CONTINGENT CLAIMS VALUATION OF "GREATER OF" BENEFITS

by

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Abstract

Pension funds are usually either defined benefit or defined contribution funds. In Australia in recent years, for a number of reasons, some pension funds have offered the "greater of" these two benefits.

Such a benefit design can be valued using contingent claims valuation techniques since it is equivalent to an option on the maximum of two random benefit amounts - one equal to a multiple of salary and the other the accumulation of a percentage of salary at an earnings rate.

This paper overviews some of the issues in applying contingent claims valuation techniques to the valuation of this style of benefit including:

- incorporation of decrements,
- the dependence of the accumulation benefit on the two stochastic state variables, salary and fund earnings rate and the resulting path- dependency,
- the numerical techniques for efficient calculation of benefit values including discrete lattice models, finite difference techniques, simulation and approximations based on bivariate log-normal assumption,
- the lack of traded assets to price salary risk and implications for valuation.

Numerical evaluation of benefit values and the assessment of computational efficiency of alternative techniques is the next stage of this research.

Introduction

Pension funds, or superannuation funds as they are referred to in Australia, have developed benefit designs which offer resignation, death and/or retirement benefits that are the greater of two alternative benefits. The two alternatives are typically the accumulation of contributions with interest and a benefit based on a multiple of service and salary. The origin and form of these benefits are discussed in more detail in Britt (1991).

Traditional deterministic actuarial valuation techniques do not handle these "greater of" benefits. The valuation of these benefits requires the use of a stochastic model. The obvious method to apply to this valuation problem is contingent claims valuation which has been developed in the finance literature. Such an approach also can allow the calculation of a market value for these pension liabilities provided traded assets are available to price the relevant risks. Such a market valuation of the liability will be different to the usual actuarial valuation which would usually contain margins in establishing contribution rates or solvency levels for a pension fund.

Market values of pension benefits are often needed. This arises from the increasing importance being placed on the valuation of the assets of such funds at market value. A natural consequence of requiring a market value of assets is to value the liabilities at a market value consistent with the basis used for the assets. Accounting standards for pension funds are increasingly based on the use of market values for assets and, in Australia, require the use of a market determined risk adjusted discount rate for the valuation of the fund liabilities (AAS25). The value of the liability so obtained can be interpreted as the equivalent of a market value for the liabilities.

There is also a need for market values of liabilities in order to assess the net value of a pension fund to the company sponsor. Some overseas and proposed Australian accounting standards (ED53 in Australia) require the above the line reporting of changes in the net value of a company pension arrangement. Such values are reported using values of the liabilities determined using a risk adjusted discount rate to value the accrued benefits. The economic value of the company, as reflected in its share price, should also reflect the market value of the liabilities rather than the actuarial value.

The valuation of actuarial liabilities using option pricing techniques has gained acceptance. The initial application of such techniques was to investment guarantees provided in maturity benefits of life insurance products as first discussed in Boyle and Schwartz (1976). More recently option pricing techniques have been adopted or proposed for the valuation of a range of actuarial liabilities. For example Wilkie (1989) discussed the use of these techniques in the valuation of pension payments from U.K. pension schemes. The contingent claims framework based on arbitrage free pricing has also been applied to the valuation of life insurance policy cash flows. As an example Manistre (1990) uses no arbitrage interest rate models as the basis for valuing the effect of interest sensitive withdrawals on the value of life

insurance liabilities.

This paper considers the valuation of "greater of" benefits in a multivariate contingent claims valuation framework. In conceptual terms the greater of benefits can be treated as equivalent to an option on the maximum of two risky assets along the lines of Johnson (1987) and Stulz (1982). There are however some additional issues to be considered when applying such techniques to this area. These are as follows:

- the incorporation of decrements of resignation, mortality and retirement into the calculations,

- the complication that the value of one of the benefits, namely the accumulation of contributions at the fund earning rate, can not be written in a simple form, at least not when the contribution rate is expressed as a percentage of salary, hence one of the risky assets is a complex security whose value is a function of both state variables,

- the numerical techniques that are appropriate for computing numerical values as efficiently as possible, including simulation, discrete lattice models and numerical solutions to partial differential equations,

and

- the incomplete markets problem which arises if these pension benefits are non-redundant claims in the absence of existing traded assets which exactly replicate the liability value.

