

Interest and Mortality Randomness in Some Annuities

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Abstract. A model is presented which can be used when interest rates and future lifetimes are random, for certain annuities. Expressions for the mean values and the standard deviations of the present values of future payment streams are obtained. These can be used in determining contingency reserves for possible adverse interest and mortality experience for collections of life annuity contracts. Several complete examples are considered. Certain boundary crossing probabilities for the stochastic process component of the model are obtained.

Keywords: Random interest rates, Joint randomness in interest and mortality, Ornstein-Uhlenbeck stochastic process, Function space integrals, Boundary crossing probabilities.

1 Introduction

Actuaries and other people involved with interest rates and investment value variations have long been aware of their random nature. During the last fifteen years many papers have been written about stochastic interest rates and the randomness of asset and liability values. More recently papers have considered randomness in both mortality (morbidity) and interest earnings in life insurance and annuities. A significant article is *Why Not Random Interest?* by J. C. Hickman (1985). P. P. Boyle (1976) considered

this dual randomness, and E. W. Frees (1990) discussed the extension of the theory of life contingencies to a random interest environment.

This paper presents a model which can be used when interest rates and future life times are random, for certain annuities. It was constructed so that mean values and standard deviations of the present values of future payment streams can be obtained, and keeps a fair level of reasonableness in the model. References about stochastic interest rates which contributed to our understanding were Boyle (1976), Panjer and Bellhouse (1980), Wilkie (1981), Giacotto (1986), and Dhaene (1989). Mathematical models for randomness in several interest functions which also aided our thinking were Ziock (1973), (1975), Miller-Hickman (1974), Dufresne (1989), Pollard (1976), and Jetton (1988). Asset and liability value variability, and immunization theory has an extensive research history, see, e.g. Boyle (1978), Shiu (1987), Beekman-Shiu (1988), and references.

The important focus of Bowers, Gerber, Hickman, Jones and Nesbitt (1986) on using multiples of standard deviations of random loss functions in premium calculations helped direct the plan of this paper.

A motivation for this study is finding a way to determine contingency reserves for possible adverse interest and mortality experience for collections of life annuity contracts. Boermeester (1956), Fretwell and Hickman (1964), and Bowers (1967) addressed this problem for mortality randomness. Fretwell and Hickman (1964), p. 56, give the following challenge: *We will leave for later development the question of determining intervals for life annuity costs with associated approximate probability statements where both time until death and the interest rate are random variables.*

The models of the paper include successively more amounts of randomness. Section 2 of this paper presents a model for annuities certain in which there are stochastic fluctuations about a fixed interest level, over the period of the annuity. In Section 3 the period of the annuity is the random future lifetime of the purchaser, and the randomness from Section 2 is maintained. Section 4 permits the basic interest level to assume values in accord with a probability distribution. Formulas for the mean values and standard deviations of the present values of future payment streams in Sections 2, 3, and 4 are derived. Several complete examples are considered, and tables of values are shown. The final section is devoted to certain boundary crossing probabilities for the stochastic process used in Section 2.

It is not suggested that Section 4's model is better than those contained in our cited references, but its structure does present one method of providing for randomness in interest rates and future lifetimes. An important feature is that one can calculate the means and standard deviations, and hence can make scientific provision for variability in the present value of future annuity payments.

2 Random Deviations from Interest δ Only, Fixed Time Interval

In this section, a constant force of interest δ is perturbed by a stochastic process. This provides one method of generating interest scenarios over time. Associated with these interest functions are annuities certain for various periods of time. We use the Ornstein-Uhlenbeck stochastic process as our means of modeling interest randomness about a fixed level. This process has the advantage that its sample functions tend to revert to the initial position, a property which seems appropriate for many interest rate scenarios. The finite dimensional distributions are normal, and the process has the Markovian property. One weakness of our model is that our random force of interest accumulation function may not be non-decreasing. However, we will demonstrate in Section 5 that the probability of negative force of interest accumulations is quite small. Admittedly, that only addresses part of the weakness. If market values of investments are utilized, it may happen that the random force of interest accumulation function may be decreasing, and indeed, may be negative for certain intervals of time.

When we compute the means and standard deviations, we are using function space integrals, i.e. averages over collections of functions which reflect the randomness over time. As a reference, see pages 166-168 of Beekman-Shiu (1988).

