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A REVIEW OF *ACTUARIAL MATHEMATICS* AND THE SOCIETY OF ACTUARIES COURSE 150

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I. INTRODUCTION

Actuarial mathematics is the expertise which distinguishes ours among the business professions. Lately there have been indications from some Society of Actuaries members that the importance of actuarial mathematics relative to other skills (communication is the skill mentioned most frequently) was on the decrease. Those who rely on our expertise probably have a much different view. The world in which we work is becoming more complex, not less. Mathematical models and techniques we use to deal with risk are becoming more mathematical, not less. While the demand for mathematical talent seems to be increasing, fewer students are preparing for careers in the mathematical disciplines[1]. The need for fundamental mathematical skills is not likely to decrease, regardless of how much our membership might wish it to do so. This implies that the nature of actuarial work is going to continue to be fundamentally mathematical, and that the competition for mathematically talented students will continue to be strong. I was glad to see Irwin Vanderhoof's letter in *The Actuary* ("Do we want to be more like accountants?", [15]). I certainly agree that the we can expect the need for technical

[1]This is reported by James A. Voytuk in "A Challenge for the Future," UME Trends, News and Reports on Undergraduate Mathematics Education (December 1989). Voytuk reports that the need for mathematically trained workers is increasing and that shortages for mathematical sciences faculty are projected.

expertise to increase. The actuarial educational system clearly plays a key role in attracting new students and in conveying the nature of the actuarial profession to the public. The textbooks used by the profession are probably the most important part of the system and the public can reasonably infer that mathematics is an important part of actuarial work based on the texts in use.

For Society of Actuaries members, actuarial mathematics currently refers to life contingencies and risk theory, which are the subjects of the Society sponsored text, *Actuarial Mathematics* ([3]). Related topics such as survival models, demography, and mathematics of actuarial tables are ancillary, but also are included in the realm of actuarial mathematics. The topics such as credibility and loss distributions, which are emphasized on examinations of the Casualty Actuarial Society, are also included in the current scope of actuarial mathematics. In addition, I would include some of the mathematics of modern finance which actuaries are helping to develop. However, the subject of this article is limited to the current cornerstone of the Society of Actuaries education: life contingencies, the sole topic of the examination titled Course 150 Actuarial Mathematics.

This article may be considered a review of the new textbook *Actuarial Mathematics* and its role in actuarial education. One of the most important points established here will not surprise readers of this review who have examination experience, either as students or teachers, with both *Actuarial Mathematics* and Jordan's *Life Contingencies* ([8]). The new life contingencies text is much more demanding and requires more time to master because of the greater complexity of the models involved. However, the scope of the life contingencies examination has actually narrowed in terms of the variety of topics. The Society of Actuaries education policies now allow students access to recent examinations and solutions, which is of great value in preparing for the examination and learning the subject. Secondary purposes of this review are to suggest some alternatives to the textbook presentation of several topics.

There are several book reviews of *Actuarial Mathematics* already in the mathematics and actuarial literature ([5], [6], [7], [9], [10], [11], [13], [14], [15]). All of these survey the topics, so it is not necessary to include a survey here. The best of these in terms of summarizing the contents is Luckner's [12]. All of these reviews are quite positive, but it is not clear if the reviewer has used the book in the classroom. Cecil Nesbitt wrote me that Jean Lemaire's reviews ([9], [10], [11]) are based on classroom use. Lemaire is very enthusiastic about *Actuarial Mathematics*, describing it as monumental and pedagogically impeccable. I share this enthusiasm and welcome the replacement of Jordan by such a superb exposition. However, it has been exceedingly difficult to prepare students for the life contingencies examination based on this text.

Part of the trouble has been the earlier, exclusive use of multiple choice questions on the Society's examination. Introduction of essay questions has helped a great deal. Also, in the past, examiners frequently set questions which were based on arcane and practically useless facts. It has been difficult to convince students that the Society is truly interested in education at times.

The most recently released examination (May 1990) is quite good in my opinion. Also, the recent publication of commutation functions to be used on examinations has allowed examiners to choose more practical problems. The recent questions are relevant to the material and do not rely on obscure minor points. This is certainly a welcome change in the Society's education policy.

