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Multiple-decrement models and corresponding conditional single-decrement models

by
James W. Daniel, A.S.A., Ph.D.
The University of Texas at Austin

Standard multiple-decrement models in the actuarial mathematics of life contingencies treat two random variables: the future lifetime T of a status and an integer variable J indicating which of m 'causes' was involved in the failure of the status. In addition, there are usually introduced some 'associated single-decrement models' or 'absolute rates of decrement' that can be used in the construction of the full multiple-decrement model if special assumptions are made such as 'UDD' or 'constant force'. A problem with these associated single-decrement models is that it is sometimes unclear what they actually mean—that is, to what sort of decrement they actually refer. Thus it may be difficult to have any sense of how to choose appropriate single-decrement models to use in the construction of a desired multiple-decrement model. Another—but minor—difficulty is that these single-decrement models often fail to behave as do many standard single-decrement models in that the probability of decrement may not tend to unity as duration increases. A strength of using associated single-decrement models is that they may be chosen quite freely and will produce a proper multiple-decrement model.

This paper introduces the alternative notion of 'corresponding conditional single-decrement models'; these models: 1) do have both an intuitive and a precise probabilistic meaning; 2) do behave as do standard single-decrement models; and 3) can easily be used to construct the full multiple-decrement model without any special assumption.

1. INTRODUCTION: THE STANDARD APPROACH.

Consider the standard multiple-decrement model for the failure of a life status subject to various causes of failure or decrement. Following [Bowers *et alia*, *Actuarial Mathematics*, Society of Actuaries, 1986, pp. 259-280], let T be the random variable giving the future time at which the status fails, and let J be the integer random variable indicating which of m causes produced the failure of the status. For ease and simplicity of presentation, only *lifelike* statuses are considered in this section—that is, statuses for which: 1) the status is intact when first observed and remains intact until some future time T, after which time it is not intact; and 2) T is a continuous-type random variable on $(0, \infty)$. The simplest such status is just the non-select (x) denoting a life aged x; equally possible are select statuses $[x]$, joint statuses such as $(x\!:\!y)$ or $([x]\!:\!y)$, and so on. Excluded in this section are statuses with mixed-type distributions for T, such as $(\overset{1}{x}: y)$ and $(x: \overline{n}|)$. These can of course be handled in much the same way, for example by allowing point masses in the densities and forces of decrement. We denote a general lifelike status by χ, since the Greek 'chi' is reminiscent of the common status (x); the notation $\chi + k$ then denotes the status χ still intact k years later—just as with the standard $[x] + k$.

As in [Bowers *et alia*, pp. 260-267], let $f(t, j)$ be the density function for the joint distribution of T and J, in terms of which the following standard symbols are defined:

$$
\begin{aligned}
{}_tq_\chi^{(j)} &= \Pr[J = j \text{ and } T \le t] = \int_0^t f(s, j)\, ds \\
h(j) &= \Pr[J = j] = \lim_{t \to \infty} {}_tq_\chi^{(j)} \\
{}_tq_\chi^{(\tau)} &= \Pr[T \le t] = \sum_{j=1}^{m} {}_tq_\chi^{(j)} \\
{}_tp_\chi^{(\tau)} &= \Pr[T > t] = 1 - {}_tq_\chi^{(\tau)} \\
\mu_{\chi+t}^{(j)} &= \frac{f(t, j)}{{}_tp_\chi^{(\tau)}} = \frac{\frac{d}{dt}\,{}_tq_\chi^{(j)}}{{}_tp_\chi^{(\tau)}} \\
\mu_{\chi+t}^{(\tau)} &= \sum_{j=1}^{m} \mu_{\chi+t}^{(j)}.
\end{aligned}
$$

Each of the preceding symbols has a probabilitstic interpretation. The first four were explicitly stated as probabilities; the fifth is the probability density function for decrement an instant after time t from cause j, conditioned on $T > t$; the last is the probability density function for decrement an instant after time t from any cause, conditioned on $T > t$.

Associated single-decrement models.