The efficient numerical computation of these values is an important practical consideration.

Valuation Approach

Britt (1991) discussed the application of option pricing theory to the valuation of "greater of" benefits and, following Wilkie (1989), suggested an adaptation of the Garman-Kohlhagen (1983) formula for currency options to determine the value of these benefits. Such a formula approach fits well with the standard actuarial approach to the valuation of pension benefits which is based on a deterministic model and which is not generally suited to the valuation of option style benefits such as these greater of benefits. Britt also used simulation to value the benefits. Bell and Sherris (1991) illustrated how a simple numerical technique could potentially be used to value these "greater of" benefits using the binomial equivalent of the Margrabe (1978) approach.

Both these approaches used a device which allowed the values to be determined using the value of one of these benefit payments as the numeraire along with a simplifying assumption about the form of the benefits as a function of the state variables. This allowed the calculation of the additional cost of these benefits in terms of the normal cost of one or the other of the benefits. Neither the Britt formula nor the Bell and Sherris approach considered the incorporation of decrements such as death and resignation benefits. The Britt simulation

approach had the advantage of allowing for such decrements but the choice of the stochastic simulation parameters was not based on economic or market based assumptions.

A general contingent claims approach to the problem offers the ability to incorporate decrements into the process, to allow for the form of the greater of benefit more exactly and also to develop a theoretical basis for the choice of parameters. The approach involves developing a partial differential equation for the benefit values and solving this subject to the appropriate boundary conditions. The partial differential equation parameters are selected using arbitrage free or equilibrium pricing assumptions. Techniques that can be used to solve the partial differential equation include simulation and finite difference or lattice approximations. In general, finite difference or lattice approaches are required if the boundary conditions involve dynamic optimal decisions based on the current value of the contingent claim. Such techniques are computationally intensive where the value of the benefit or the boundary conditions are path-dependent. Simulation is likely to be computationally more efficient for such path-dependent problems but does not capture any optimal dynamic aspects of the valuation.

Form of Benefit

This paper assumes that the benefit to be valued takes the following form. For a member who joined a fund at age x, which will be taken as time 0, the benefit on exit from the fund at time s when the life is aged x+s, for cause of exit death, resignation or retirement, will be the greater of:

$X(s)$ = a fixed multiple of final salary for each year of service or part thereof

and

$Y(s)$ = the accumulation of contributions at the fund earning rate to time s

The contributions will usually be a percentage of salary which will mean that the accumulation benefit will be a function of salary. In the case that the contribution is a fixed amount or a percentage of the fund assets then this added complication would not arise.

Note that this is a simplification of this form of benefit found in practice. The greater of benefit will often only apply on retirement rather than on earlier exit, the benefit on death will usually be a fixed multiple of final salary rather than a greater of style benefit and salary is usually defined as an average salary rather than the salary at the date of exit. With the exception of this last complication the form of the benefit considered here can be readily adapted to any practical situation.

If the basis of the benefit X were a multiple of average salary then the problem of path-dependency arises since the value of the benefit will depend on the realised values of salary used to compute the average salary. This problem is identical to that which arises in

the valuation of so-called Asian or exotic options which have payoffs as functions of the average value of an asset. The theoretical approach is not changed but the derivation of closed form analytical solutions, which is often possible in the no path-dependency case, will no longer be possible and valuation will require the use of a numerical technique or simulation.

It will be assumed that the X benefit is based on final salary at the date of exit. The benefit as a multiple of final salary is then:

$X(s) = s \times k \times S(s)$

where s is service in years
   k is the salary multiple
   S(s) is the salary at time s.

The benefit in the form of an accumulation of contributions is a function of salary and the fund earning rate since the contribution rate is usually expressed as a percentage of salary.