Let $R(s) = \delta + V(s)$, $0 \leq s \leq n$, be the variable forces of interest. Let $X(t) = \int_0^t V(s)ds$, $0 \leq t \leq n$. Then the force of interest accumulation function is $\int_0^t R(s)ds = \delta t + X(t)$, $t \geq 0$. We will assume that $\{X(t), 0 \leq t \leq n\}$ is an Ornstein-Uhlenbeck (O.U.) stochastic process. This process is both Gaussian and Markovian. In the notation of Beekman-Shiu (1988),

equations (3.3) and (3.4), its mean function

$$m(\tau) = E[X(\tau)] \equiv 0,$$

its autocorrelation function

$$\begin{aligned} C(s, t) &= E[X(s) - m(s)][X(t) - m(t)] \\ &= \beta^2 e^{-\kappa|s-t|}/(2\kappa), \text{ and} \\ X(t) &= \int_0^t \beta e^{-\kappa(t-\tau)} dZ(\tau), \end{aligned}$$

where $\{Z(\tau), 0 \leq \tau < \infty\}$ is the Wiener stochastic process, and κ and β are positive constants. We assume that $X(0) = 0$. The conditional mean and variance functions are

$$\begin{aligned} E[X(t) | X(s) = x] &= xe^{-\kappa(t-s)}, \text{ and} \\ \text{Var}[X(t) | X(s) = x] &= A(s, t) \\ &= \beta^2 [1 - e^{-2\kappa(t-s)}]/(2\kappa), \end{aligned}$$

for $t > s$.

We will denote the unconditional variance $\beta^2/(2\kappa)$ by σ^2 . An estimation procedure for κ is given in (3.23) of Beekman-Shiu (1988). As applied to certain U.S. Treasury bill returns, $\kappa \doteq 0.17$, and $e^{-\kappa} \doteq 0.84$. The sample variance could serve to estimate σ^2 .

The random present value of a future payment stream $b(t)$, $0 \leq t \leq n$, would be $\int_0^n b(t) \exp\{-\int_0^t R(s) ds\} dt$. When we use $b(t) \equiv 1$, $0 \leq t \leq n$, we write

$$\begin{aligned} \bar{a}_{\bar{n}|R} &= \int_0^n \exp\{-\int_0^t R(s) ds\} dt \\ &= \int_0^n \exp\{-\delta t - X(t)\} dt. \end{aligned}$$

The expected value of $\bar{a}_{\bar{n}|R}$ would be

$$E\{\bar{a}_{\bar{n}|R}\} = \int_{C_0[0, n]} \int_0^n \exp\{-\delta t - X(t)\} dt dX$$

where the function space integral is over $C_0[0, n]$, the set of continuous functions on $[0, n]$ which vanish at the time origin. By Fubini's Theorem, we can exchange orders of integration:

$$E\{\bar{a}_{\bar{n}|R}\} = \int_0^n \int_{C_0[0, n]} \exp\{-\delta t - X(t)\} dX dt.$$

See p. 99 of Beekman (1974) – note the typos: the 2nd and 3rd integrals should have the orders of integration $d\rho d\mu(x)$ and $d\mu(x) d\rho$, respectively. By using Theorem 1, p. 96, and Example 1, p. 97 of Beekman (1974), and then the moment generating function for a normal variate (see, e.g., pages 110-111 of Hogg and Craig (1978)),

$$\begin{aligned} E\{\bar{a}_{\bar{n}|R}\} &= \int_0^n e^{-\delta t} \int_{C_0[0, n]} e^{-X(t)} dX dt \\ &= \int_0^n e^{-\delta t} \int_{-\infty}^{\infty} e^{-x} \exp\{-x^2/[2A(0, t)]\} \{2\pi A(0, t)\}^{-\frac{1}{2}} dx dt \\ &= \int_0^n e^{-\delta t} \exp[A(0, t)/2] dt. \end{aligned}$$

Note that if $\beta^2 \rightarrow 0$ (corresponding to no randomness in interest), this reduces to $\bar{a}_{\bar{n}|\delta}$, as it should. The interchange of the limit and the integral is justified by Lebesgue's dominated convergence theorem.

We will evaluate

$$E\{\bar{a}_{\bar{n}|R}\} = \int_0^n e^{-\delta t} \exp\{\sigma^2[1 - e^{-2\kappa t}]/2\} dt$$

when $\kappa = 0.17$, $\delta = 0.05, 0.06, 0.07, 0.08$, $\sigma = 0.0200, 0.0100, 0.0050, 0.0025$, and $n = 5, 10, 20, 30$. These are displayed in Table 1.