In terms of the forces of decrement $\mu_{\chi+t}^{(j)}$ one can also define [Bowers *et alia*, pp. 271-278] the so-called 'absolute rates of decrement':

$$
\begin{aligned}
{}_tp_\chi'^{(j)} &= e^{-\int_0^t \mu_{\chi+s}^{(j)}\, ds}, \\
{}_tq_\chi'^{(j)} &= 1 - {}_tp_\chi'^{(j)}.
\end{aligned}
$$

The m different functions ${}_tp_\chi'^{(j)}$ are usually viewed as defining m associated single-decrement models, in that ${}_tp_\chi'^{(j)}$ is viewed as analogous to ${}_tp_\chi$ in a standard single-decrement setting. These associated single-decrement models that were defined from the multiple-decrement model can of course instead be used in the reverse manner: so long as all the ${}_tp_\chi'^{(j)}$ decrease from 1 and at least one decreases to 0 as t increases from 0 to infinity, knowledge of all the ${}_tp_\chi'^{(j)}$ for all t easily produces the $\mu_{\chi+t}^{(j)}$, then $\mu_{\chi+t}^{(\tau)}$, then ${}_tp_\chi^{(\tau)} = \exp(-\int_0^t \mu_{\chi+s}^{(\tau)}\, ds)$, and finally the basic density $f(t, j) = {}_tp_\chi^{(\tau)} \mu_{\chi+t}^{(j)}$ from which the multiple-decrement model is defined.

The associated single-decrement data is often given in tabular form as values of the discrete probabilities $q'^{(j)}_{\chi+k}$. Unfortunately, the corresponding discrete data $q^{(j)}_{\chi+k}$ for the multiple-decrement model cannot be computed exactly from the $q'^{(j)}_{\chi+k}$ without some additional special assumption on the form of, say, ${}_tq^{(j)}_\chi$ for all t. For example, either assuming that all the $\mu^{(j)}_{\chi+t}$ are constant within each integer interval $k < t < k+1$ or assuming that all the ${}_tq^{(j)}_\chi$ are linear within each such interval leads to the following scheme—only the fourth step of which actually depends on the special assumption—for computing the $q^{(j)}_{\chi+k}$ from the $q'^{(j)}_{\chi+k}$.

$$p'^{(j)}_{\chi+k} = 1 - q'^{(j)}_{\chi+k}$$

$$p^{(\tau)}_{\chi+k} = \prod_{j=1}^{m} p'^{(j)}_{\chi+k}$$

$$q^{(\tau)}_{\chi+k} = 1 - p^{(\tau)}_{\chi+k}$$

$$q^{(j)}_{\chi+k} = \frac{\ln p'^{(j)}_{\chi+k}}{\ln p^{(\tau)}_{\chi+k}} q^{(\tau)}_{\chi+k}$$

The preceding paragraph noted one of the common difficulties in using the absolute rates in a discrete single-decrement model to construct the discrete multiple-decrement model—the need to make special assumptions. Another commonly noted difficulty is that the absolute rates $q'^{(j)}_{\chi+k}$ are difficult to interpret: they do not actually equal the probability of any particular meaningful event associated with the multiple-decrement model—which makes it rather difficult to know how to choose a set of absolute rates to use in constructing a multiple-decrement model appropriate to a given real situation. [Absolute rates are also criticized for not behaving like true single-decrement probabilities in that not all of the ${}_tp'^{(j)}_\chi$ need tend to 0 as t tends to infinity; this is also true, however, of true probabilities ${}_tp_\chi$ associated with important non-lifelike statuses such as $\chi = (\overline{x{:}y})$ and should not be viewed a serious concern.] Absolute rates can *intuitively* be viewed as probabilities for some *imaginary* independent causes which, had they been combined to form a multiple-decrement model, would have produced the original multiple-decrement model. *Note, however, that this does not mean that those imaginary independent causes are in any sense like the true causes in the multiple-decrement model; it need not make sense to view one as 'death-like', another as 'retirement-like', and so on in an employee-benefits multiple-decrement model, for instance.*

Another interpretation for absolute rates comes from the relation

$${}_tq'^{(j)}_\chi = {}_tq^{(j)}_\chi + \int_0^t {}_sp^{(\tau)}_\chi \, [\mu^{(\tau)}_{\chi+s} - \mu^{(j)}_{\chi+s}] \, {}_{t-s}q'^{(j)}_{\chi+s} \, ds.$$

This leads to the intuitive interpretation that ${}_tq'^{(j)}_\chi$ represents the fraction of people either who *do* decrement for cause j by time t (the fraction ${}_tq^{(j)}_\chi$) or who *would have* had they not decremented for another cause (the fraction given by the integral)—if it's truly possible to understand what is meant by the number of people who would have been disabled had they not died, for example. In simple examples in which ${}_tq_{\chi}^{(j)}$ represents the probability of decrement by death, however, it is easy to show that ${}_tq_{\chi}^{\prime(j)}$ usually does *not* equal the fraction of people that die regardless of other events.