In this case Y can be considered as the equivalent of the value of a notional security that has a negative continuous dividend equal to the contribution rate times salary. The capital value of this notional security grows at the fund earning rate. If the fund is invested in a diversified portfolio, as is generally the case for such pension funds, then the growth in the capital value of the notional security will be the same as the growth rate for the index value of a diversified portfolio. There is therefore the need to value a security whose capital value grows like a market index and whose dividend is a percentage of salary. Note that the form of this benefit implies that it is a function of the history of the salary state variable and that it is path- dependent. This path-dependency indicates that numerical techniques will be required to accurately compute the value of the greater of benefit.

Theoretical Value of Benefits

Allowing for decrements
At first sight the allowance for benefit payments on death, resignation or retirement might not be obvious. This situation is the equivalent of early exercise of an American style option in option pricing. In the option case the early exercise is assumed to be based on a rational decision based on the payoff from exercising the option and the then current price of the option. Thus the early exercise of an American option is an optimal dynamic decision. In the pension fund case the death of a life and receipt of the greater of benefit is not assumed to be dependent on the payoff of the option. If it is assumed that the same applies for withdrawal and early retirement then incorporation of decrements into the calculation turns out to be relatively straightforward. To illustrate the procedure consider a simple benefit in the form of a fixed death benefit payable on death in the pension fund.

The standard calculation of actuarial liability values is based on the assumption that interest rates are non-stochastic. In this case the value of a death benefit on a life of current age x

of fixed amount D payable on death would be given by the solution to the non-homogeneous first order linear differential equation

$$\frac{dV}{dt} = r(t) V - {}^{d}\mu_{x+t} (D-V)$$

$$= \left[r(t) + {}^{d}\mu_{x+t} \right] V - {}^{d}\mu_{x+t} D$$

where $r(t)$ is the force of interest or instantaneous interest rate at time t and ${}^{d}\mu_{x+t}$ is the force of mortality or instantaneous death rate at age $x+t$.

The solution to this differential equation is

$$V(0) = \int_{0}^{\bullet} v_{i}(s) \, v_{d}(s) \, {}^{d}\mu_{x+s} D \, ds$$

where

$$v_{i}(s) = \exp \int_{0}^{\bullet} -r(u) \, du$$

and

$$v_{d}(s) = \exp \int_{0}^{\bullet} -{}^{d}\mu_{x+u} \, du$$

Notice that this value takes the form of the integral (or sum) of the expected payments at each time t multiplied by discount functions which allow for both interest and mortality to time t.

The value can also be expressed as

$$V(0) = \int_{0}^{\bullet} q(s) \, v_{i}(D, s) \, ds = E_{s} [v_{i}(D, s)]$$

where $q(s) = v_{d}(s) {}^{d}\mu_{x+s}$ is the probability density of the random variable time till death s for a life aged x (Bowers et al, 1986) and $v_{i}(D,s)$ is the present value of the benefit assumed paid with certainty at time s allowing for interest only and ignoring mortality. E_{s} is the expectation

operator with respect to the random variable s = age at death.

In general, where the decrement rate is independent of the state variables and the value of the benefit then the value can be written in this form. This is most likely to be the case for pension benefits. The only exception would be for the withdrawal assumption and the early retirement assumption where the decision to withdraw or retire could be assumed to be based on a dynamic optimisation decision. Such behaviour could be modelled approximately by expressing the withdrawal or retirement rate as a function of the state variables. If this was to be done then the partial differential equation for the benefit value would incorporate the decrements directly.

The implementation of the contingent claims approach used in this paper is based on the traditional life contingencies approach for incorporating decrements which is to assume that decrement rates vary only by age. The benefit value for each age at exit is then derived by treating it as a European style option. The value of the benefit can then be derived by calculating the expected value of the conditional expected benefit values with respect to the time to exit random variable.

Allowing for stochastic interest rates
So far the interest rate has been assumed fixed and known. If we let the interest rate be a random variable then an analytical formula for the value of the benefit can be determined by using a term structure model. This is an area of considerable current research in both the finance and actuarial literatures. Two relatively simple models that produce an analytical formula are those of Vasicek (1977) and Cox, Ingersoll, and Ross (1985b). These are also covered in Manistre (1990). See also Boyle (1978). The Vasicek model is derived from arbitrage free considerations only whereas the Cox, Ingersoll and Ross model is based on an underlying equilibrium model with restrictions on preferences and the stochastic assumptions used to represent the economy. A brief summary of the Cox, Ingersoll and Ross results is given in Appendix One.

