GENERAL FINANCIAL TRANSACTIONS
AND APPLICATIONS TO THE THEORY OF
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Abstract: The theory of Teichroew, Robicheck, and Montalbano is concerned with assigning internal rates of return to arbitrary financial projects . Their work was carried out in a discrete model. Our goal is to establish the main results in a more general setting, where cash flows are given by signed measures.

This paper contains a summary of results. Detailed proofs will appear in a subsequent publication.

General Transactions

The most familiar description of a financial transaction is by the *discrete model.* A transaction T is identified with a finite sequence

$$T = \{ c_0, c_1, \dots , c_N \} \tag{1}$$

where c_i denotes the payment at time t. For many purposes it is convenient to consider the *continuous model* , in which a transaction is described by a continuous function c_t, which denotes the periodic *rate* of payment at time t. A comprehensive approach, synthesizing the discrete and continuous models, and thereby allowing for mixed and more general cases was given in [4, §X] .The idea, borrowed from probability theory, is simply to describe the transaction by a function G(t) of bounded variation defined on $(- \infty , \infty)$ such that $\lim_{t \to -\infty} G(t) = 0$. G(t) denotes the total

amount received up to time t and will be called the *total flow function*. We normalize by assuming that G is right continuous, so G(t) denotes the amount received up to *and including* time t.

In the discrete case (1), G is a step function,

$$
G(t) = \begin{cases} 0, & \text{for } -\infty < t < 0 \\ \sum_{i=0}^{k} c_i, & \text{for } 0 \leq k \leq t < k+1 \leq n \\ \sum_{i=0}^{n} c_i, & \text{for } t > n \end{cases}
$$

In the continuous case,

$$
G(t) = \int_{-\infty}^{t} c_s \, ds
$$

There are many practical situations where this general point of view is useful. For example, in the typical life insurance policy, the benefits can be viewed as a continuous transaction while the premium payments are discrete. Taken together they form a mixed case which can be conveniently described by the more general model.

It is well known that G induces a signed measure μ on the Borel sets of the real line, where $\mu[(\,a,b)]$ is $G(b) - G(a)$. For a Borel set A, $\mu(A)$ denotes the total amount of funds received during the time period represented by A. This gives us another point of view and for many purposes it is more convenient to use μ in place of G. In this paper however, we will confine ourselves to the use of the total flow function.

These ideas appear in a paper by R. Norberg [3]. In that work, all payments are nonnegative, which corresponds to an increasing G, or a positive measure. It is true of course that any function of bounded variation is the difference of two increasing functions, so for many purposes it is sufficient to consider only the increasing case. However, for the main application in this paper, it seems necessary to adopt the more

general viewpoint which allows for both positive and negative payments in the same transaction.

Given such a G, let $C(t) = G(t) - G(t-)$, the actual payment at time t. Note that in the continuous case, $C(t) = 0$ for all t.

We review some definitions from [5]. An *accumulation function* for our purposes will be a positive valued function of two variables, $a(s,t)$ satisfying

$$a(s,t) = a(s,r) \, a(r,t), \quad \text{for all } s,t, \text{ and } r. \tag{2}$$

It is an easy consequence of this property that

$$a(t,t) = 1, \quad \text{and} \quad a(s,t) = a(t,s)^{-1}, \quad \text{for all } s,t.$$

We think of $a(s,t)$ as denoting the amount at time t resulting from an investment of 1 at time s. Equivalently, one can view $a(s,t)^{-1}$ as the price at time s of a 1-unit, pure discount bond maturing at time t. This is the approach taken in [3], where it is shown that (2) follows from a "no arbitrage" hypothesis.

Given a transaction with total flow function G, and an accumulation function a, the *value of the transaction at time t* is defined by

$$\text{Val}(t) = \int_{-\infty}^{\infty} a(s,t) \, dG(s) \tag{3}$$

This represents the single payment at time t which would be equivalent to all the payments of the transaction, assuming growth of capital according to the given accumulation function. One of the main goals in [3] is to provide an axiomatic justification of formula (3).