Table 4 of P. Boyle (1976) contains some values for comparison. We must provide some ideas and notation from Boyle's paper, before presenting the comparative values. The one year rate of return in year t is a random variable \tilde{r}_t , and $\tilde{x}_t = 1 + \tilde{r}_t$. It is assumed that the $\{\tilde{x}_t\}$ are independent random variables. The random discounting factor $\tilde{v}_t = (1 + \tilde{r}_t)^{-1}$. The value of unit payments per annum at the ends of the next n years is denoted and defined by

$$\tilde{a}_{\bar{n}|} = \tilde{v}_1 + \tilde{v}_1 \tilde{v}_2 + \dots + (\tilde{v}_1 \tilde{v}_2 \dots \tilde{v}_n).$$

New random variables \tilde{z}_m are defined by

$$\tilde{z}_m = \prod_{k=1}^m \tilde{v}_k.$$

Table 1: $E\{\bar{a}_{n|R}\}$

	n = 5			n = 10		
$\delta \backslash \sigma$	0.0100	0.0050	0.0025	0.0100	0.0050	0.0025
0.05	4.424096	4.424012	4.423991	7.869655	7.869454	7.869404
0.06	4.319804	4.319723	4.319703	7.520059	7.519869	7.519822
$\delta \backslash \sigma$	0.0200	0.0100	0.0050	0.0200	0.0100	0.0050
0.07	4.219160	4.218846	4.218768	7.192597	7.191878	7.191698
0.08	4.121406	4.121101	4.121025	6.884296	6.883615	6.883445

	n = 20			n = 30		
$\delta \backslash \sigma$	0.0100	0.0050	0.0025	0.0100	0.0050	0.0025
0.05	12.642915	12.642537	12.642443	15.538045	15.537559	15.537437
0.06	11.647221	11.646877	11.646792	13.912256	13.911828	13.911721
$\delta \backslash \sigma$	0.0200	0.0100	0.0050	0.0200	0.0100	0.0050
0.07	10.764566	10.763317	10.763005	12.538356	12.536842	12.536463
0.08	9.977813	9.976673	9.976388	11.367823	11.366475	11.366138

Thus, $\bar{a}_{n|} = \bar{z}_1 + \bar{z}_2 + \dots + \bar{z}_n$. The expected value and variance of $\bar{a}_{n|}$ are

$$\sum_{m=1}^n E(\bar{z}_m),$$

and

$$\sum_{m=1}^n \text{Var}(\bar{z}_m) + 2 \sum_{j < k} \text{Cov}(\bar{z}_j, \bar{z}_k).$$

In obtaining the values for Table 4, Boyle assumes that \bar{x}_t is lognormally distributed with $\log \bar{x}_t$ having mean μ and various σ^2 . It is also assumed that $\{\bar{x}_t\}$ are identically distributed. One consequence is that

$$E\left(\prod_{k=1}^m \bar{v}_k\right) = \exp(-m\mu + m\sigma^2/2),$$

and

$$E(\bar{a}_{n|}) = \sum_{t=1}^n \exp(-t\mu + t\sigma^2/2).$$

In Table 4, for $\mu = 0.06$, and $\sigma = 0.01$, and $n = 5, 10, 20$, and 30 , the mean values of $\bar{a}_{\bar{n}|R}$ are 4.192, 7.298, 11.306, and 13.506 respectively.

We now wish to compute the second moment of $\bar{a}_{\bar{n}|R}$. By using Fubini's Theorem, a suitable split of the range of integration, Theorem 1, p. 96, Example 2, p. 97, and the definition of $v(\tau)$, $0 \leq \tau < \infty$, p. 89 of Beekman (1974),

$$\begin{aligned}
 E\{(\bar{a}_{\bar{n}|R})^2\} &= \int_{C_0[0,n]} \left[\int_0^n e^{-\delta t - X(t)} dt \right]^2 dX \\
 &= \int_{C_0[0,n]} \int_0^n e^{-\delta s - X(s)} ds \int_0^n e^{-\delta t - X(t)} dt dX \\
 &= \int_0^n \int_0^n \int_{C_0[0,n]} e^{-\delta s - X(s)} e^{-\delta t - X(t)} dX ds dt \\
 &= \int_0^n e^{-\delta t} \int_0^t e^{-\delta s} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\exp\left\{-x - y - \frac{x^2}{2A(0,s)} - \frac{[y - xe^{-\kappa(t-s)}]^2}{2A(s,t)}\right\}}{[(2\pi)^2 A(0,s)A(s,t)]^{\frac{1}{2}}} dx dy ds dt \\
 &+ \int_0^n e^{-\delta t} \int_t^n e^{-\delta s} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\exp\left\{-x - y - \frac{x^2}{2A(0,t)} - \frac{[y - xe^{-\kappa(s-t)}]^2}{2A(t,s)}\right\}}{[(2\pi)^2 A(0,t)A(t,s)]^{\frac{1}{2}}} dx dy ds dt.
 \end{aligned}$$