Thus the problems with the associated single-decrement models and their absolute rates are three: 1) it's hard to interpret them and hence to know how to select them in constructing a multiple-decrement model from absolute rates; 2) special assumptions have to be made to construct the multiple-decrement probabilities when only discrete absolute rates are avaiable such as in tabular data; and 3) they need not behave as do single-decrement probabilities associated with lifelike statuses (not really a serious problem). These difficulties can be surmounted by using 'corresponding conditional single-decrement models'.

2. CORRESPONDING CONDITIONAL SINGLE-DECREMENT MODELS.

Suppose that you are an employee-benefits actuary. Although you may not know a numerical value, you can at least consider the probability that an employee dies in the next year *when it is given that the employee will not otherwise terminate employment during the year*. That is, you can consider the *conditional* probability of decrement from death *conditioned* on not decrementing from the other causes. We will show that such probabilities behave as do single-decrement-model probabilities and that they can be used to easily construct the full multiple-decrement model without special assumptions.

Consider again the full multiple-decrement model from Section 1. The general status χ need not be assumed to be lifelike, in the sense that the future-lifetime random variable T can be allowed to be of mixed or continuous type—that is, to have point masses in its probability density function. Although it is possible to allow positive probability that the status never fail (that is, that $T = \infty$), for convenience the analysis here is restricted to the case in which ${}_tq_{\chi}^{(\tau)}$ tends to 1 as t tends to infinity.

(1) **Definition.** The ***jth corresponding conditional probability of decrement*** ${}_t\widetilde{q}_{\chi}^{(j)}$ is defined as the conditional probability

$${}_t\widetilde{q}_{\chi}^{(j)} = \Pr[J = j \text{ and } T \leq t \mid \text{NOT}(J \neq j \text{ and } T \leq t)],$$

and the associated survival probability ${}_t\widetilde{p}_{\chi}^{(j)}$ and force $\widetilde{\mu}_{\chi+t}^{(j)}$ are defined by

$${}_t\widetilde{p}_{\chi}^{(j)} = 1 - {}_t\widetilde{q}_{\chi}^{(j)},$$
$$\widetilde{\mu}_{\chi+t}^{(j)} = \frac{-\frac{d}{dt}\,{}_t\widetilde{p}_{\chi}^{(j)}}{{}_t\widetilde{p}_{\chi}^{(j)}}.$$

The intuitively clear meaning of ${}_t\tilde{q}_x^{(j)}$ and its technical definition as a probability avoids one of the problems associated with ${}_tq_x'^{(j)}$. The questions that remain concern its computation, behavior, and usefulness.

(2) **Theorem.** The *jth* corresponding conditional probabilities ${}_t\tilde{q}_x^{(j)}$ and ${}_t\tilde{p}_x^{(j)}$ can be evaluated from the multiple-decrement probabilities as

$$ {}_t\tilde{q}_x^{(j)} = \frac{{}_tq_x^{(j)}}{1-\sum_{k\neq j} {}_tq_x^{(k)}} = \frac{{}_tq_x^{(j)}}{{}_tp_x^{(\tau)} + {}_tq_x^{(j)}} $$

$$ {}_t\tilde{p}_x^{(j)} = \frac{{}_tp_x^{(\tau)}}{1-\sum_{k\neq j} {}_tq_x^{(k)}} = \frac{{}_tp_x^{(\tau)}}{{}_tp_x^{(\tau)} + {}_tq_x^{(j)}} $$