Life contingencies has served as the common, fundamental educational experience for Society of Actuaries members. Jordan's *Life Contingencies* provided those of us who were examined on its contents not only a common language but a common experience as well. Jordan was used long after it became technically inadequate, perhaps because its value in these regards was enough to overcome its technical inadequacies. The Society of Actuaries sponsored successor *Actuarial Mathematics* by Bowers *et al* continues the Jordan tradition, providing a common experience and language. It also overcomes many of Jordan's inadequacies. We are very fortunate to have such a fine textbook to rely upon for the fundamental education of members of the Society of Actuaries. First Jordan and now Bowers *et al* have provided a necessary ingredient for the successful development of actuarial science programs at many universities and the subsequent introduction of many students to actuarial careers supported by the Society of Actuaries (contrast this to the situation with regard to casualty actuarial science[2]).

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[2]The Casualty Actuarial Society recently published *Foundations of Casualty Actuarial Science* [1]. From the preface, we find that it is "intended as an introduction to casualty actuarial concepts and practices..(for).. members and students of the CAS, university and college students ... ." It took nine authors, and numerous other actuaries twenty years to produce this text. It consists of nine individually authored chapters, with no exercises. The lack of a

II. THE NEW BOOK AND THE EXAM

An important thesis of this paper it that *Actuarial Mathematics* differs drastically from *Life Contingencies* with regard to its suitability for self-study. This is an important consideration for actuaries who are supervising an actuarial student who is preparing to write the life contingencies examination. The new approach requires more complex models and relies on probability and statistics as well as calculus and numerical analysis to a much greater extent than the former approach. As a result, students must prepare a much more complex body of material than was the case when Jordan was the basis of the Society's life contingencies examination. For example, a current problem typically involves means and variances of a present value random variable, whereas it formerly would involve only the mean. It is simply going to take more time for a student to master the material. The advantage of learning life contingencies in an academic setting is even greater than it was with Jordan's text. Students working on their own in an office, with no help from anyone who has passed a recent life contingencies examination, are at an even greater disadvantage. The new approach, which is absolutely required for modern actuarial work, relies on mathematical and statistical sophistication well beyond that required for Jordan. For example, in addition to calculus and numerical analysis found in *Life*

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casualty text has been a significant barrier to the inclusion of casualty topics in university and college curricula. I hope the new casualty text proves to be useful in development of casualty actuarial science courses at universities and colleges, and perhaps within the Society of Actuaries Associateship syllabus.

Contingencies, students of *Actuarial Mathematics* also must be prepared to use probability theory and mathematical statistics as they learn life contingencies. Of course, the Society's curriculum has prepared students for this, but now students must be able to use this material on the life contingencies examination.

SOME MEASURES OF THE DIFFERENCE

	Jordan	Bowers, *et al*
Number of		
Pages	390	624
Chapters	16	19
Ounces	26	75
Exercises	405	507
US dollars	25	65
SOA Life Contingencies Syllabus Number of		
Pages	390	353
Chapters	16	10
Ounces	26	?
Exercises	405	338

This table gives some interesting numbers for the two texts. Of course, it does not indicate the level of sophistication, but nevertheless shows that in there is some crude support for the notion that the current examination is actually easier to prepare for than the Jordan based examination.

Another development in the user's favor is an ample supply of good ancillary material. For examples we have the manual by Crofts *et al* ([4]) and, recently, the study guide by Batten ([2]). Perhaps even more important is the Society's new policy of releasing examinations and solutions. The supplemental materials and the Society's increased interest in education have been of great benefit, especially to those students who have to study alone. Indeed, students have frequently mentioned that the manuals, examinations and solutions have helped clarify certain text topics. The next two sections deal with some of these minor technical issues.

III. COMPLETE AND APPORTIONABLE ANNUITIES

The *Actuarial Mathematics* treatment of complete annuities-immediate and apportionable annuities-due is completely deterministic. Students are supposed to develop the present value random variables for these annuities as solutions to exercise 5.31. That is, the concept of present value random variable is dropped in this section. I have found that students need a great deal of help with this. Perhaps leaving this as an exercise is asking too much of the student. Here is an approach which is consistent with the earlier random variable approach and takes no more space to present than the deterministic discussion.