The general arbitrage free valuation results of Harrison and Kreps (1979) and Harrison and Pliska (1981) provide a framework for deriving the value of the greater of benefit. The value of a fixed benefit payment of amount D at time s, denoted by V, will be assumed to be a function of only the spot or instantaneous interest rate r and time t. There is only one state variable, the interest rate, in this valuation problem.

If it is assumed that the spot interest rate follows the stochastic differential equation

$$dr = \mu(r,t)dt + \sigma(r,t)dZ$$

where dZ is a standardised Wiener process, then from the application of Ito's Lemma the value of D, V(r,t) follows the stochastic differential equation

$$dV = (V_t + \mu(r,t)V_r + \tfrac{1}{2}\{\sigma(r,t)^2\}V_{rr})dt + \sigma(r,t)V_r dZ$$

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where subscripts denote partial differentials of V. In this paper subscripts will usually denote partial differentials with respect to the variable(s) in the subscript. There are no cash flows associated with the benefit D before it is paid and the boundary condition is $V(r,t) = D$ for $t = s$. The value required is $V(r(0),0)$.

If M is an instantaneously riskless money market account accumulating at rate r then

$$M(s) = \exp \int_0^s r(u)\, du$$

and $M(0)/M(s)$ represents the amount that should be invested to accumulate to a unit amount at time s. This is not the same as $V(r(0),0)$ with $D = 1$, unless $\sigma(r,t) = 0$, since this is assumed not observable at time 0 and is to be derived.

If the market is arbitrage free then there exists an equivalent probability measure under which the process $V(r,t)/M(t)$ is a martingale. If this probability measure is unique then the market is said to be complete and the value of the fixed payment D can be written uniquely as

$$E^*[V(r,s)/M(s)]$$

where expectations are taken with respect to the equivalent probability measure.

The change to the drift term of $V(r,t)$ to ensure that $V(r,t)/M(t)$ is a martingale involves setting the expected return on V to the instantaneous riskless rate and the value of V is then calculated as the discounted expected value of $V(r,s)$ with respect to these altered dynamics of V. The discounting is carried out using the ratio of money market account values $M(0)/M(s)$.

This procedure is the 'risk-neutral' valuation approach used in option pricing. It can also be considered as the certainty equivalent valuation approach discussed in finance texts.

If risk free government bonds are available with maturities corresponding to the dates of exit and payment of benefit cash flows then it is not necessary to use a term structure model since the price of interest rate risk and hence the discount rate for the benefit cash flows can be derived directly from the traded bonds.

Greater of Benefit
The approach that is considered in this paper requires the value of the greater of benefit payable for each age at exit. These values can be determined by treating the greater of benefit for each age at exit as a European style option. This allows the calculation of the expected value of the benefit using risk neutral dynamics which is then present valued using risk free government bond rates.

The greater of benefit can be valued using an arbitrage free contingent claims approach on the assumption that the value of the benefit is a function of two state variables. These are the earnings rate on the contributions in the fund and the growth rate in salary. The earnings rate

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on the fund will be taken as the earnings rate on a diversified portfolio that is representative of the asset portfolio of a pension fund. The other complication is that the boundary condition will include the path-dependent value of a hypothetical security dependent on these same two state variables.

These two state variables are the only sources of uncertainty considered. Interest rate uncertainty will no longer be incorporated for ease of exposition. The two sources of uncertainty will in effect be assumed to dominate any interest rate uncertainty. As already mentioned there is some justification for setting aside interest rate uncertainty if it is assumed that marketable bonds are traded for the required maturities so that interest rate risk can be priced in the discounting of the expected benefit payments. A market based spot rate for a bond with the same maturity as the benefit payment would then be used instead of the ratio of values of the money market account to discount expected benefit payments.

