

Evaluation of the Rollover Option

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Abstract

In order to persuade its customer with a maturing Guaranteed Investment Contract to roll it over for another term, an insurance company may have to provide him with an incentive in the form of a call option. If the exercise price of this option is close to the forward price of the underlying zero coupon bond, there is a very simple formula for determining the amount of annual interest that should be charged throughout the term of the new contract to pay for the option. The formula is: Multiply by 0.4 the standard deviation of the interest rate of the underlying zero coupon bond at the exercise date as estimated at the contract commitment date.

I. Introduction

In a recent paper [M], John Mereu discussed the quantification of interest rate risk. At the end of [M], he wrote: "The presence of inertia in the exercise of ... options makes the application of option pricing theory in an insurance setting difficult." The purpose of the present paper is to supplement John's paper by examining a special case in which there is no inertia in option exercise because the insurance company promises the customer to exercise the option for him.

Consider an insurance company which issues Guaranteed Investment Contracts (GIC). Suppose that a certain contract will mature in a few weeks from now. The company contacts the owner of the contract and asks him to roll over the contract for another term. The company proposes that, if the customer commits himself now to reinvest the proceeds from his current GIC in a new GIC, the interest rate for the new GIC will be the maximum of today's interest rate and the interest rate on the day when the current GIC matures. Thus the customer is given some kind of an option (or an interest rate floor), which will be exercised automatically by the insurance company. What is the value of this option? In this paper we shall present a simple formula for valuing the option.

A similar situation occurs when the company tries to attract business from its competitors. Even if a customer agrees to transfer his money from another company, it may take several weeks for the money to arrive. To get the business, the company may need to guarantee that the new customer's GIC be credited at the maximum of today's interest rate and the interest rate on the day when the money is received. Again, an option is given to the customer.

Before we proceed further, let us review some financial terminology. A *forward contract* is an agreement between two parties at time τ for delivery of an asset at a later

time t at a price specified at time τ . Payment for the asset takes place at time t and no intermediate payments are made. The price specified at the initial time τ is called *delivery price* or *contract price*; however, if the price is such that the forward contract has zero value at time τ , it is called *forward price*. The party that agrees to buy the underlying asset at time t for the contract price is said to assume a *long position*. The other party agreeing to sell the asset at time t for the contract price is said to assume a *short position*.

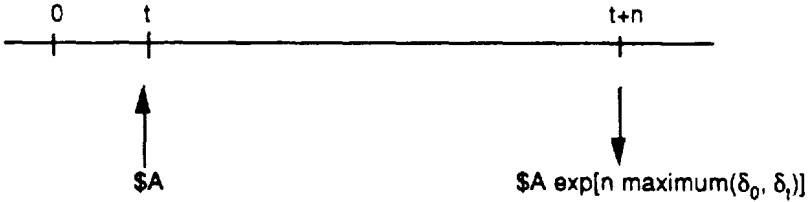
A *call option* on an asset gives its holder the right to buy the underlying asset by a certain date for a certain price. A *put option* gives its holder the right to sell the underlying asset by a certain date for a certain price. The date specified in the contract is known as the *exercise date* or *maturity date*. The price in the contract is known as the *exercise price*. A *European* option can only be exercised on the maturity date itself, while an *American* option can be exercised at any time up to the maturity date. The terms "European" and "American" do not refer to the location of the option or the exchange.

Note that an option gives its holder the right to do something, but he is not obliged to exercise this right. On the other hand, the holder of a forward contract is obligated to buy or sell the underlying asset.

II. Formulation of the Problem

Let today's force of interest for an n -year GIC or zero-coupon bond be denoted as δ_0 . Assume that the customer's old GIC will mature for $\$A$ at time t , $t > 0$, or the customer's money, $\$A$, will arrive at time t (t is around 20 days or $20/365$ year). Let the force of interest for an n -year GIC or zero-coupon bond at time t be denoted by δ_t , which is a random variable as viewed from time 0. The customer is guaranteed by the insurance company to receive, at time $t + n$, the amount

$$A \left[e^{\text{maximum}(\delta_0, \delta_t)} \right]^n = A e^{n \text{maximum}(\delta_0, \delta_t)}.$$



As the customer is given an option, the insurance company should know how much this option is worth. In this paper we attempt to provide a simple answer to this question.