Complete Annuities-Immediate

Consider an immediate annuity of 1 per year payable at the end of each year $(k, k+1]$, should (x) survive to age $x+k+1$, together with a death benefit b_t paid at the moment of death. Just after an annuity payment the death benefit is zero; thereafter it gradually increases. Just before an annuity payment, its value is 1. Thus the complete annuity-immediate is exactly like an immediate annuity in each year which the annuitant survives. In the year of death, it pays a death benefit at the moment of death. The moment of death $T(x)$ occurs in year $K(x)+1$, so that $T(x) = K(x) + S$ with $0 < S \le 1$. The death benefit is defined to be the accumulated value of a continuous payment annuity, paid at an annual rate of payment c, over the period the annuitant lived in the year of death. Thus $b_{T(x)} = c\bar{s}_{\overline{S|}}$. Since the death benefit is required to tend to the annuity payment of 1, then we see that $c\bar{s}_{\overline{1|}} = 1$ and hence $c = \delta/i$. Therefore, we can describe the present value random variable $\mathring{Y}_x$ for the complete annuity-immediate (with one payment per year) as follows:

$$\mathring{Y}_x = a_{\overline{K(x)|}} + \left(\frac{\delta}{i}\right)\bar{s}_{\overline{T(x)-K(x)|}}\,v^{T(x)}$$

where $T(x) > 0$ is the moment of death, and $K(x) \ge 0$ is the integral number of years (x) survives. By using the formulas for $a_{\overline{k|}}$ and $\bar{s}_{\overline{t-k|}}$, we see that this simplifies:

$$\begin{aligned}\mathring{Y}_x &= a_{\overline{K(x)|}} + \left(\frac{\delta}{i}\right)\bar{s}_{\overline{T(x)-K(x)|}}\,v^{T(x)} \\ &= \frac{1 - v^{K(x)}}{i} + \frac{\delta}{i}\,\frac{(1+i)^{T(x)-K(x)} - 1}{\delta}\,v^{T(x)} \\ &= \frac{1 - v^{T(x)}}{i}\end{aligned}$$

An intuitive explanation goes like this: Suppose a pension benefit is a complete annuity-immediate. Contributions to the fund are made continuously but annuity payments are discrete, 1 at the end of each year (x) that survives. Assume (x) is the only annuitant. The fund earns interest continuously at a force of interest δ. At the moment of death, the accumulated contributions are immediately paid in behalf of (x).

The generalization to m-thly payment annuities is straightforward. Replace $K(x)$ by the duration in years, $K(x) + J_m(x)$ of the number of complete m-ths that (x) lives. Thus, $J_m(x) = j/m$ where $K(x) + j/m < T(x) \le K(x) + (j+1)/m$ for some integer j between 0 and $m-1$[3]. The death benefit increases from 0 just after an annuity payment to $1/m$ just before the next scheduled payment. If c denotes the rate of payment, then $c\bar{s}_{\overline{1/m}|} = 1/m$ and hence $c = \delta/i^{(m)}$. The present value random variable is

$$\mathring{Y}_x^{(m)} = a^{(m)}_{\overline{K(x)+J_m(x)}|} + \left(\frac{\delta}{i^{(m)}}\right) \bar{s}_{\overline{T(x)-K(x)-J_m(x)}|}\, v^{T(x)}$$

$$= \frac{1 - v^{T(x)}}{i^{(m)}}$$

This development has the advantage of allowing for the variance of present values without a significant increase in the complexity of the presentation. All of the text results with

[3]The text defines J_m differently in exercise 5.14. My definition is the analog of $K(x)$. That is, $mJ_m(x)$ counts the number of m-ths (x) survives in the year of death just as $K(x)$ counts the number of years (x) survives.

regard to expected values follow directly from the definition of the present value random variable. Variances results follow as well. For example, by taking expected values, multiplying by $i^{(m)}$ and noting that $E[\mathring{Y}_x^{(m)}]$ is denoted by $\mathring{a}_x^{(m)}$, we obtain (5.9.4):

$$1 = i^{(m)}\mathring{a}_x^{(m)} + \bar{A}_x$$

The formula for the variance of the m-thly complete annuity-immediate is

$$\text{Var}[\mathring{Y}_x^{(m)}] = \left(\frac{\delta}{i^{(m)}}\right)^2 \text{Var}[\frac{1 - v^{T(x)}}{\delta}] = \left(\frac{1}{i^{(m)}}\right)^2 \left({}^2\bar{A}_x - \bar{A}_x{}^2\right) .$$

This should be compared to the variance of the continuous payment annuity:

$$\text{Var}[\bar{Y}_x] = \left(\frac{1}{\delta}\right)^2 \left({}^2\bar{A}_x - \bar{A}_x{}^2\right) \qquad (5.3.14)$$

The development of the present value random variable for the n-year m-thly complete annuity-immediate is similar.