Assume that the salary S follows the stochastic differential equation

$$dS = \mu_s S dt + \sigma_s S dZ_s$$

and that F the value of an amount credited with the fund earnings rate follows the stochastic differential equation

$$dF = \mu_f F dt + \sigma_f F dZ_f$$

with $dZ_s dZ_f = \rho_{fs} dt$

where ρ_{fs} is the instantaneous correlation coefficient between the standardised Weiner processes dZ_s and dZ_f. It is assumed that dZ_s and dZ_f generate the only uncertainty allowed in the model.

This implies that the values of S and F are bivariate log-normally distributed.

It is possible to write the stochastic differential equation for F as

$$dF = \mu_f F dt + \sigma_f F(\rho_{fs} dZ_s + \sqrt{(1-\{\rho_{fs}\}^2)} dZ^*)$$

where dZ_s and dZ^* are independent Wiener processes. This result is useful for the numerical evaluation of the greater of benefit and for simulation of the processes.

The value of the benefit will be a function of F,S and t and Ito's lemma gives

$$dV = V_s dS + V_f dF + V_t dt + \frac{1}{2}\{V_{ss}\sigma_s^2 + V_{ff}\sigma_f^2 + 2V_{sf}\sigma_s\sigma_f\rho_{sf}\}dt$$

$$= \{\mu_s S V_s + \mu_f F V_f + V_t + \frac{1}{2}(V_{ss}\sigma_s^2 + V_{ff}\sigma_f^2 + 2V_{sf}\sigma_s\sigma_f\rho_{sf})\}dt$$
$$+ \sigma_s S dZ_s + \sigma_f F dZ_f$$

Note that the state variables F and S are not traded assets. If we assume that the benefit is a redundant contingent claim then this implies that traded securities exist that provide perfect hedges against the uncertainty that arises from S and F. Denote the value of these assets by A and C respectively with dynamics

$$dA = \mu(S,t)dt + \sigma(S,t)dZ_s$$

$$dC = \mu(F,t)dt + \sigma(F,t)dZ_f$$

Applying Ito's lemma to A and C and allowing for any expected cash flows on the securities gives the form of μ and σ in each of these expressions. Assuming that the rate of cash flow on these assets is $D(S,t)$ and $D(F,t)$ respectively then

$$\mu(S,t) = \mu_s SA_s + A_t + \tfrac{1}{2}A_{ss}\sigma_s^2 S^2 + D(S,t)$$
$$\sigma(S,t) = A_s\sigma_s S$$

$$\mu(F,t) = \mu_f FC_f + C_t + \tfrac{1}{2}C_f\sigma_f^2 F^2 + D(F,t)$$
$$\sigma(F,t) = C_f\sigma_f F$$

It is also assumed that an instantaneously riskless bond is traded and the value of the bond is denoted by B with dynamics

$$dB = rBdt$$

The benefit can be valued by deriving a partial differential equation for V using the no-arbitrage valuation approach. This is solved subject to the appropriate boundary conditions. For a fixed age at benefit payment s the boundary condition is that

$$V = \max\{X,Y\}$$

where
$$X = ksS$$
and
$Y(A,C)$ is the solution of the stochastic differential equation

$$dY = Y_A dA + Y_C dC + Y_t dt + ySdt +$$
$$\tfrac{1}{2}\{Y_{AA}\{A_s\sigma_s S\}^2 + Y_{CC}\{C_f\sigma_f F\}^2 + 2Y_{AC}\rho_{sf}A_s\sigma_s SC_f\sigma_f F\}dt$$

where y is the contribution rate as a fraction of salary S. It is assumed that y is a fixed and known rate set by the actuary to the fund.