For ease of notation, let I be the first integral. By integration on y first, and using pp. 110-111 of Hogg and Craig (1978), we find that

$$\begin{aligned}
 I &= \int_0^n e^{-\delta t} \int_0^t e^{-\delta s} \int_{-\infty}^{\infty} \frac{\exp\left\{-x - \frac{x^2}{2A(0,s)} - xe^{-\kappa(t-s)} + \frac{A(s,t)}{2}\right\}}{[2\pi A(0,s)]^{1/2}} dx ds dt \\
 &= \int_0^n e^{-\delta t} \int_0^t e^{-\delta s} e^{A(s,t)/2} \int_{-\infty}^{\infty} \frac{\exp\left\{-x(1 + e^{-\kappa(t-s)}) - \frac{x^2}{2A(0,s)}\right\}}{[2\pi A(0,s)]^{1/2}} dx ds dt \\
 &= \int_0^n e^{-\delta t} \int_0^t e^{-\delta s + A(s,t)/2} \exp\left\{\frac{A(0,s)}{2} [1 + e^{-\kappa(t-s)}]^2\right\} ds dt.
 \end{aligned}$$

Let J be the second integral. In a similar manner,

$$J = \int_0^n e^{-\delta t} \int_t^n e^{-\delta s} \int_{-\infty}^{\infty} \frac{\exp\left\{-x - \frac{x^2}{2A(0,t)} - xe^{-\kappa(s-t)} + \frac{A(t,s)}{2}\right\}}{[2\pi A(0,t)]^{\frac{1}{2}}} dx ds dt$$

$$\begin{aligned}
&= \int_0^n e^{-\delta t} \int_t^n e^{-\delta s + A(t,s)/2} \int_{-\infty}^{\infty} \frac{\exp\{-x(1 + e^{-\kappa(s-t)}) - \frac{x^2}{2A(0,t)}\}}{[2\pi A(0,t)]^{\frac{1}{2}}} dx ds dt \\
&= \int_0^n e^{-\delta t} \int_t^n e^{-\delta s + A(t,s)/2} \exp\left\{\frac{A(0,t)}{2} [1 + e^{-\kappa(s-t)}]^2\right\} ds dt.
\end{aligned}$$

Therefore,

$$\text{Var}\{\bar{a}_{\bar{n}|R}\} = I + J - \left\{ \int_0^n e^{-\delta t} e^{A(0,t)/2} dt \right\}^2.$$

Note that if $\beta^2 \rightarrow 0$, or $n \rightarrow 0$, $\text{Var}\{\bar{a}_{\bar{n}|R}\} \rightarrow 0$, as it should. One can demonstrate that $I = J$, and that fact was used in creating Table 2.

Table 2: $[\text{Var}\{\bar{a}_{\bar{n}|R}\}]^{\frac{1}{2}}$

	n = 5			n = 10		
$\delta \backslash \sigma$	0.0100	0.0050	0.0025	0.0100	0.0050	0.0025
0.05	0.024173	0.012086	0.006043	0.046536	0.023267	0.011633
0.06	0.023482	0.011740	0.005870	0.044105	0.022052	0.011026
$\delta \backslash \sigma$	0.0200	0.0100	0.0050	0.0200	0.0100	0.0050
0.07	0.045635	0.022815	0.011407	0.083686	0.041837	0.020918
0.08	0.044347	0.022171	0.011085	0.079449	0.039719	0.019859

	n = 20			n = 30		
$\delta \backslash \sigma$	0.0100	0.0050	0.0025	0.0100	0.0050	0.0025
0.05	0.070094	0.035045	0.017522	0.078813	0.039404	0.019700
0.06	0.064025	0.032011	0.016005	0.070386	0.035190	0.017592
$\delta \backslash \sigma$	0.0200	0.0100	0.0050	0.0200	0.0100	0.0050
0.07	0.117407	0.058694	0.029345	0.126690	0.063333	0.031663
0.08	0.108009	0.053995	0.026996	0.114783	0.057380	0.028685

3 Random Deviations from Interest δ , and Random Mortality

In this section we replace the fixed period of the annuity by a random period modeling the future lifetime of the annuitant. The interest randomness is

preserved. Formulas for the mean values and standard deviations of the present values of future streams are obtained.

If T denotes the random future lifetime of (x) , then T has a probability density function of ${}_t p_x \mu_{x+t}$, $0 \leq t < \infty$. Assume that T is independent of the process $\{X(t), 0 \leq t < \infty\}$.

The random present value of a future payment stream $b(t)$, $0 \leq t \leq T$, would be $\int_0^T b(t) \exp\{-\int_0^t R(s) ds\} dt$. For $b(t) \equiv 1, \forall t$, we let

$$\bar{a}_{T|R} = \int_0^T \exp\{-\int_0^t R(s) ds\} dt.$$

By the independence assumption,

$$E[\bar{a}_{T|R}] = E_R E_T [\bar{a}_{T|R} | R].$$

By Theorem 3.1 on page 62 of Bowers et al (1986),

$$E_T [\bar{a}_{T|R} | R] = \int_0^\infty e^{-\delta t - X(t)} {}_t p_x dt.$$

With the use of Fubini's Theorem, and Section 2,

$$\begin{aligned} E_R E_T [\bar{a}_{T|R} | R] &= \int_0^\infty e^{-\delta t} {}_t p_x E_X \{e^{-X(t)}\} dt \\ &= \int_0^\infty e^{-\delta t} {}_t p_x \exp\{A(0, t)/2\} dt \\ &= \int_0^\infty e^{-\delta t} {}_t p_x \exp\{\sigma^2 [1 - e^{-2\kappa t}]/2\} dt. \end{aligned}$$

Note that as $\sigma^2 \rightarrow 0$, this reduces to \bar{a}_x , as it should.