At time t, the market value of the customer's account is

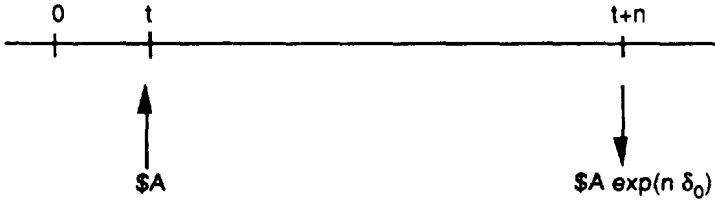
$$\frac{A e^{n \text{ maximum}(\delta_0, \delta_t)}}{e^{n \delta_t}} = A e^{n \text{ maximum}(\delta_0 - \delta_t, 0)}$$

$$= \text{maximum}[A e^{n(\delta_0 - \delta_t)}, A]. \quad (2.1)$$

Consider a forward contract for the purchase, at time t, of A dollars of n-year zero-coupon bonds at the force of interest δ_0 ; i.e., one undertakes to pay A at time t and will receive

$$A e^{n \delta_0}$$

at time t + n.



At time t, the market value of the long position in this forward contract is

$$\frac{A e^{n \delta_0}}{e^{n \delta_t}} - A = A e^{n(\delta_0 - \delta_t)} - A. \quad (2.2)$$

Rewriting (2.1) as

$$A + \text{maximum}[A e^{n(\delta_0 - \delta_t)} - A, 0], \quad (2.3)$$

we can interpret the option given by the insurance company to the customer as a

European call option for assuming a long position in this forward contract; the exercise price of this call option is 0 and the exercise date is t which is also the maturity date of the forward contract. On the other hand, by rewriting (2.1) as

$$A + [Ae^{n(\delta_0 - \delta_t)} - A] + \text{maximum}[0, A - Ae^{n(\delta_0 - \delta_t)}], \quad (2.4)$$

the customer can be viewed as holding a long position in the forward contract and a European *put* option on the forward contract, while the insurance company is assuming a short position in the forward contract.

Now, our problem becomes the evaluation of the call option. Furthermore, since the insurance company does not receive any money until time t , we should determine the value of the option at time t as seen from time 0, i.e., **determine the forward price of the call option**. Having determined the forward price of the option, we then convert it into a force of interest over the term of the GIC. It turns out that there is a very simple approximation formula for this force of interest: it is the standard deviation of the force-of-interest random variable δ_t as viewed from time 0 divided by the square root of 2π .

III. Black's Formula

The option-pricing theory of F. Black and M. Scholes [BS] has been described as the most important single advance in the theory of financial economics in the 1970's. Our strategy for obtaining a solution to the problem is to modify the Black-Scholes formula so that it can be used to determine the forward price of a European call option on a non-dividend-paying security.

Let $S(\tau)$ denote the price at time τ of a security which pays no dividends, $\tau \geq 0$; assume that it is a geometric Brownian motion, i.e., it satisfies the stochastic differential equation

$$\frac{dS}{S} = \mu dt + \sigma dW, \quad (3.1)$$

where μ and σ are constants and $W(\cdot)$ is a Wiener process. This means that the security price is assumed to be distributed lognormally over any time period. Let r denote the riskfree force of interest, which is assumed to be constant. Black and Scholes [BS] show that the value of the European call option on the security at time τ with exercise price X and exercise date t ($\tau \leq t$) is given by

$$c(\tau, t; X) = S(\tau)N(d_1) - Xe^{-r(t-\tau)}N(d_2), \quad (3.2)$$

where

$$N(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-z^2/2} dz,$$

$$d_1 = \frac{\log_e \frac{S(\tau)}{Xe^{-r(t-\tau)}} + \frac{\sigma^2(t-\tau)}{2}}{\sigma\sqrt{t-\tau}} \quad (3.3)$$

and

$$d_2 = d_1 - \sigma\sqrt{t-\tau}. \quad (3.4)$$

For $\tau \leq t$, let $F_S(\tau, t)$ denote the forward price of the forward contract on the security committed at time τ for transaction at time t . Assume that the security can be stored at no cost; with no dividends to be paid, the cost-of-carry relation