The partial differential equation for V is derived as follows. Express V as a function of asset values A and C rather than the underlying state variables.

$$dV = V_A dA + V_C dC + V_t dt +$$
$$\tfrac{1}{2}\{V_{AA}\{A_s\sigma_s S\}^2 + V_{CC}\{C_f\sigma_f F\}^2 + 2V_{AC}\rho_{sf}A_s\sigma_s SC_f\sigma_f F\}dt$$

Construct a portfolio of traded assets A, C and B with proportions m of A, n of C and (1-m-n) of B. The hedge portfolio is required to replicate V and the return on the portfolio follows the stochastic differential equation

$$dV = n(dA/A)V + m(dC/C)V + (1-n-m)rVdt$$

Select m and n so that
$$n = V_A A/V$$
$$m = V_C C/V$$
and equate the two expressions and divide by dt to get the partial differential equation for V
$$V_t + \tfrac{1}{2}\{V_{AA}\{A_s\sigma_s S\}^2 + V_{CC}\{C_f\sigma_f F\}^2 + 2V_{AC}\rho_{sf}A_s\sigma_s SC_f\sigma_f F\}$$
$$= rV - rV_A A - rV_C C$$

In this partial differential equation the mean returns on the traded assets A and C do not appear. The no-arbitrage requirement results in the risk free rate r being substituted for the mean returns on both the hedge assets.

A numerical technique is required to solve this partial differential equation. The use of these numerical techniques is discussed in Hull and White (1990). There is an additional consideration in this problem not normally found in the contingent claims in financial markets. This is that the benefit Y is function of the history of the state variable S so that the boundary condition for age at exit s is not a simple function of the then current values of the state variables.

A more direct alternative approach is to use the contingent claim general arbitrage free valuation results. The procedure is to transform the drift terms on the replicating asset dynamics to the risk free rate. These transformed dynamics are used to calculate the expected value of the benefit E[max{X,Y}] for each age at death and this expected value is present valued using the market determined risk free spot rate for bonds maturing on the date of the benefit payment. These spot rates are assumed available from market data on traded bonds as mentioned earlier. The expected value of these benefit payments with respect to the age at death random variable is then calculated to obtain the value of the benefit.

It is important to recognise that the drift terms on the replicating assets A and C are set to the risk free rate and not those on the state variables S and F. The benefit payments are defined in terms of the state variables and in practice the parameters of these state variables would be estimated from available data. The transformed parameters for the state variables are determined from the relationship between the drift terms on the assets A and C and those for S and F.

In practice the cost of the benefit is expressed as a percentage of the salary. To do this it is also necessary to calculate the present value of 1% of the salary. Since it is assumed in the

contingent claims framework that markets are complete this implies that the asset A provides a complete hedge for salary uncertainty and the future level of salary can be related to the asset value A. The dynamics of A are used to value a stream of payments of 1% of the salary S.

Numerical Techniques

The valuation of the benefit in practice will require the use of a numerical technique either to numerically solve the partial differential equation for the benefit value or to evaluate the expected value of the benefit using the contingent claims valuation technique. The contingent claims approach is implemented using a discrete time lattice representation of the underlying state variables. Some of the standard numerical techniques used to solve the partial differential equation are equivalent to using the lattice approach.

An important practical consideration is the selection of an efficient computational technique since values are required for a range of current ages and for each current age a value is required for each age at death in order to calculate the expected value of the benefit allowing for decrements. In order to numerically evaluate the benefit value it will be necessary to calculate the expected value for a set range of ages at death and the usual practice in actuarial calculations would be to do this for integral ages up to and including the final age for retirement.

The technique used should satisfy the following requirements:
- rapid convergence to the solution of the differential equation as the discrete time interval tends to zero,
- values which converge to the unique continuous time complete markets value.
- simple and efficient numerical computation of values.

On these grounds Hull and White (1990) demonstrate that the explicit finite difference method with a transformation of the state variable process to ensure time and state independent volatility meets most of these criteria for the single state variable case. The problem with this method is that the numerical solution does not necessarily converge. They demonstrate the equivalence of this method to a trinomial lattice approach as well as a binomial lattice approach. In the two state variable case the transformation suggested is one which produces constant volatility parameters for both state variables and with the transformed variables uncorrelated. The resulting lattice has nine branches from each node.