The retrospective value at time t, which we also call the *balance function*, is similarly defined by

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$$B(t) = \int_{-\infty}^{t} a(s,t)\, dG(s)$$

$B(t)$ denotes the accumulated amount at time t, including any payment due at that time.

TRM Theory

We review the theory of Teichroew, Robichek and Montalbano (henceforth denoted by TRM) originally introduced in [8] and [9]. For more details, see [6] (note however that the our notation has changed somewhat), or for a very extensive discussion see [7]. In addition, Kellison's text contains a brief summary [2, §5.9]. For definiteness, suppose we have an *investment project*, that is, a transaction as given in (1) for which $c_0 < 0$. (The conclusions below have corresponding dual statements in the case that T is a *financing project*, one for which $c_0 > 0$.) For any deposit rate d and investment rate r, define inductively the balance function $B_{r,d}$, on the set $\{0,1,\ldots,N\}$ as follows.

$$B_{r,d}(0) = c_0$$

$$B_{r,d}(k+1) = \begin{cases} B_{r,d}(k)(1+d) + c_{k+1}, & \text{if } B_{r,d}(k) \geq 0 \\ B_{r,d}(k)(1+r) + c_{k+1}, & \text{if } B_{r,d}(k) \leq 0 \end{cases}$$

Following is a main result of the theory. It is readily established by induction on the duration n.

THEOREM 1: For fixed d and a positive integer $n \leq N$, $B_{r,d}(n)$ is a *strictly decreasing* function of r. Moreover, $B_{r,d}(n)$ becomes negative for sufficiently large values of r.

The theorem can be generalized somewhat. Suppose that T is *not* an investment project. The conclusion is clearly no longer true as stated, since for sufficiently high values of d, $B_{r,d}(k)$, for $k = 0,1,2,\ldots,n$, will be nonnegative and independent of r. However, for those values of d which are small enough to allow for a negative value of $B_{r,d}(k)$, for some r and some k between 0 and n, the same result holds.

The first statement in the theorem is intuitively clear as it essentially says the following. Consider two individuals A and B. They begin with the same balance in their bank accounts and they make exactly the same deposits and withdrawals. They both receive exactly the same amount of interest on positive balances. However, A is charged a higher rate than B on overdrafts. It is evident that at the end of any given time period. A will always have a balance which is less than or equal to that of B. Moreover, if any overdraft occured during that period, which is certain if they both began with a negative balance, then A will have a balance which is *strictly* less than that of B.

Theorem 1 allows us to define, for any investment project T, the quantity $I_d(T)$, the TRM internal rate of return corresponding to the deposit rate d. Namely, $I_d(T)$ equals the unique zero of the function $r \rightarrow B_{r,d}(N)$, or -1, if $B_{r,d}(N) \leq 0$ for all r.

We would like to derive Theorem 1 in the general setting which we described in the first section. For this purpose we will place some further restrictions on G. We assume the following:

Transactions are of bounded duration. That is, there exist points a and b such that

$$G(t) = 0 \text{ for } t < a \text{ and } G(t) = G(b) \text{ for } t > b \tag{4}$$

There is a constant K > 0 such that for all t,

$$\limsup_{h \to 0+} \left| \frac{G(t+h)-G(t)}{h} \right| \leq K \text{ and } \limsup_{h \to 0+} \left| \frac{G(t-h)-G(t-)}{h} \right| \leq K \tag{5}$$

Condition (5) is satisfied in the discrete case where we can take K = 0, and in the continuous case where we can take $K = \sup\{c_t : a \leq t \leq b\}$.

The first problem is to arrive at a suitable definition of an investment project. We do so as follows. A transaction with total flow function G is an *investment project* if there exists a point t_0 such that

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(i) $G(t_0) < 0$.

(ii) G is nonincreasing on $(-\infty, t_0)$.

In other words, up to some point there is a steady outflow of funds without anything coming in. Similarly, by reversing the inequality in (i) and replacing nonincreasing by nondecreasing in (ii), we obtain the definition of a *financing project*.