$$F_S(\tau, t) = S(\tau)e^{r(t-\tau)} \quad (3.5)$$

holds. Substituting (3.5) into (3.2) and (3.3) yields

$$c(\tau, t; X) = e^{-r(t-\tau)}[F_S(\tau, t)N(d_1) - XN(d_2)] \quad (3.6)$$

and

$$d_1 = \frac{\log_e \frac{F_S(\tau, t)}{X} + \frac{\sigma^2(t-\tau)}{2}}{\sigma\sqrt{t-\tau}}, \quad (3.7)$$

respectively.

We remark that Black [B, p. 176] has derived (3.6) by assuming that, for a fixed transaction date t , the fractional change in the forward price $F_S(\cdot, t)$ over any time interval is distributed log-normally, with a known variance rate σ^2 . Formula (3.6) can also be found in [H, p. 147], [Mt, §10.3] and [W]; also see [J].

Now, let us consider a forward contract on the call option to be committed at time τ for transaction at time t , $\tau \leq t$. Let $F_c(\tau, t; X)$ denote its forward price. Note that the transaction date of the forward contract on the call option, the exercise date of the call option and the transaction date of the forward contract on the security are identical (and equal to t). Applying the cost-of-carry relation

$$F_c(\tau, t; X) = c(\tau, t; X)e^{r(t-\tau)}, \quad (3.8)$$

we have the formula for the forward price of the call option:

$$F_c(\tau, t; X) = F_S(\tau, t)N(d_1) - XN(d_2). \quad (3.9)$$

Note that the riskfree force of interest r does not appear in (3.9), (3.7) and (3.4).

Furthermore, if $F_S(\tau, t) = X$, then we have $d_1 = -d_2$ and (3.9) becomes

$$\begin{aligned} F_c(\tau, t; F_S(\tau, t)) &= F_S(\tau, t)[N(d_1) - N(d_2)] \\ &= F_S(\tau, t)[2N(d_1) - 1] \\ &= F_S(\tau, t)[2N(\sigma\sqrt{t}/2) - 1]. \end{aligned} \quad (3.10)$$

[We remark that, by *put-call parity*, expression (3.10) also gives the forward price of the put option with exercise date t and exercise price $F_S(\tau, t)$.] As the forward price $F_S(\tau, t)$ should be known at time τ , to apply (3.10) all we need to know is the standard deviation σ .

Formula (3.10) is simpler than the classical Black-Scholes formula (3.2).

IV. A Solution

Let us return to our problem. We would like to use (3.9) or (3.10) to determine the value of the call option given by the insurance company to its customer. For $x \leq y$, let $P(x, y)$ denote the price at time x of a zero coupon bond that pays 1 at time y .



Note that $P(y, y) = 1$. In terms of the notation introduced in Section II, $P(0, n) = \exp(-n\delta_0)$ and $P(t, t + n) = \exp(-n\delta_t)$.

To apply the results in the last section, consider the security as a zero coupon bond that pays 1 at time $t + n$; then for $0 \leq \tau \leq t$

$$S(\tau) = P(\tau, t + n)$$

and

$$F_S(\tau, t) = P(\tau, t + n)/P(\tau, t)$$

(t and n are fixed). Define the forward force of interest $f(\tau)$ by the equation

$$P(\tau, t + n)/P(\tau, t) = e^{-nf(\tau)}. \quad (4.1)$$

If the riskfree force of interest r is constant and $f(\tau)$ is a Brownian motion satisfying the stochastic differential equation

$$df(\tau) = m d\tau + s dW(\tau), \quad 0 \leq \tau \leq t, \quad (4.2)$$

where m and s are constants, we can apply (3.9), (3.7) and (3.4) with $\tau = 0$, $\sigma = ns$,