For the second moment,

$$E[\bar{a}_{T|R}^2] = E[\{\int_0^T \exp\{-\int_0^t R(s) ds\} dt\}^2].$$

In the notation of Theorem 3.1 [Bowers et al (1986)],

$$\begin{aligned} Z(t) &= [\int_0^t \exp\{-\int_0^v R(u) du\} dv]^2, \text{ and} \\ Z'(t) &= 2[\int_0^t \exp\{-\int_0^v R(u) du\} dv] \exp\{-\int_0^t R(u) du\} \end{aligned}$$

$$\begin{aligned}
&= 2\left[\int_0^t \exp\{-\delta v - X(v)\} dv\right] e^{-\delta t - X(t)}. \text{ Thus,} \\
E[\bar{a}_{\overline{T}|R}^2] &= E_R E_T[Z(T)|R] \\
&= E_R\left\{\int_0^\infty 2 \int_0^t e^{-\delta v - X(v)} dv e^{-\delta t - X(t)} {}_t p_x dt\right\} \\
&= 2 \int_0^\infty \int_0^t E_R\{e^{-X(v)} e^{-X(t)}\} e^{-\delta v - \delta t} {}_t p_x dv dt \\
&\quad \text{by Fubini's Theorem} \\
&= 2 \int_0^\infty \int_0^t e^{-\delta v - \delta t} \exp\left\{\frac{A(v, t)}{2} + \frac{A(0, v)}{2} [1 + e^{-\kappa(t-v)}]^2\right\} {}_t p_x dv dt
\end{aligned}$$

by Section 2.

Remarks. As usual, $\text{Var}[\bar{a}_{\overline{T}|R}] = E[\bar{a}_{\overline{T}|R}^2] - \{E[\bar{a}_{\overline{T}|R}]\}^2$. These applications of Theorem 3.1 (loc. cit.) prove very helpful in our later numerical work.

Example. Let us now assume a Makeham law applies for the randomness in survival. Then $\mu_x = A + Bc^x$ for suitable constants A, B , and c , and

$${}_t p_x = \exp[-At - mc^x(c^t - 1)]$$

for $m = B/\ln c$. Thus,

$$E_R E_T[\bar{a}_{\overline{T}|R}] = \int_0^\infty e^{-\delta t} \exp\{\sigma^2[1 - e^{-2\kappa t}]/2\} \exp[-At - mc^x(c^t - 1)] dt.$$

In particular, we will use the values for A, B , and c given on page 72 of Bowers et al (1986), which were used in an Illustrative Life Table for ages 13-110. Thus, $A = 0.0007$, $B = 0.00005$, and $c = 10^{0.04}$. Also, we choose $\kappa = 0.17$. Since ${}_t p_x = 0$ for $t > 110 - x$,

$$\begin{aligned}
E_R E_T[\bar{a}_{\overline{T}|R}] &= \int_0^{110-x} e^{-\delta t} \exp\{\sigma^2[1 - e^{-0.34t}]/2\} \cdot \\
&\quad \exp[-0.0007t - 0.000543(10^{0.04x})(10^{0.04t} - 1)] dt.
\end{aligned}$$

To continue the example, we will let $x = 65, 70, 75$, and 80 , $\delta = 0.05, 0.06, 0.07, 0.08$, and $\sigma = 0.0200, 0.0100, 0.0050$, and 0.0025 . The results appear in Table 3.

Table 3: $E[\bar{a}_{T|R}]$

	x = 65			x = 70		
$\delta \backslash \sigma$	0.0100	0.0050	0.0025	0.0100	0.0050	0.0025
0.05	9.997377	9.997093	9.997021	8.506024	8.505792	8.505734
0.06	9.270577	9.270317	9.270252	7.973056	7.972842	7.972788
$\delta \backslash \sigma$	0.0200	0.0100	0.0050	0.0200	0.0100	0.0050
0.07	8.628516	8.627566	8.627329	7.494350	7.493560	7.493362
0.08	8.057140	8.056268	8.056050	7.061512	7.060779	7.060595

	x = 75			x = 80		
$\delta \backslash \sigma$	0.0100	0.0050	0.0025	0.0100	0.0050	0.0025
0.05	7.015673	7.015492	7.015447	5.594024	5.593891	5.593858
0.06	6.645794	6.645626	6.645584	5.352478	5.352353	5.352322
$\delta \backslash \sigma$	0.0200	0.0100	0.0050	0.0200	0.0100	0.0050
0.07	6.308186	6.307557	6.307400	5.128616	5.128143	5.128024
0.08	5.998107	5.997518	5.997371	4.919888	4.919440	4.919329