The next problem is to arrive at a suitable definition of the balance function. It is now much more convenient to work with forces of interest rather than rates. Let δ and ρ refer to the *forces* of interest for the deposit rate and the investment rate respectively. We seek an appropriate definition of $B_{\rho, \delta}$. (For convenience we use the same letter B as in the discrete model. Note that the function $B_{r,d}$ described above would now be denoted by $B_{\ln(1+r), \ln(1+d)}$.) In the continuous case, it is not difficult to give the appropriate definition. Namely, $B_{\rho, \delta}$ is the solution to the differential equation,

$$\frac{d}{dt} B_{\rho, \delta}(t) = \begin{cases} \delta B_{\rho, \delta}(t) + c_t, & \text{if } B_{\rho, \delta}(t) \geq 0 \\[2mm] \rho B_{\rho, \delta}(t) + c_t, & \text{if } B_{\rho, \delta}(t) \leq 0 \end{cases}$$

$$B_{\rho, \delta}(a) = 0.$$

where c_t is a continuous function on the interval [a, b]

In the general case we are faced with the following difficulty. We want to describe a situation in which accumulation is at a force of interest δ when balances are negative, and at a force of interest ρ when balances are positive. Therefore, the rate of accumulation depends on the balances, but these in turn depend on the rate of accumulation. We resolve this circularity in the discrete case via the inductive definition. In the continuous case we resolve it via a differential equation, which can be viewed as a continuous version of an inductive definition. In the case of a general transaction we proceed as follows.

Fix δ and ρ and a transaction with total flow function G. Let P and N be two Lebesgue measurable subsets of the real line such that $\nu(P \cap N) = 0$, where ν denotes Lebesgue measure. Define an accumulation function $a_{P,N}$ by

$$a_{P,N}(s,t) = e^{(\delta\nu[P \cap (s,t)] + \rho\nu[N \cap (s,t)])}$$

and let $B^{P,N}$ denote the corresponding balance function. The resulting accumulation is at a force of interest δ for points in P and at a force of interest ρ for points in N. There can be points in the intersection which may at first glance seem contradictory, but a moments reflection shows that it is essential to allow for this possibility. For example, suppose that in the discrete model a positive payment changes the balance from negative to positive. At that point of time, accumulation is at a force of interest δ if one looks forward, but at a force of interest ρ if one looks backwards.

Given such P and N, let

$$P_1 = \{t: B^{P,N}(t) > 0\} \cup \{t: B^{P,N}(t) - C(t)\} > 0$$

$$N_1 = \{t: B^{P,N}(t) < 0\} \cup \{t: B^{P,N}(t) - C(t)\} < 0$$

We see that P_1 consists of all times at which the balance is positive, or was positive just before the withdrawal at that time, and we similarly interpret N_1

In order to achieve the goal which we described above we want to choose P and N, so that $P_1 \subseteq P$ and $N_1 \subseteq N$. While it is far from obvious that this can be done, it is possible with the assumptions made above. The details are somewhat complicated and will not be given here. Moreover, although there may be many possible choices for such P and N, the resulting balance function is *unique*, and we can denote it unambiguously by $B_{\rho,\delta}$. We are then able to prove the following general version of Theorem 1.

THEOREM 1': Suppose we have an investment project satisfying (4) and (5). Then, for fixed δ and t, $B_{\rho,\delta}(t)$ is a strictly decreasing function of ρ which becomes negative for sufficiently large ρ.

Fixing δ we can then define l_δ as the unique zero of the function which assigns $B_{\rho,\delta}(b)$ to ρ, or $-\infty$ if all values of $B_{\rho,\delta}(b)$ are nonpositive.

We are also able to demonstrate the relationship between l_δ and the Arrow-Levhari internal rate of return defined in [1]. For the discrete case, this was announced in [6] and a detailed proof appears in [7, Theorem 5.5] We are able to give a definition of the Arrow-Levhari rate in our general setting, and show that it is equal to $\sup_\delta \{l_\delta\}$.

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