$X = P(0, n)$ and

$$F_S(0, t) = P(0, t + n)/P(0, t)$$

to obtain the formula

$$F_c(0, t; P(0, n)) = [P(0, t+n)/P(0, t)]N(d_1) - P(0, n)N(d_2). \quad (4.3)$$

Formula (4.3) gives a solution to the problem posed in Section II. The value of the call option per dollar of investment received at time t is

$$F_c(0, t; P(0, n))/P(0, n) = \{P(0, t+n)/[P(0, t)P(0, n)]\}N(d_1) - N(d_2).$$

We now convert this call option value as a force of interest Δ over the term of the GIC, i.e., the interest rate spread that the insurance company needs in order to provide the option. Let δ denote the maximum of δ_0 and δ_t . For each dollar the insurance company receives from the customer at time t , it has to pay him back $\$e^{n\delta}$ at time $t+n$. The cost of the option at time t is $F_c/P(0, n)$; therefore the company has only $1 - F_c/P(0, n)$ to invest. (We assume that there is no other cost for the insurance company.) Thus we have the equation

$$\{1 - [F_c/P(0, n)]\}e^{n(\delta+\Delta)} = 1e^{n\delta},$$

from which we get

$$\begin{aligned} \Delta &= -\frac{\log_e \left[1 - \frac{F_c(0, t; P(0, n))}{P(0, n)} \right]}{n} \\ &= -\frac{\log_e \left\{ 1 - \frac{P(0, t+n)}{P(0, t)P(0, n)} N(d_1) + N(d_2) \right\}}{n}. \end{aligned} \quad (4.4)$$

Formula (3.10) is simpler than (3.9). Rewriting (3.10) in terms of zero coupon bond prices, we have, with $\tau = 0$ and $\sigma = ns$,

$$F_c(0, t; P(0, t+n)/P(0, t))/[P(0, t+n)/P(0, t)] = 2N(ns\sqrt{t/2}) - 1. \quad (4.5)$$

Suppose that $P(0, n)$ is the same as (or very close to) the forward price $P(0, t+n)/P(0, t)$ or

that the insurance company would guarantee the forward rate $f(0)$, not δ_0 , as the interest rate floor, then we can apply (4.5). If t is small, the spot price $P(0, n)$ should be close to the forward price $P(0, t + n)/P(0, t)$. It follows from (4.5) that the option value, expressed as a force of interest, is given by the formula

$$\begin{aligned} \Delta &= -(1/n)\log_e[2 - 2N(ns\sqrt{t/2})] \\ &= -(1/n)\log_e[2N(-ns\sqrt{t/2})]. \end{aligned} \tag{4.6}$$

We have programmed (4.6) in *APL*. Listed on the next page are two *APL* functions and some numerical values of Δ . The *CmplmNrml* function approximates the complement of the normal distribution

$$1 - N(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-z^2/2} dz$$

with absolute error less than 7.5×10^{-8} [AS, p. 932, 26.2.17]. The values of Δ are given for $n = 1, 3, 5, 10, 15, 25$, $s = 0.005, 0.01, 0.015, 0.02, 0.025$ and $t = 15/365, 20/365, 25/365, 30/365$. For example, for a 5-year GIC with $t = 20/365$, the five values of Δ corresponding to the five standard deviation assumptions are 0.04674749713%, 0.09360407434%, 0.1405698622%, 0.18764497% and 0.2348295071%.

▽ Option

- [1] $rt \leftarrow (15 \ 20 \ 25 \ 30 \ + \ 365) \ * \ 0.5$
- [2] $s \leftarrow 0.005 \ 0.01 \ 0.015 \ 0.02 \ 0.025$
- [3] $n \leftarrow 1 \ 3 \ 5 \ 10 \ 15 \ 25$
- [4] $- \ (e \ 2 \ * \ CmplmNrm1 \ 0.5 \ * \ n \ * \ s \ * \ rt) \ + \ n \ * \ ((ps), \ prt)pl$

▽

▽z ← CmplmNrm1 x , t

- [1] $t \leftarrow + \ 1 \ + \ 0.2316419 \ * \ x$
- [2] $z \leftarrow t \ * \ .31938153 \ + \ t \ * \ -.356563782 \ + \ t \ * \ 1.781477937 \ + \ t \ * \ -1.821255978 \ + \ t \ * \ 1.330274429$
- [3] $z \leftarrow z \ * \ (+ \ -0.5 \ * \ x \ * \ 2) \ + \ (o2) \ * \ 0.5$