PROOF: The second formula follows directly from the first. The first is defined as a conditional probability; to calculate a conditional probability $\Pr[A \mid B]$ one divides $\Pr[A \text{ and } B]$ by $\Pr[B]$. In our case, the event $[A \text{ and } B]$ is the same as the event $[A]$ alone, since the condition $J = j$ in $[A]$ makes the condition $J \neq j$ false and hence makes the event $[B]$ automatically true. So $\Pr[A \text{ and } B] = \Pr[A] = {}_tq_x^{(j)}$ is our numerator term. Our event $[B]$ is actually $[\text{ NOT}(C)]$, so we need $1 - \Pr[C]$ for our denominator. Now $[C]$ occurs in any of the $m-1$ disjoint cases $J = k$ and $T \leq t$ for $1 \leq k \leq m$ and $k \neq m$, each of which has probability ${}_tq_x^{(k)}$. The sum of these $m-1$ probabilities is $\sum_{k\neq j} {}_tq_x^{(k)} = {}_tq_x^{(\tau)} - {}_tq_x^{(j)}$, and so our denominator becomes $1 - \sum_{k\neq j} {}_tq_x^{(k)} = 1 - ({}_tq_x^{(\tau)} - {}_tq_x^{(j)}) = {}_tp_x^{(\tau)} + {}_tq_x^{(j)}$. This makes the quotient for the conditional probability exactly as claimed above. ∎

Since Theorem 2 shows how to calculate the corresponding conditional probabilities of decrement, it is easy to see how they behave. In the formula above for ${}_t\tilde{q}_x^{(j)}$, the numerator clearly equals 0 for $t = 0$ and is non-decreasing as t increases. Since the denominator in the formula is $1 - \sum_{k\neq j} {}_tq_x^{(k)}$ and each ${}_tq_x^{(k)}$ is non-decreasing, that denominator is non-increasing. Thus the quotient defining ${}_t\tilde{q}_x^{(j)}$ is non-decreasing. Note also that each ${}_tq_x^{(j)}$ must increase to a positive limit—not necessarily 1—as t tends to infinity. Since ${}_tp_x^{(\tau)}$ tends to 0, the quotient defining ${}_t\tilde{q}_x^{(j)}$ must tend to 1 and thus ${}_t\tilde{p}_x^{(j)}$ to 0. Moreover, all the ${}_t\tilde{p}_x^{(j)}$ tend to 0 at the same rate, since

$$ \frac{{}_t\tilde{p}_x^{(j)}}{{}_t\tilde{p}_x^{(k)}} = \frac{{}_tp_x^{(\tau)} + {}_tq_x^{(k)}}{{}_tp_x^{(\tau)} + {}_tq_x^{(j)}} $$

tends to $h(k)/h(j)$, which is positive. This proves the following.

(3) **Theorem.** The corresponding conditional single-decrement models described by the corresponding conditional probabilities ${}_t\tilde{q}_x^{(j)}$ and ${}_t\tilde{p}_x^{(j)}$ be-

have as do true single-decrement models, in the sense that, for each j, ${}_t\tilde{q}_x^{(j)}$ increases from 0 to 1 and ${}_t\tilde{p}_x^{(j)}$ decreases from 1 to 0 as t increases to infinity, allowing ${}_t\tilde{q}_x^{(j)}$ to serve as a cumulative distribution function. Moreover, all the ${}_t\tilde{p}_x^{(j)}$ decrease to 0 at the same rate, in that ${}_t\tilde{p}_x^{(j)}/{}_t\tilde{p}_x^{(k)}$ tends to $h(k)/h(j)$.

(4) **Example.** Suppose that the multiple-decrement model has constant forces $\mu_{x+t}^{(j)} = .01j$ for $j = 1, 2$. Then it is easy to see that

$$\begin{aligned} {}_tp_x^{(\tau)} &= e^{-.03t} \\ {}_tq_x^{(j)} &= \frac{j}{3}(1 - e^{-.03t}) \\ {}_tp_x'^{(j)} &= e^{-.01jt} \\ {}_t\tilde{p}_x^{(1)} &= \frac{3e^{-.03t}}{1 + 2e^{-.03t}} \\ {}_t\tilde{p}_x^{(2)} &= \frac{3e^{-.03t}}{2 + e^{-.03t}}. \end{aligned}$$

Note that the ${}_t\tilde{p}_x^{(j)}$ share other properties besides what Theorem 3 shows. For example, if ${}_tp_x^{(\tau)} = 0$ for some t, then the same is true for all the ${}_t\tilde{p}_x^{(j)}$. Similarly, point masses in ${}_tp_x^{(\tau)}$ are inherited by the ${}_t\tilde{p}_x^{(j)}$.