In a similar manner,

$$E[\bar{a}_{T|R}^2] = 2 \int_0^{110-x} \int_0^t e^{-\delta v - \delta t} \cdot \exp\{\sigma^2[1 - e^{-0.34(t-v)}]/2 + \sigma^2[1 - e^{-0.34v}]/2[1 + e^{-0.17(t-v)}]^2\} \cdot \exp[-0.0007t - 0.000543(10^{0.04x})(10^{0.04t} - 1)] dv dt.$$

We then let $x = 65, 70, 75,$ and $80,$ $\delta = 0.05, 0.06, 0.07, 0.08,$ and $\sigma = 0.0200, 0.0100, 0.0050,$ and $0.0025,$ and calculated the second moments. The standard deviations were computed in the usual way. The results appear in Table 4.

4 Randomness in $\delta,$ Deviations from $\delta,$ and Mortality

Instead of a fixed level $\delta,$ we will now consider a random level Δ with distribution function $P[\Delta \leq \delta] = F_{\Delta}(\delta), 0 \leq \delta \leq L$ for some upper limit

Table 4: $\{\text{Var}[\bar{a}_{T|R}]\}^{1/2}$

	x = 65			x = 70		
$\delta \backslash \sigma$	0.0100	0.0050	0.0025	0.0100	0.0050	0.0025
0.05	4.045069	4.044869	4.044819	3.921975	3.921796	3.921752
0.06	3.574495	3.574316	3.574271	3.527891	3.527729	3.527689
$\delta \backslash \sigma$	0.0200	0.0100	0.0050	0.0200	0.0100	0.0050
0.07	3.177656	3.177013	3.176852	3.187913	3.187326	3.187179
0.08	2.839882	2.839299	2.839154	2.892203	2.891669	2.891535

	x = 75			x = 80		
$\delta \backslash \sigma$	0.0100	0.0050	0.0025	0.0100	0.0050	0.0025
0.05	3.661435	3.661281	3.661243	3.277316	3.277190	3.277158
0.06	3.349754	3.349613	3.349578	3.046110	3.045994	3.045965
$\delta \backslash \sigma$	0.0200	0.0100	0.0050	0.0200	0.0100	0.0050
0.07	3.074991	3.074475	3.074346	2.838123	2.837691	2.837583
0.08	2.830978	2.830505	2.830386	2.649732	2.649332	2.649231

L. Assume that Δ is independent of T , and $\{X(t), 0 \leq t < \infty\}$. Then the expected values of $\bar{a}_{T|R}$ and $\bar{a}_{T|R}^2$ will involve three operations.

$$\begin{aligned}
 E[\bar{a}_{T|R}] &= E_{\Delta} E_R \{ E_T [\bar{a}_{T|R} | R, \Delta] | \Delta \} \\
 &= \int_0^L \int_0^{\infty} e^{-\delta t} e^{A(0,t)/2} p_x dt dF_{\Delta}(\delta).
 \end{aligned}$$

$$\begin{aligned}
 E[\bar{a}_{T|R}^2] &= E_{\Delta} E_R \{ E_T [\bar{a}_{T|R}^2 | R, \Delta] | \Delta \} \\
 &= \int_0^L \int_0^{\infty} 2 \int_0^t e^{-\delta v} e^{-\delta t} e^{A(v,t)/2} \exp\left\{ \frac{A(0,v)}{2} [1 + e^{-\kappa(t-v)}]^2 \right\} dv p_x dt dF_{\Delta}(\delta).
 \end{aligned}$$

As usual,

$$\text{Var}[\bar{a}_{T|R}] = E[\bar{a}_{T|R}^2] - \{E[\bar{a}_{T|R}]\}^2.$$

As an example, assume that $\Delta = 0.05, 0.06, 0.07$, and 0.08 with probabilities $0.10, 0.50, 0.20$, and 0.20 respectively. Then we obtain the following

tables.