▽

0.0004044557119	0.0004670396698	0.0005221804863	0.0005720342094
0.0008090738898	0.0009342963167	0.001044632463	0.001144394424
0.001213855612	0.001401771034	0.00156735704	0.001717081774
0.001618800906	0.001869463866	0.002090354281	0.00229009634
0.002023909802	0.002337374856	0.002613624244	0.002863438203
0.0004046185373	0.0004672570114	0.0005224523468	0.0005723605913
0.0008097274425	0.0009351680163	0.001045722331	0.001145702481
0.001215327322	0.001403733759	0.001569810855	0.001720026744
0.001621418432	0.001872954635	0.002094718468	0.002295334103
0.002028001029	0.002342831037	0.002620445722	0.002871625284
0.0004047819604	0.0004674749713	0.0005227248489	0.0005726876406
0.0008103818589	0.0009360407434	0.001046813413	0.001147011957
0.001216800617	0.001405698622	0.001572267433	0.00172297517
0.001624038946	0.0018764497	0.002099088437	0.002300579288
0.002032097555	0.002348295071	0.002627277949	0.002879826314
0.0004051909295	0.0004680203717	0.0005234067066	0.0005735059784
0.0008120194729	0.0009382248502	0.001049544218	0.001150289644
0.001220488576	0.001410617912	0.001578418746	0.001730359123
0.001630601073	0.001885203913	0.002110036371	0.002313722406
0.002042359787	0.002361987196	0.002644403156	0.002900387448
0.0004056002057	0.0004685662073	0.0005240891444	0.0005743250568
0.000813659051	0.0009404119411	0.001052279164	0.001153572749
0.001224182984	0.001415547082	0.001584583827	0.00173776114
0.001637178358	0.001893981393	0.002121016759	0.002326908117
0.002052651495	0.002375724589	0.002661591507	0.002921031455
0.000406419511	0.0004696590142	0.0005254555897	0.0005759652628
0.0008169439149	0.0009447948785	0.001057761262	0.001160154979
0.001231590897	0.001425434753	0.001596954904	0.001752618873
0.001650377975	0.001911605535	0.002143074012	0.002353406125
0.002073322541	0.002403333896	0.002696155734	0.00296256542

We now present a formula even simpler than (4.6). Put

$$w = ns\sqrt{t}/2; \tag{4.7}$$

in most practical situations, w is a small positive number. Let $v(z)$ denote the normal density function $(2\pi)^{-1/2}\exp(-z^2/2)$. Then

$$\begin{aligned} 2N(-w) &= 1 - 2 \int_0^w v(z) dz \\ &= 1 - 2wv(0) = 1 - 2w/\sqrt{2\pi}. \end{aligned}$$

Thus

$$\log_e[2N(-w)] = -2w/\sqrt{2\pi},$$

from which we obtain

$$\Delta = \frac{s\sqrt{t}}{\sqrt{2\pi}}. \tag{4.8}$$

Note that the numerator $s\sqrt{t}$ is the (conditional) standard deviation of the force-of-interest random variable δ_t as seen from time 0; the number s is an annualized standard deviation. Below is a table of values of Δ according to (4.8) for $s = 0.005, 0.01, 0.015, 0.02, 0.025$ and $t = 15/365, 20/365, 25/365, 30/365$.

0.000404370316	0.0004669266216	0.0005220398332	0.0005718659852
0.0008087406321	0.0009338532433	0.001044079666	0.00114373197
0.001213110948	0.001400779865	0.0015661195	0.001715597955
0.001617481264	0.001867706487	0.002088159333	0.002287463941
0.00202185158	0.002334633108	0.002610199166	0.002859329926

On comparing these numbers with those on the previous page, we see that (4.8) is a good approximation formula. The term of the GIC, n , does not appear in (4.8). Retracing the steps above more carefully, we can in fact prove that

$$\Delta \geq s\sqrt{t}/\sqrt{2\pi}. \tag{4.9}$$

V. A More General Framework

We have used the Black-Scholes formula to derive the option-pricing formulas (4.4), (4.6) and (4.8). Are the assumptions valid? A criticism that comes to mind immediately is that the riskfree force of interest r is assumed to be constant over time. Indeed, I wrote [C, p. 146]: "The risks considered in this paper are due to long-term interest rate fluctuations. It is not realistic to assume that the short-term (riskless) interest rate will remain fixed while the long-term rates fluctuate." However, because we have formulated the problem in terms of forward prices, r does not appear in the final formulas. Thus one can hope that (4.8) (or something as simple) may be valid in a more general framework, in which the riskfree interest rate is stochastic. This turns out to be true.

