Society of Actuaries* 41 (1989): 231-265; Discussion, 267-279.