ACTUARIAL RESEARCH CLEARING HOUSE
1993 VOL. 1

RUIN THEORY BEYOND CHAPTER 12

Hans U. Gerber, University of Lausanne

(Summary of the talk given at the 27th Actuarial Research Conference, University of Iowa, August 7, 1992)

1. Ruin theory and complex numbers

In section 12.6 of *Actuarial Mathematics* the probability of ruin is calculated for the case where the claim amount distribution is a mixture of exponential distributions. The resulting probability of ruin is of the form

$$\psi(u) = C_1 e^{-r_1 u} + \cdots + C_n e^{-r_n u} .$$

A combination of exponential distributions is formally like a mixture of exponential distributions, except that now some of the "weights" may be negative. The family of combinations of exponential distributions is quite rich and is a useful tool to model the claim amount distribution. Then the probability of ruin is still of the same form, but now some of the C_k's and r_k's may be complex numbers. Thus the actuary should learn what his pocket calculator already knows: how to do calculations in complex mode!

Now suppose a claim amount distribution that is gamma, with shape parameter n and scale parameter β . This distribution is in the closure of the family of combinations of exponential distributions. So it is not surprising that the probability of ruin is still of the same form. The r_k's are the solutions of the equation

$$\sum_{k=1}^{n} \left(\frac{\beta}{\beta-r}\right)^k = n(1+\theta) ,$$

where θ is the relative security loading, and the C_k's are given by the formula

$$C_k = \frac{\theta(\beta-r_k)}{(1+\theta)(n+1)r_k - \beta\theta}$$

(Gerber, 1992). For a first illustration the reader should revisit exercise 12.18 of *Actuarial Mathematics*. For a second we assume n=3 , β =1, θ =.5 . Then

r_1 = .178258 $\quad$ C_1 = .721398

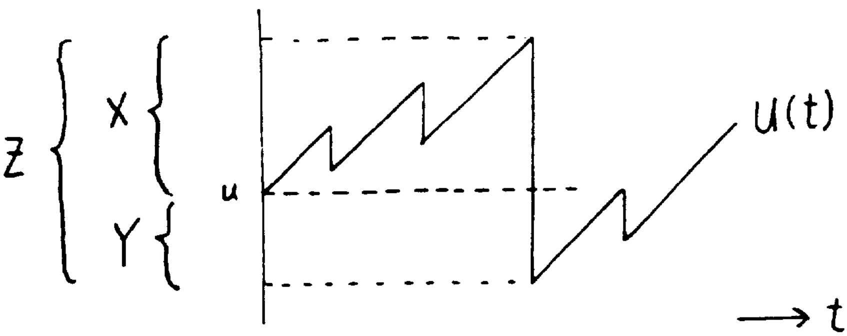
r_2 = 1.299760 + .424938 i , $\quad$ C_2 = -.027366 - .019551 i

r_3 = 1.299760 - .424938 i , $\quad$ C_3 = -.027366 + .019551 i

2. More insight for Theorem 12.4

We are interested in the probability of the event that the surplus will ever fall below the initial level, and if it occurs, how it will occur. We denote by u+X the surplus immediately before it drops below u for the first time, by u-Y the surplus immediately afterwards, and by Z=X+Y the amount of the corresponding claim. Dufresne and Gerber (1988) show that the defective joint pdf of X and Y is the function $\frac{\lambda}{c}\, p(x+y)$, which reveals a surprising symmetry. By integrating once we obtain the common (defective) pdf of X and Y, and in particular Theorem 12.4. Furthermore, the (proper) pdf of Z is $\frac{1}{\mu}\, z\, p(z)$, which confirms the intuitive idea that the distribution of Z must be different from the distribution of an arbitrary claim amount.

Generalizations and related results can be found in forthcoming papers by Dickson (1992) and Reis (1992).



3. The surplus process "enriched" by a Wiener process

Dufresne and Gerber (1991) add an independent Wiener process (variance 2D per unit time) to the surplus process of chapter 12. Because of the oscillating nature of the sample paths, the probability of ruin with no initial surplus is now one, and for a positive initial surplus, ruin can occur in two ways, by oscillation or because of a claim. The probability of survival satisfies a certain defective renewal equation, from which the following result is obtained:

$$1-\Upsilon(u) = \sum_{n=0}^{\infty} (1-a)\, a^n \, H_1^{*(n+1)} * H_2^{*n}(u)$$

with

$$a = \frac{\lambda p_1}{c}\, , \quad H_1(x) = 1 - e^{-\left(\frac{c}{D}\right)x}$$

$$H_2(x) = \frac{1}{p_1}\int_0^x [1-P(y)]\, dy$$

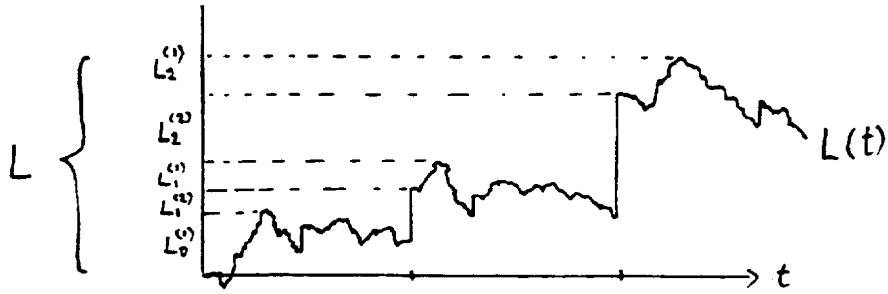
This generalizes *Beekman's convolution formula*, which is the special case D=0.

The formula has a probabilistic interpretation in terms of the distribution of the maximal aggregate loss L. This random variable can be decomposed as follows:

$$L = L_0^{(1)} + L_1^{(2)} + L_1^{(1)} + \ldots + L_N^{(2)} + L_N^{(1)} \quad .$$

Here N is the number of record highs of the aggregate loss process that are caused by a claim. Evidently, N has a geometric distribution; surprisingly, its parameter does not depend on D. If we denote by M_k the kth record high that is caused by claim ($M_0=0$), we can decompose $M_k - M_{k-1}$ as $L_{k-1}^{(1)} + L_k^{(2)}$, where $M_{k-1} + L_{k-1}^{(1)}$ is the intermediate record high. The common distribution function of the $L_k^{(1)}$'s is $H_1(x)$ as defined earlier; surprisingly, it is independent of the specific form of the compound Poisson process. The common distribution of the $L_k^{(2)}$'s is $H_2(x)$ as defined earlier; surprisingly, it is the same as in the classical case (D=0).

The definitions are illustrated in the following figure, which shows a typical sample path of the aggregate loss process, and which generalizes figure 12.6 of *Actuarial Mathematics*.



__References__

Dickson, D.C.M. (1992). On the distribution of the surplus prior to ruin. Forthcoming in Insurance: Mathematics & Economics.

Dufresne, F. and H.U. Gerber (1988). The surpluses immediately before and at ruin, and the amount of the claim causing ruin. Insurance: Mathematics & Economics 7, 193-199.

Dufresne, F. and H.U. Gerber (1991). Risk theory for the compound Poisson process that is perturbed by diffusion. Insurance: Mathematics & Economics 10, 51-59.

Gerber, H.U. (1992). On the probability of ruin for infinitely divisible claim amount distributions. Forthcoming in Insurance: Mathematics & Economics.

Reis, A.E. dos (1992). How long is the surplus below zero? Forthcoming in Insurance: Mathematics & Economics.