Merton [Me; H, p. 304, §12.2] has generalized the Black-Scholes formula to the case where the riskfree rate is stochastic. We can start with Merton's generalization to come up with a formula similar to (4.8). Jamshidian [Ja], assuming that the riskfree force of interest $r(\cdot)$ evolves according to the mean-reverting Gaussian process

$$dr(\tau) = a[r_0 - r(\tau)]d\tau + \beta dW(\tau), \quad (5.1)$$

has derived a Black-Scholes type call option formula for zero coupon bonds, from which we can obtain the approximation formula

$$\Delta \approx \frac{\sigma_P}{n\sqrt{2\pi}}, \quad (5.2)$$

where σ_P is the standard deviation of the logarithm of the n -year bond price random variable $P(t, t+n)$ as seen from time 0,

$$\sigma_P^2 = \text{Var}[\log_e P(t, t+n) | r(0)].$$

It is proved in [Ja] that

$$\sigma_P = \beta \sqrt{[(1 - e^{-2at})/2a](1 - e^{-an})/a}. \quad (5.3)$$

Now, if the speed of mean reversion a is close to zero, then

$$(1 - e^{-2a})/2a = t$$

and

$$(1 - e^{-an})/a = n;$$

hence (5.2) becomes $\Delta = \beta\sqrt{t}/\sqrt{(2\pi)}$, which is similar to (4.8).

There are two problems with an interest rate model as defined by (5.1): interest rates may become negative and the initial yield curve cannot be prescribed exogenously.

Jamshidian [J1, J2] and Hull and White [HW] have shown that (5.1) can be generalized in such a way that the initial yield curve can be taken as exogenously given. In a subsequent paper, we shall show that (5.2) holds in such a framework.

One may be surprised that, in these option-pricing formulas, the direction of interest rate movements does not seem to matter. The constant m in the stochastic differential equation (4.2) does not appear in the formulas. One of my criticisms of the model presented in [CI] is that "the option-pricing formula is independent of" the underlying trend of the bond yield movements [CI, p. 147]. This same criticism seems to be applicable here. However, this is not the case. The forward price $P(0, t + n)/P(0, t)$ is a market forecast of how interest rates may move.

VI. Executive Summary

Formula (4.8) assesses the force of interest which should be charged (continuously) throughout the term of the GIC to pay for the option given to the customer. It is straightforward to use. At the time when the customer commits himself to roll over the maturing GIC for another term, the actuary looks forward and estimates the standard deviation of the force of interest of the new GIC. He then divides the standard deviation value by $\sqrt{(2\pi)}$. The result is the interest rate spread needed to pay for the option. There is one caution: for (4.8) or (5.2) to be applicable, the exercise price and the forward price

of the underlying zero coupon bond have to be close; this should normally be the case, as the time between commitment date and exercise date is usually quite short.

Some people might find it strange that the constant π appears in a formula that deals with interest rates and options. The following anecdote was told by the distinguished English mathematician and logician Augustus de Morgan (1806-1871) [DM, p. 285; Ca, p. 160]:

More than thirty years ago I had a friend, now long gone, who was a mathematician, but not of the higher branches: he was, *inter alia*, thoroughly up in all that relates to mortality, life assurance, &c. One day, explaining to him how it should be ascertained what the chance is of the survivors of a large number of persons now alive lying between given limits of number at the end of a certain time, I came, of course upon the introduction of π , which I could only describe as the ratio of the circumference of a circle to its diameter. "Oh, my dear friend! that must be a delusion; what can the circle have to do with the numbers alive at the end of a given time?" — "I cannot demonstrate it to you; but it is demonstrated." — "Oh! stuff! I think you can prove anything with your differential calculus: figment, depend upon it."

I wonder if some twentieth century actuaries might say: "What can the circle have to do with interest rate guarantees? I think you can prove anything with your stochastic calculus."