Apportionable Annuities-Due

The apportionable annuity-due is an annuity-due which pays 1 at the beginning of each year, but requires a partial refund of the annual payment if the annuitant should die during the year. The refund is paid at the moment of death. A story which describes this annuity involves discrete payments of 1 at the beginning of each year while (x) is alive, in lieu of continuous payments at a rate of c per year until the date of death. Each year that (x) survives the payment of 1 exactly pays for the continuous cash flow if $1 = c\bar{a}_{\overline{1|}}$. Therefore $c = \delta/d$. In the year of death, $(K(x), K(x)+1]$, the beginning year payment of 1 is too much since the annuitant lives only a fraction S of the year of death. The moment of death is $T(x) = K(x) + S$, where $0 < S < 1$. The excess payment's value at the moment of death is $c\bar{a}_{\overline{1-S}|} = \frac{1 - v^{1-S}}{d}$, which is refunded on the annuitant's behalf. Thus the present value random variable, $Y_x^{(1)}$, for the annual payment apportionable annuity-due is

$$Y_x^{(1)} = \ddot{a}_{\overline{K(x) + 1}|} - \left(\frac{\delta}{d}\right) \bar{a}_{\overline{K(x) + 1 - T(x)}|} \; v^{T(x)}$$

$$= \frac{1 - v^{T(x)}}{d}$$

$$= \ddot{a}_{\overline{T(x)}|}.$$

The development for n-year, m-thly payment apportionable annuities-due is analogous.

Perhaps the complete annuity-immediate and the apportionable annuity-due are not important enough to warrant more space than the authors already spend on the subject. However, as the development given here shows, very little more would be needed to develop the present value random variables in place of the deterministic variables.

IV. CHAPTER 9: MULTIPLE DECREMENT MODELS

There are a three points to be discussed with regard to multiple decrement models. I learned of the first two from a paper that was submitted to the *Transactions of the Society of Actuaries*, which I was assigned to review. Unfortunately, I do not know who the author was because of the Committee on Papers' policy which does not permit the reviewer to know the author's identity until the paper is published[4]. The paper shows that *Actuarial Mathematics* has too many definitions (two) of $\mu_x^{(j)}$ and too few definitions (zero) of $p_x^{(j)}$. The third topic is the text's use of discrete distributions for the lifetime random variable, conditional on the cause of failure, $T(x)|J=j$. Thus, topics of this section are

1. The symbol $p_x^{(j)}$;
2. The definition of $\mu_x^{(j)}$; and
3. The construction of multiple decrement tables from a set of single decrement tables, some of which may be discrete.

Definition of ${}_tp_x^{(j)}$

Some actuaries have said (I may have told students this), that the symbol ${}_tp_x^{(j)}$ doesn't make sense. *Actuarial Mathematics* does not define the symbol. I learned that it can be defined in a way that makes sense. Probably, what was meant is that it is a mistake to define ${}_tp_x^{(j)}$ to be $1 - {}_tq_x^{(j)}$. The appropriate definition is ${}_tp_x^{(j)} = h(j) - {}_tq_x^{(j)}$ where h is the probability density function of the random variable $J(x)$, the cause of death[5]. This definition gives ${}_tp_x^{(j)}$ the following meaning: ${}_tp_x^{(j)}$ is the probability that (x) fails due to cause j after t years. An

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[4]The author is S. David Promislow, FSA. I learned this during a discussion after this review was presented at the Twenty-fifth Actuarial Research Conference which was held at the University of Western Ontario, August 23 - 25, 1990.

[5]h does not have an x attached to denote its dependence on the attained age. The *Actuarial Mathematics* presentation of the stochastic multiple decrement model is essentially a select model.

equivalent way to say this is that ${}_tp_x^{(j)}$ is the probability that (x) survives t years and dies of cause j. Since the sum of $h(j)$ over all causes is 1, then

$$\sum {}_tp_x^{(j)} = \sum h(j) - \sum {}_tq_x^{(j)} = 1 - {}_tq_x^{(\tau)} = {}_tp_x^{(\tau)}$$

and the ${}_tp_x^{(j)}$ sum to ${}_tp_x^{(\tau)}$ as required. This is a perfectly good, and rather useful, definition for ${}_tp_x^{(j)}$. It is due to David Promislow, as I noted earlier. As elegant as this idea is, it is not on the syllabus, so students must use it with care. The Examination Committee will continue to regard the symbol as undefined.