The literature on discrete time approximations for the continuous time dynamics include Boyle (1990), Boyle, Evnine and Gibbs (1989) and Rajasingham (1990). As discussed in Rajasingham it is not necessary for the multivariate lattice to complete markets in order to ensure convergence to the continuous time value. The important consideration should be efficiency and rapid convergence for practical applications of the technique. For exposition purposes the complete markets approach is still the most appropriate. It is however an open question as to which of the many possible lattice structures and associated choices of jumps and jump probabilities in the multivariate problem have the most rapid convergence.

In the complete markets case with two state variables the lattice structure will require three branches. Hence the lattice for S and F would be as follows:

Values for S_{t+1}, F_{t+1}

$$[\delta_{1s}S_t, \ \delta_{1f}F_t] \quad \text{w.p. p}$$

$$/$$

$$[S_t, F_t] - \quad [\delta_{2s}S_t, \ \delta_{2f}F_t] \quad \text{w.p. q}$$

$$\backslash$$

$$[\delta_{3s}S_t, \ \delta_{3f}F_t] \quad \text{w.p. (1-p-q)}$$

Each of the lattice branches can be written in terms of an increase at the risk free rate plus a jump. He (1990) uses equal probabilities of 1/3 for each branch and selects the jumps to have the required means, variances and correlations. This is done by expressing the jumps as a function of a basis of uncorrelated discrete processes. The basis has the property that the means are zero, the variances are unity and the processes are uncorrelated i.e they are orthogonal martingales. In the two state variable case this is represented as

Jump sizes
State Variable

One	Two	
$\surd(3/2)$	$1/\surd 2$	w.p. 1/3
0	$-2/\surd 2$	w.p. 1/3
$-\surd(3/2)$	$1/\surd 2$	w.p. 1/3

This ensures convergence to the continuous time complete markets value. For n time intervals the number of nodes on the lattice will be $(n+2)(n+1)/2$.

The Boyle, Evnine and Gibbs (1989) approach does not use the complete markets representation. Each state variable can go up or down resulting in four branches in the lattice from each node. They assume the state variables are multivariate log-normally distributed and select the jumps and probabilities by equating the moment generating functions for the discrete time and continuous time distributions. The jump sizes are chosen using an extension of the Cox, Ross and Rubinstein single state variable approach so that the jump size is $\exp(+\sigma\surd h)$ for an up jump and $\exp(-\sigma\surd h)$ for a down jump where h is the interval size. The probabilities are no longer equal. The lattice is structured so that it recombines at nodes allowing for more efficient numerical calculations. The number of nodes on the lattice for n time intervals is $(n+1)^2$.

If the multivariate lognormal distribution assumption is not used then the lattice structure is constructed by selecting jump parameters determined from the equivalent martingale basis. For four branches from each node on the lattice the equivalent orthogonal martingale basis with equal probabilities is as follows.

Jump sizes
State Variable
One   Two
 1   1   w.p. 1/4
 1   -1   w.p. 1/4
-1   1   w.p. 1/4
-1   -1   w.p. 1/4

As already indicated it is not necessary to use the complete markets lattice for computational purposes and increasing the number of branches has the potential to reduce the number of calculations required to obtain a given accuracy since the number of steps for calculation in the discrete approximation can be reduced. Increasing the number of branches can aid numerical calculation but the issue of convergence to the complete markets value needs consideration. This has been handled currently by assuming multivariate lognormal continuous time distributions for the state variables or by generating jumps with equal probabilities from an orthogonal martingale basis. The best technique is yet to be determined and Boyle (1990) indicates this is an area of current research.

In the "greater of" benefits case the lattice will not recombine since the benefit value is path-dependent. This leads to a very large number of nodes and computational difficulties unless a coarse grid is used.

Use of Simulation
Simulation was used by Britt (1991) to estimate the value of the greater of benefit. This technique involves a sampling of the total possible paths for the continuous time multivariate state variables. Monte Carlo techniques should be used to speed the computation. The conditions for the use of the simulation technique apply in the greater of benefits case since it can be assumed that there are no optimal dynamic decisions involved in the pension benefit payment. Decrement rates can be modelled as state dependent and readily incorporated in the simulation approach.