Table 5: $E_{\Delta}E_R\{E_T[\bar{a}_{\bar{T}|R}|R, \Delta]|\Delta\}$

$\sigma \backslash x$	65	70	75	80
0.0100	8.971793	7.747998	6.485480	5.245158
0.0050	8.971544	7.747791	6.485317	5.245036

Table 6: $\{\text{Var}[\bar{a}_{\bar{T}|R}]\}^{1/2}$

$\sigma \backslash x$	65	70	75	80
0.0100	3.463658	3.414022	3.246349	2.961359
0.0050	3.463485	3.413866	3.246213	2.961247

5 Several Probabilities for the Ornstein-Uhlenbeck Process

In Section 2, it is observed that we want $\delta t + X(t) \geq 0$, $t \geq 0$ for non-negative force of interest accumulations. So we want the O.U. measure for the set of paths which stay above the line $-\delta t$, for $t \geq 0$. In order to utilize some earlier research on a similar problem, we will seek the O.U. measure for the set of paths which stay above the line $-a - \delta t$, $a > 0$, for $t \geq 0$. By the symmetry of the O.U. paths over $[0, Q]$ starting with $X(0) = 0$, we can seek the measure of the paths which stay below the line $a + \delta t$, $t \geq 0$. Or equally good, we seek the measure of the paths which exceed $a + \delta t$ for some t values in $[0, Q]$.

Remark. The following theorem generalizes Theorem 3 of Beekman-Fuelling (1979), by replacing the boundary function $A\sigma e^{\delta t}$, $t \geq 0$, by a function $f(t)$

where $f(t)$ is continuous on $[0, Q]$, and $f(0) > 0$. Its proof parallels that of Theorems 1, 2, and 3 of the cited paper. It is included because it presents the ideas in a more unified manner, and for the more general boundary function.

Theorem 1 Let $\{X(t), 0 \leq t < \infty\}$ be an O.U. process with transition density function ($s < t$)

$$\begin{aligned} p(x, s; y, t) &= \frac{\partial}{\partial y} P\{X(t) \leq y \mid X(s) = x\} \\ &= [2\pi A(s, t)]^{-\frac{1}{2}} \exp\left\{-\frac{\{y - x \exp[-\kappa(t - s)]\}^2}{2A(s, t)}\right\}, \end{aligned}$$

where

$$A(s, t) = \sigma^2[1 - \exp(-2\kappa(t - s))],$$

and

$$\sigma^2 > 0, \kappa > 0.$$

Let

$$F(Q) = P\{\max_{0 \leq t \leq Q} [X(t) - f(t)] \geq 0 \mid X(0) = 0\},$$

where $f(t)$ is continuous on $[0, Q]$, and $f(0) > 0$. Then $F(Q) = G(1)$, where $G(t)$, $t \geq 0$, is the solution of the integral equation

$$\int_0^t \psi\{|h(t) - h(s)|/(t - s)^{\frac{1}{2}}\} dG(s) = \psi\{h(t)/t^{\frac{1}{2}}\}$$

where

$$h(x) = \sigma^{-1}[x + (e^{2\kappa Q} - 1)^{-1}]^{\frac{1}{2}} f(\ln\{1 + x(e^{2\kappa Q} - 1)\})^{\frac{1}{2}}$$

and

$$\psi(x) = (2\pi)^{-\frac{1}{2}} \int_x^\infty \exp(-u^2/2) du, x \geq 0.$$

Proof. Let $\{Y(t), 0 \leq t < \infty\}$ be a second O.U. process with variance and covariance parameters of 1, that is, $\sigma^2 = \kappa = 1$. Then

$$F(Q) = P\{\max_{0 \leq t \leq \kappa Q} [Y(t) - f(t)/\sigma] \geq 0 \mid Y(0) = 0\}.$$

This follows from Theorem 2 of Part II of Beekman (1976). We now transform O.U. probabilities into probabilities concerning the Wiener process

$\{W(t), t \geq 0\}$, where $E\{W(t)\} = 0$ for all $t \geq 0$, and $\text{Covar}\{W(s), W(t)\} = \text{minimum}(s, t)$, for $s \geq 0, t \geq 0$. From page 138 of Rosenblatt (1962),

$$L\{Y(t), 0 \leq t < \infty\} = L\{e^{-t}W(e^{2t}), 0 \leq t < \infty\},$$

where L stands for probability law. Thus

$$\begin{aligned} L\{Y(t), t \geq 0 | Y(0) = 0\} &= L\{e^{-t}W(e^{2t}), t \geq 0 | W(1) = 0\} \\ &= L\{e^{-t}[W(e^{2t}) - W(1)], t \geq 0 | W(1) = 0\} \\ &= L\{e^{-t}W(e^{2t} - 1), t \geq 0\}, \end{aligned}$$

since the distributions of increments are stationary in time and $W(0) = 0$; see page 94 of Rosenblatt (1962). Therefore, one obtains

$$F(Q) = P\{\max_{0 \leq t \leq \kappa Q} [e^{-t}W(e^{2t} - 1) - f(t)/\sigma] \geq 0\}.$$