Two definitions of $\mu_x^{(j)}$

Actuarial Mathematics defines the same symbol, $\mu_{x+t}^{(j)}$, in two different ways[6]. The first definition is

$$\mu_{x+t}^{(j)} = \frac{f(t,j)}{1 - G(t)} \qquad (9.2.10)$$

where $f(t,j)$ is the joint density[7] of $T(x)$ and $J(x)$. And $G(t)$ is the (marginal) cumulative distribution of $T(x)$. This can also be written

$$\mu_{x+t}^{(j)} = \frac{1}{{}_tp_x^{(\tau)}} \frac{\partial}{\partial t} {}_tq_x^{(j)}. \qquad (9.2.11)$$

The second definition of the force of decrement is

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[6]I learned of the existence of the two definitions from David Promislow's paper ([14]), while it was under review by the Committee on Papers.

[7]There is no x adorning the joint density function, evidently because this is essentially a select model.

$$\mu_x^{(j)} = \lim_{h \to 0} \frac{l_x^{(j)} - l_{x+h}^{(j)}}{h l_x^{(\tau)}} = - \frac{1}{l_x^{(\tau)}} \frac{dl_x^{(j)}}{dx} \qquad (9.4.5)$$

To compare the two definitions at the same age, substitute $y = x + t$ for x in the second definition:

$$\mu_y^{(j)} = \lim_{h \to 0} \frac{l_y^{(j)} - l_{y+h}^{(j)}}{h l_y^{(\tau)}} = - \frac{1}{l_y^{(\tau)}} \frac{dl_y^{(j)}}{dy} \quad \text{where } y = x + t$$

The two definitions are equivalent if there is no selection present in any of the decrements. To see this, it is useful to consider the single decrement case first. In the single decrement case, the two definitions are

$$\mu_{x+t} = \frac{1}{{}_t p_x} \frac{\partial}{\partial t} {}_t q_x \qquad (3.2.19)$$

and

$$\mu_y = - \frac{1}{l_y} \frac{dl_y}{dy} \quad \text{where } y = x+t.$$

The last is a slight rearrangement of

$$\mu_{x+t} = - \frac{s'(x+t)}{s(x+t)} \quad . \qquad (3.2.13)$$

These two are equivalent provided the lifetime distribution of (x), denoted $T(x)$, is the same as the distribution of remaining life after age x lifetime of a person age 0, conditional on survival to age x, denoted $[T(0)|T(0) > x] - x$. This is just what is meant by a model which does not reflect selection of the life at age x. Written in terms of the distribution functions, this becomes:

$$\Pr[T(x) \le t] = \Pr[T(0) \le x+t \mid T(0) > x]$$

$$= \frac{\Pr[x < T(0) \le x+t]}{\Pr[T(0) > x]}$$

or, in actuarial symbols,

$$_tq_x = \frac{_{x+t}q_0 - {_xq_0}}{_xp_0} = \frac{l_x - l_{x+t}}{l_x} = \frac{s(x) - s(x+t)}{s(x)}$$

and so $\quad s(x+t) = s(x)[1 - {_tq_x}]$.

Thus, beginning with the second definition, we find that

$$-\frac{s'(x+t)}{s(x+t)} = -\frac{1}{s(x+t)}\frac{\partial}{\partial t}s(x+t)$$

$$= -\frac{1}{s(x)[1 - {_tq_x}]}\frac{\partial}{\partial t}s(x)[1 - {_tq_x}]$$

$$= \frac{1}{_tp_x}\frac{\partial}{\partial t}{_tq_x}$$

which is the first definition. In the single decrement setting the two definitions are equivalent, provided there is no selection. In order to emphasize the difference, it would be a good idea to use a notation such as $\mu_x(t)$, even if we are not using a select table. We would then show that $\mu_x(t) = \mu_0(x+t) = \mu_{x+t}(0)$. For select mortality these could be three distinct values. This notation is introduced Chapter 8 apparently because it is notationally more efficient when multiple lives are involved.