While of little practical importance, it's interesting to note that straightforward calculus produces the following relationship among forces of decrement:

$$\tilde{\mu}_{x+t}^{(j)} = {}_t\tilde{q}_x^{(j)}\mu_{x+t}^{(\tau)} + {}_t\tilde{p}_x^{(j)}\mu_{x+t}^{(j)}.$$

3. CONSTRUCTING MULTIPLE-DECREMENT MODELS.

Theorem 2 showed how the corresponding conditional single-decrement models described by ${}_t\tilde{q}_x^{(j)}$ and ${}_t\tilde{p}_x^{(j)}$ can be constructed from the multiple-decrement model. Consider now the reverse.

Working with continuous data.

Suppose that the corresponding conditional single-decrement probabilities ${}_t\tilde{q}_x^{(j)}$ and ${}_t\tilde{p}_x^{(j)}$ are known for all t and the task is to construct the multiple-decrement probabilities ${}_tp_x^{(\tau)}$ and ${}_tq_x^{(j)}$ for all t. The formula in Theorem 2 is easily solved for ${}_tq_x^{(j)}$ to give

$${}_tq_x^{(j)} = \frac{{}_t\tilde{q}_x^{(j)}}{1 - {}_t\tilde{q}_x^{(j)}}\,{}_tp_x^{(\tau)} = \omega^{(j)}\,{}_tp_x^{(\tau)}$$

if $\omega^{(j)}$ is used to denote the odds ratio ${}_t\widetilde{q}_x^{(j)}/(1 - {}_t\widetilde{q}_x^{(j)}) = {}_t\widetilde{q}_x^{(j)}/{}_t\widetilde{p}_x^{(j)}$ corresponding to the probability ${}_t\widetilde{q}_x^{(j)}$. But then

$$ {}_tp_x^{(\tau)} = 1 - {}_tq_x^{(\tau)} = 1 - \sum_{j=1}^{m} {}_tq_x^{(j)} = 1 - \sum_{j=1}^{m} \omega^{(j)}\,{}_tp_x^{(\tau)} $$

can be solved for ${}_tp_x^{(\tau)}$ to give

$$ {}_tp_x^{(\tau)} = \frac{1}{1 + \sum_{j=1}^{m} \omega^{(j)}} = \frac{1}{1 + \omega^{(\tau)}} $$

with $\omega^{(\tau)} = \sum_{j=1}^{m} \omega^{(j)}$. So the multiple-decrement model can be constructed from the corresponding conditional probabilities ${}_t\widetilde{q}_x^{(j)}$ and ${}_t\widetilde{p}_x^{(j)}$ as follows:

$$ (5) \qquad \begin{aligned} {}_tp_x^{(\tau)} &= \frac{1}{1 + \sum_{j=1}^{m} \frac{{}_t\widetilde{q}_x^{(j)}}{{}_t\widetilde{p}_x^{(j)}}} = \frac{1}{1 + \sum_{j=1}^{m} \omega^{(j)}} = \frac{1}{1 + \omega^{(\tau)}} \\ {}_tq_x^{(j)} &= \frac{{}_t\widetilde{q}_x^{(j)}}{{}_t\widetilde{p}_x^{(j)}}\,{}_tp_x^{(\tau)} = \omega^{(j)}\,{}_tp_x^{(\tau)} \end{aligned} $$

Does this mean that a proper multiple-decrement model will result from applying Equation 5 to an arbitrary set of cumulative distribution functions chosen to serve as ${}_t\widetilde{q}_x^{(j)}$? Not necessarily. Recall from Theorem 3 that the ${}_t\widetilde{p}_x^{(j)}$ must all tend to 0 at the same rate; this restricts the choice of the ${}_t\widetilde{p}_x^{(j)}$. To construct a multiple-decrement model with m causes by first constructing corresponding conditional single-decrement models, proceed as follows:

1. Choose the values $h(j)$ for eventual failure from cause j, being careful that the $h(j)$ sum to 1.
2. Choose cumulative distribution functions ${}_t\widetilde{q}_x^{(j)}$ in a consistent manner so that ${}_t\widetilde{p}_x^{(j)}/{}_t\widetilde{p}_x^{(k)}$ tends to $h(k)/h(j)$ as t tends to infinity.
3. Construct the multiple-decrement model using Equation 5.
4. **Verify that the constructed ${}_t\widetilde{q}_x^{(j)}$ are non-decreasing.**

The fourth step above is essential, since it is possible for some ${}_t\widetilde{q}_x^{(j)}$ produced in this manner to have a decreasing region; this is the difficulty in using corresponding conditional probabilities **for all t** to construct the multiple-decrement probabilities. But there is a better way to perform the construction.

Working from tabular data.

In practice, models based on discrete tabular data are more commonly used than are those with analytical formulas for the various probabilities as a function of t, as discussed above. We saw in Section 1 that, if the absolute rates of decrement were given only in tabular form for each year as $q'^{(j)}_{x+k}$, then special assumptions had to be made in order to compute the multiple-decrement probabilities. This is not necessary when using corresponding conditional models, however; for example, setting $t = k$ in Equation 5 immediately produces discrete data from discrete data.

Typically, however, discrete data are displayed in multiple-decrement tables that give the values of $q_{\chi+k}^{(j)}$ rather than of ${}_k q_{\chi}^{(j)}$. These can in fact easily be produced from similar tables for $\tilde{q}_{\chi+k}^{(j)}$. How? Since $\chi + k$ denotes a status just as does χ, and since $\tilde{q}_{\chi+k}^{(j)}$ is shorthand for the probability ${}_t\tilde{q}_{\chi+k}^{(j)}$ with $t = 1$, we can simply use Equation 5 with $t = 1$ and χ replaced by $\chi + k$ for each k to compute the discrete multiple-decrement probabilities $q_{\chi+k}^{(j)}$ for each k. This leads to the following procedure.

(6) **Theorem.** Suppose that for the lifelike status χ and for each non-negative integer k, the corresponding conditional single-decrement one-year probabilities

$$\tilde{q}_{\chi+k}^{(j)} = \Pr[J = j \text{ and } T \leq 1 \mid \text{NOT}\,(J \neq j \text{ and } T \leq 1)]$$

and $\tilde{p}_{\chi+k}^{(j)} = 1 - \tilde{q}_{\chi+k}^{(j)}$ are known, where J denotes $J(\chi + k)$ and T denotes $T(\chi + k)$. Then the multiple-decrement one-year probabilities $p_{\chi+k}^{(\tau)}$ and $q_{\chi+k}^{(j)}$ can be computed as follows:

$$p_{\chi+k}^{(\tau)} = \frac{1}{1 + \sum_{j=1}^{m} \frac{\tilde{q}_{\chi+k}^{(j)}}{\tilde{p}_{\chi+k}^{(j)}}}$$

$$q_{\chi+k}^{(j)} = \frac{\tilde{q}_{\chi+k}^{(j)}}{\tilde{p}_{\chi+k}^{(j)}}\, p_{\chi+k}^{(\tau)}.$$

Theorem 6 is the main result of this paper—it shows how to use the intuitively meaningful corresponding conditional one-year probabilities of decrement from cause j given non-decrement from the other causes in order to construct the full multiple-decrement model.

(7) **Example.** Suppose that there are three causes of decrement and that the corresponding conditional one-year probabilities $\tilde{q}_{\chi+k}^{(j)}$ are constant with respect to k and equal .01, .02, and .03, respectively. Then Theorem 6

produces the multiple-decrement probabilities as

$$p_{x+k}^{(\tau)} = \frac{1}{1 + \frac{.01}{.99} + \frac{.02}{.98} + \frac{.03}{.97}} = .9422$$

$$q_{x+k}^{(1)} = \frac{.01}{.99} p_{x+k}^{(\tau)} = .0095$$

$$q_{x+k}^{(2)} = \frac{.02}{.98} p_{x+k}^{(\tau)} = .0192$$

$$q_{x+k}^{(3)} = \frac{.03}{.97} p_{x+k}^{(\tau)} = .0291$$

for all k.

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