Although it is an open question as to whether simulation using Monte Carlo techniques is likely to be a more efficient calculation method for the benefit values than the numerical techniques based on discrete approximations or lattice models it appears probable that, with the path-dependency of this valuation problem, this will be the case. One approach would be to implement a modified version of the option on the maximum formula suggested by Britt (1990) as an estimate of the required expected value and to use the control variate technique suggested by Boyle (1977).

It is interesting to note that the simulation technique has not been discussed much in the multivariate contingent claims literature. It is a technique which is used in practice in the valuation of mortgage backed securities where prepayment rates are modelled as state dependent.

Conclusion

This paper discusses the conceptual issues in applying arbitrage free contingent claims valuation techniques to the valuation of "greater of" benefits in pension funds. It suggests a technique for incorporating decrements that is related to the stochastic approach to life contingencies found in actuarial texts. This allowance for decrements is based on the assumption that the decrements are independent of the state variables. Decrements can be directly incorporated into the partial differential equation for the benefit value and can be allowed to be state dependent. The assumption that these benefits are redundant claims whose value depends on state variables which are priced in traded asset markets allows an arbitrage free valuation approach. In practice this is not likely to be the case and a market price of risk for the state variables will need to be incorporated and estimated.

Path-dependency is a feature of the benefit value since the accumulation of contributions as a percentage of salary at the fund earning rate will result in the value being a function of the path of the salary state variable. This results in computational advantages in using a simulation approach to solve the partial differential equation for the benefit value. Numerical techniques which use finite differences or lattices can not take advantage of the more efficient techniques usually used to implement them in practice because of this path-dependency.

Efficient computation of these values is a significant practical issue since the number of calculations in valuing these benefits for fund members is equivalent to that required in the evaluation of a very large number of options. The factors involved in determining the most efficient computational technique have been discussed. This remains an issue for further research.

Appendix One

For the Cox, Ingersoll and Ross model, the spot rate r is assumed to follow the partial differential equation

$$dr = k(\mu - r)dt + \sigma\sqrt{r}dZ$$

where Z is a standardised Weiner process and the variance of the spot rate is proportional to the level of interest rates. The condition $2k\mu/\sigma^2 \geq 1$ is required to ensure non-negative interest rates. r has a conditional non-central chi-squared distribution.

Application of Ito's lemma gives

$$dV = (V_t + k(\mu-r)V_r + \tfrac{1}{2}\{\sigma^2 r\}V_{rr})dt + \sigma\sqrt{r}V_r dZ$$

and in general the values of payments dependent on the short interest rate r are (locally) perfectly correlated and their dynamics can be written as

$$dV = \mu(r,t)Vdt + \sigma(r,t)VdZ$$

For there to be no arbitrage the instantaneous expected returns on payments at different dates will take the form

$$\mu(r,t) = r + m(r,t) \times \sigma(r,t)$$

where r is the instantaneous risk free rate and m(r,t) is the risk premium factor which is the instantaneous market price of interest rate risk.

Cox, Ingersoll and Ross assume that $m(r,t)=m\sqrt{r}/\sigma$ and $\sigma(r,t)=\sigma\sqrt{r}V_r/V$ so that

$$rV + mrV_r = (V_t + k(\mu-r)V_r + \tfrac{1}{2}\{\sigma^2 r\}V_{rr})$$

which is a partial differential equation for the value of any payment whose value depends only on the instantaneous interest rate r and t. With boundary condition $V(r,s)=D$ this partial differential equation has a closed form solution given by

$$V(r,s) = D.A(s).\exp\{-B(s)r\}$$

where $A(s) = [\{\Phi_1\exp(\Phi_2 s)\}/\{\Phi_2(\exp(\Phi_1 s)-1) + \Phi_1\}]^{\Phi_3}$

   $B(s) = [\{\exp(\Phi_1 s)-1\}/\{\Phi_2(\exp(\Phi_1 s)-1) + \Phi_1\}]$

   $\Phi_1 = [(k+m)^2 + 2\sigma^2]^{\frac{1}{2}}$ $\Phi_2 = [k + m + \Phi_1]/2$ $\Phi_3 = 2k\mu/\sigma^2$.

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