In the multiple decrement setting the same care must be taken to avoid misunderstanding. The issue involves selection (or the lack of it) in the decrements. The difference between stochastic and deterministic approaches is not involved. In order for the two definitions to be equivalent the analog of

$$s(x+t) = s(x)[1 - {}_tq_x] = s(x)\,{}_tp_x$$

is required. Let $s_j(y)$ denote the probability that a life age 0 dies of cause j after age y. The requirement for no selection in the decrements is that

$$s_j(x+t) = s(x)[h(j) - {}_tq_x^{(j)}] = s(x)\,{}_tp_x^{(j)} \; .$$

As a result, we have

$$s'_j(x+t) = s(x)\,\frac{\partial}{\partial t}\,[h(j) - {}_tq_x^{(j)}] = s(x)\,\frac{\partial}{\partial t}\,{}_tp_x^{(j)} \; .$$

Introducing this notation into the second definition, we obtain, using $y = x + t$,

$$\mu_y^{(j)} = -\frac{1}{l_y^{(\tau)}}\,\frac{dl_y^{(j)}}{dy} = -\frac{s'_j(x+t)}{s(x+t)}$$

$$= -\frac{s(x)}{s(x+t)}\,\frac{\partial}{\partial t}\,[h(j) - {}_tq_x^{(j)}]$$

which is equivalent to the first definition by the same argument we gave in the single decrement case:

$$\mu_{x+t}^{(j)} = \frac{1}{{}_tp_x^{(\tau)}}\,\frac{\partial}{\partial t}\,{}_tq_x^{(j)}$$

As in the single decrement model, it appears that a better notation would be $\mu_x^{(j)}(t)$ which distinguishes the two variables x and t.

Construction of Multiple Decrement Tables

The theory of multiple decrement models starts with two random variables, $T(x)$ and $J(x)$, and their joint probability density function. The associated single decrement tables are defined by using the forces of decrement $\mu_{x+t}^{(j)}$ as if each were a force in a single decrement table, called the associated single decrement table. The survival function in the single associated single decrement table corresponding to cause j is defined by ${}_tp_x^{\prime(j)} = \exp(-\int_0^t \mu_{x+s}^{(j)} ds)$. Because the sum of the forces is the force of failure corresponding to $\mu_{x+t}^{(\tau)}$, then

$${}_tp_x^{(\tau)} = {}_tp_x^{\prime(1)}\,{}_tp_x^{\prime(2)} \cdots {}_tp_x^{\prime(m)}$$

Each of the associated single decrement models is a survival model (except perhaps for some which may fail to satisfy ${}_\infty p_x^{\prime(j)} = 0$). So each has all of the concepts associated with a survival model; the $'(j)$ superscript is used on all of them to keep things straight. The exception is the force of failure in the $'(j)$ model which is simply the same function used to define the table, $\mu_{x+t}^{\prime(j)} = \mu_{x+t}^{(j)}$.

In order to use a multiple decrement model in applications, an actuary would collect observations of $T(x)$ and $J(x)$ for an appropriate sample of lives. The collected data would be used to estimate parameters in the model (perhaps maximum likelihood methods would be used, for example). Then resulting survival functions could be used to calculate expected present values, variances, confidence intervals, *etc.* The process of collecting the data is costly. Often actuaries avoid the procedure by selecting from various sources single decrement survival functions and then deriving the multiple decrement model which has them as its associated single decrement tables. The derivation is done one age at a time. Here is an example, similar to Example 9.7, page 277.

Suppose that we choose a mortality table satisfying the uniform distribution of decrements assumption for decrement 1.

Then ${}_tp_x^{\prime(j)} = 1 - tq_x^{\prime(j)}$ where $q_x^{\prime(j)}$ is from the known mortality single decrement table. Suppose that the second decrement is lapse of an insurance policy and that we are working with policies having semiannual premiums. A policy lapses during $(x,x+1]$ with known probability $q_x^{\prime(2)}$ with 60% of the lapses occurring at age $x+0.5$ and 40% occurring at age $x+1$. Thus

$$ {}_tp_x^{\prime(2)} = \begin{cases} 1 & \text{for } 0 \le t < 0.5 \\ 1 - 0.6q_x^{\prime(2)} & \text{for } 0.5 \le t < 1 \\ 1 - q_x^{\prime(2)} & \text{for } t = 1 \end{cases} $$