De Morgan had also published an actuarial book [De]. At the end of its Preface, dated August 3, 1838, he wrote:

I have endeavoured, as much as possible, to free the chapters of this work which relate to insurance offices from mathematical details, and to make them accessible to all educated persons. Whether they act by producing conviction, or opposition, a step is equally gained: nothing but indifference can prevent the public from becoming well acquainted with all that is essential for it to know on a subject, of which, though some of the details may be complicated, the first principles are singularly plain.

As I am not able to find the constant π in De Morgan's book [De], I think that I should also not have it in the option-pricing formulas. Since $\sqrt{(2\pi)} = 2.5066$, we can change (4.8) and (5.2) to

$$\Delta = s\sqrt{t}/2.5$$

and

$$\Delta = \sigma_P/(2.5n),$$

respectively. Indeed, in view of inequality (4.9), these would give better approximations in many cases.

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References

- [AS] Abramowitz, M., and Stegun, I.A. *Handbook of Mathematical Functions*. Washington, D.C.: National Bureau of Standards, 1964. Reprinted by Dover, New York.
- [BC] Bhattacharya, S., and Constantinides, G.C. *Theory of Valuation: Frontier of Modern Financial Theory*, Volume 1, Totowa, N.J.: Rowman & Littlefield, 1989.
- [B] Black, F. "The Pricing of Commodity Contracts," *Journal of Financial Economics*, 3 (1976): 167-179.
- [BS] Black, F., and Scholes, M. "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81 (1973): 637-659. Reprinted in J.L. Bicksler and P.A. Samuelson (ed.) *Investment Portfolio Decision-Making*, Lexington, Mass.: D.C. Heath, 1974, 181-198.
- [Ca] Castellanos, D. "The Ubiquitous π ," *Mathematics Magazine*, 61 (1988), 67-98, 148-163.
- [C] Clancy, R.P. "Options on Bonds and Applications to Product Pricing," *Transactions of the Society of Actuaries*, 37 (1985): 97-130; Discussion 131-151.

- [CIR] Cox, J.C., Ingersoll, J.E., Jr., and Ross, S.A. "A Theory of the Term Structure of Interest Rates," *Econometrica*, 53 (1985): 385-407. Reprinted in [BC], 129-151.
- [De] De Morgan, A. *An Essay on Probabilities, and on Their Application to Life Contingencies and Insurance Offices*. London: Longman, Orme, Brown, Green, & Longmans, 1838.
- [DM] De Morgan, A. *A Budget of Paradoxes*, Vol. I, 2nd ed., edited by D.E. Smith. London: Open Court, 1915.
- [H] Hull, J. *Options, Futures, and Other Derivative Securities*. Englewood Cliffs, N.J.: Prentice-Hall, 1989.
- [HW] Hull, J., and White A. "Pricing Interest-Rate Derivative Securities," *Review of Financial Studies*, 3 (1990).
- [Ja] Jamshidian, F. "An Exact Bond Option Formula," *Journal of Finance*, 44 (1989): 205-209.
- [J1] Jamshidian, F. "Closed-Form Solution for American Options on Coupon Bonds in the General Gaussian Interest Rate Model." Technical Report, Merrill Lynch Capital Markets, 1989.
- [J2] Jamshidian, F. "Bond and Option Evaluation in the General Gaussian Interest Rate Model." Technical Report, Merrill Lynch Capital Markets, 1990.
- [J] Jarrow, R.A. "The Pricing of Commodity Options with Stochastic Interest Rates," *Advances in Futures and Options Research*, 2 (1987): 19-45.
- [M] Mereu, J. A. "A Guide to Quantifying C-3 Risk," *Transactions of the Society of Actuaries*, 41 (1989).
- [Me] Merton, R.C. "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science*, 4 (1973): 141-183. Reprinted in [BC], 229-271, and as Chapter 8 in [Mt].
- [Mt] Merton, R.C. *Continuous-Time Finance*. Oxford: Basil Blackwell, 1990.
- [W] Whaley, R.E., "On Valuing American Futures Options," *Financial Analysts Journal*, 42 (May/June 1986): 49-59. Reprinted in *CFA Readings in Derivative Securities*, The Institute of Chartered Financial Analysts (1988): 194-204.