Note that this survival function is discrete: lapses occur at only two times $t = 0.5$ and $t = 1.0$. We have not used models of this type earlier and the only explicit mention of this change is in the discussion of Example 9.7 page 277. The probability density function in the '(2) table is of the discrete type and the general formula

$$ q_x^{(j)} = \int_0^1 {}_tp_x^{(\tau)}\mu_{x+t}^{(j)}\, dt \qquad (9.2.16) $$

can be used to calculate $q_x^{(2)}$ only if an appropriate adjustment is made. There is no problem for $j = 1$. We have, following the discussion on page 278 based on the general formula (9.2.16):

$$ q_x^{(1)} = \int_0^1 {}_tp_x^{\prime(1)}\, {}_tp_x^{\prime(2)}\mu_{x+t}^{(1)}\, dt \; . $$

Because of the uniform distribution of decrements assumption in the '(1) table, the '(1) probability density function which appears in the integral is ${}_tp_x^{\prime(1)}\mu_{x+t}^{(1)} = q_x^{\prime(1)}$ for $0 \le t \le 1$.

Hence, $q_x^{(1)} = q_x^{\prime(1)}\int_0^1 {}_tp_x^{\prime(2)}dt$. The integral of ${}_tp_x^{\prime(2)}$ over $[0,1]$ is especially easy because ${}_tp_x^{\prime(2)}$ is a step function; its integral is $0.5 + 0.5(1 - 0.6q_x^{\prime(2)})$. Hence,

$$q_x^{(1)} = q_x^{\prime(1)}[0.5 + 0.5(1 - 0.6q_x^{\prime(2)})]$$

$$= q_x^{\prime(1)}[1 - 0.3q_x^{\prime(2)}].$$

For $j = 2$, we know that, conditional on lapse during the year $(x, x+1]$, lapse occurs at 0.5 or 1 with probabilities 0.6 and 0.4, respectively. The integral equation which we would use if the '(2) table were a continuous type model is

$$q_x^{(2)} = \int_0^1 {}_tp_x^{\prime(1)}\, {}_tp_x^{\prime(2)} \mu_{x+t}^{(2)} dt.$$

Since '(2) is the discrete type, it must be replaced by a summation over the possible values $T^{\prime(2)}$ can take in (0, 1]. Let $g(t)$ denote the conditional distribution of $T^{\prime(2)}$, conditional on $T^{\prime(2)} < 1$. Thus $g(0.5) = 0.6$ and $g(1) = 0.4$. When $T^{\prime(2)}$ is a continuous type random variable, $g(t) = {}_tp_x^{\prime(2)} \mu_{x+t}^{(2)} / q_x^{\prime(2)}$. This relation helps in remembering the discrete version; use $q_x^{\prime(2)} g(t)$ in place of ${}_tp_x^{\prime(2)} \mu_{x+t}^{(2)}$ and summation in place of integration:

$$q_x^{(2)} = q_x^{\prime(2)} \sum_t {}_tp_x^{\prime(1)} g(t) = q_x^{\prime(2)}[0.6({}_{0.5}p_x^{\prime(1)}) + 0.4({}_1p_x^{\prime(1)})]$$

Now we use the '(1) table to complete the calculation:

$$q_x^{(2)} = q_x^{\prime(2)}[0.6(1 - 0.5q_x^{\prime(1)}) + 0.4(1 - q_x^{\prime(1)})]$$

$$q_x^{(2)} = q_x^{\prime(2)}[1 - 0.7q_x^{\prime(1)}]$$

This completes the derivation of the annual probabilities for age

x in the multiple decrement model. This calculation would be performed for each age x. Analogous calculations would be required if we needed probabilities like ${}_tq_x^{(j)}$ for fractional values of t.

A general formula for $q_x^{(j)}$ in terms of the associated single table distributions is

$$q_x^{(j)} = q_x^{\prime(j)} \mathrm{E}[{}_Sp_x^{\prime(-j)}]$$

where the expectation is over the distribution of S, the random variable $T^{\prime(j)}$, conditioned on $T^{\prime(j)} < 1$, and ${}_tp_x^{\prime(-j)}$ is the product of all survival functions, except the j-th, of the associated single decrement models. This corrects the text's formula (9.2.16) so that it applies when S has a discrete distribution.

V. Brief Comments

The beginning life contingencies student is faced with many new, important, complex ideas. The manner of presentation is very important. The style, typesetting, notation, and layout, for example, are especially important when the ideas are so complicated. This may not be apparent to a casual reader, but I know from many years of teaching that these little things matter a great deal to students. A few of the text's practices in these areas are worth mentioning.

Labels

Equations are labeled sequentially within each section. Thus (5.3.8) is the label on the eighth equation in Section 3 of

Chapter 5. There are no section numbers on the pages. This means you cannot page through the text scanning only for Section 5.3 in order to pursue a reference to (5.3.8). I am adapting to this. Fortunately there are a lot of numbered equations. A random sample indicated that about 50% of the pages have an equation number. So it is usually possible to tell what section you are in.

The labels for examples, tables, and figures are sequential throughout each chapter. This is not consistent with equation labeling. Of course there are not many of these, relative to the number of equations. But it is nevertheless annoying to search for an example, table or figure, page by page. It would better if the authors had used the page number when making a reference to them, or had labelled the examples, tables and figures in the same way equations are labeled. I recommend to students that they write the page number of the example, table or figure by the reference the first time they look it up. And references to an equation in a section different from the reference should be similarly marked.

Layout

The typesetting, layout and general appearance are superb. The original printing had some typos, but not a large number given the number of opportunities to err in formulas, tables, *etc.* And the Society staff and authors have meticulously noted and corrected them in later printings. The ample space surrounding equations, and the wide margins for notes make the text more expensive, but the expense is very well justified. The layout makes this a very friendly looking book.

Material Not on the Society Syllabus

There are four chapters which are not on the examination syllabus:

Chapter 16: Special Annuities and Insurances

Chapter 17: Advanced Multiple Life Theory

Chapter 18: Population Theory

Chapter 19: Theory of Pension Funding

The creation of an advanced life contingencies examination using this material is a good idea. Former students have reported that some of this material has been useful in their work. Especially highly valued is Section 16.5 Variable Products.

The Titles of the Text and the Examination

The title of the new text is misleading. Except for the introductory material (Chapters 1 and 2) and risk theory (Chapters 11 and 12), the text subject is life contingencies. Specifically, it covers the applications of life contingencies to life insurance and pension plan related employee benefits. The mathematics of the casualty risks is not covered. This vast area, which underlies the actuarial work related to health insurance, workers compensation insurance, property insurance and liability insurance, is not covered at all. My point here is not that these should be included, but rather that the title seems to imply that text's scope is broader than it actually is. The Course 150 examination covers Chapters 3 - 10, 14, 15, Appendix 4 and Appendix 5. In addition Chapters 1 and 2 are recommended reading for Course 150. There is nothing on the examination which we would consider to be other than a life contingencies topic. It seems appropriate to change the title of this examination to Life Contingencies.

Appendices

The authors have included seven valuable appendices. Especially useful are the symbol index, the list of useful formulas and the tables based on the Illustrative Life Tables. I usually spend thirty minutes of the first day of class explaining the text organization, carefully noting the appendices. The students and I have found them very useful. Their inclusion is another result of the authors' thoughtfulness and understanding of students' needs.

V. CONCLUSION

The Society of Actuaries will continue to benefit greatly from the efforts of Newton L. Bowers, Jr., Hans U. Gerber, James C. Hickman, Donald A. Jones and Cecil J. Nesbitt which resulted in such a fine text. The importance of such a text in terms of actuarial education, the professionism that it conveys to other business individuals, and the coherence it provides for communication among actuaries is enormous. The introduction of random variables to fundamental actuarial education has been accomplished with accuracy, elegance and care.

From a student's view point the introduction of this material into the examination syllabus has resulted in an increase in the level of complexity of the models involved. On the other hand, there is evidence that the Examination and Education Committee is of much greater assistance to students than it used to be. The scope of the examination (in terms of the number of pages, chapters, topics or exercises) is narrower than is was formerly. And there is an ample supply of supplemental material. Thus, the subject of the examination has become more complex, but students have more assistance than was available in the past.

There are minor criticisms of the text's treatment of multiple decrement models (the force of decrement and conditional failure times) as well as the presentation of complete and apportionable annuities.

The layout of the book makes it very attractive and pleasant to use.

I recommend basing an elective Society of Actuaries course on the material of Chapters 16-19.

I also suggest that the subject of the Course 150 examination is life contingencies, not actuarial mathematics. The Course 150 title should be changed to accurately reflect the scope of the examination.

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