Transform the variable t to $\kappa\Delta$, so that

$$F(Q) = P\{\max_{0 \leq \Delta \leq Q} [e^{-\kappa\Delta}W(e^{2\kappa\Delta} - 1) - f(\kappa\Delta)/\sigma] \geq 0\}.$$

Next, let

$$x = (e^{2\kappa\Delta} - 1)/(e^{2\kappa Q} - 1).$$

This yields

$$F(Q) = P\{\max_{0 \leq \Delta \leq 1} [(1+x(e^{2\kappa Q}-1))^{-\frac{1}{2}}W((e^{2\kappa Q}-1)x) - f(\ln[x(e^{2\kappa Q}-1)+1]^{\frac{1}{2}})]/\sigma \geq 0\}.$$

We now use the property that distributions for the process $\{W(u), u \geq 0\}$ and the process $\{\theta^{\frac{1}{2}}W(u/\theta), u \geq 0\}$ are the same. Thus

$$F(Q) = P\{\max_{0 \leq \Delta \leq 1} [W(x) - (x + (e^{2\kappa Q} - 1)^{-1})^{\frac{1}{2}}f(\ln[x(e^{2\kappa Q} - 1) + 1]^{\frac{1}{2}})]/\sigma \geq 0\}.$$

The probability can be determined through Theorem 1 and section 4 of Park-Schuermann (1976). Since $\kappa Q > 0$,

$$\sigma^{-1}(x + (e^{2\kappa Q} - 1)^{-1})^{\frac{1}{2}}f(\ln[x(e^{2\kappa Q} - 1) + 1]^{\frac{1}{2}})$$

as a function $h(x)$ is continuous on $[0, 1]$, and $h(0) > 0$. This also uses the properties of $f(t), 0 \leq t \leq Q$. With the time end-point of 1 replaced

by t , the Wiener probability is denoted by $G(t)$ and satisfies the integral equation

$$G(t) = \psi[h(t)/t^{\frac{1}{2}}] + \int_0^t \Phi\{[h(t) - h(s)]/(t - s)^{\frac{1}{2}}\}dG(s),$$

where $\Phi(x) = 1 - \psi(x)$. Because $h(0) > 0$ implies that $G(0) = 0$, $G(t) = \int_0^t dG(s)$, and the above equation becomes

$$G(t) = \psi[h(t)/t^{\frac{1}{2}}] + G(t) - \int_0^t \psi\{[h(t) - h(s)]/(t - s)^{\frac{1}{2}}\}dG(s).$$

Upon canceling the $G(t)$ terms, the Theorem's conclusion is reached.

Example 1. $f(t) = A\sigma e^{\delta t}$, $t \geq 0$. Then

$$\begin{aligned} f(\ln\{1 + x(e^{2\kappa Q} - 1)\}^{\frac{1}{2}}) &= A\sigma e^{\delta \ln\{1 + x(e^{2\kappa Q} - 1)\}^{\frac{1}{2}}} \\ &= A\sigma\{1 + x(e^{2\kappa Q} - 1)\}^{\delta/2}, x \geq 0. \end{aligned}$$

Also

$$h(x) = A\{1 + x(e^{2\kappa Q} - 1)\}^{\delta/2}[x + (e^{2\kappa Q} - 1)^{-1}]^{\frac{1}{2}}, x \geq 0.$$

Example 2. $f(t) = a + bt$, $t \geq 0$, $a > 0$. Then

$$f(\ln\{1 + x(e^{2\kappa Q} - 1)\}^{\frac{1}{2}}) = a + b \ln\{1 + x(e^{2\kappa Q} - 1)\}^{\frac{1}{2}}, x \geq 0.$$

$$\begin{aligned} h(x) &= \sigma^{-1}[x + (e^{2\kappa Q} - 1)^{-1}]^{\frac{1}{2}}\{a + b \ln[1 + x(e^{2\kappa Q} - 1)]^{\frac{1}{2}}\} \\ &= [x + (e^{2\kappa Q} - 1)^{-1}]^{\frac{1}{2}}\left\{\frac{a}{\sigma} + \frac{b}{2\sigma} \ln[1 + x(e^{2\kappa Q} - 1)]\right\} \text{ for } x \geq 0. \end{aligned}$$

We will now choose some specific values for "a" and "b" in Example 2, and provide a table of values of $F(Q)$ for various Q values. We choose $b = 0.06$, and let "a" equal various constants.

Table 7: $F(Q)$ Values

a	Q = 1	Q = 5	Q = 10
0.0250	0.000000	0.000000	0.000000
0.0200	0.000000	0.000000	0.000000
0.0150	0.000009	0.000018	0.000018
0.0100	0.000809	0.001022	0.001022
0.0010	0.541455	0.543956	0.543956
0.0001	0.941373	0.941727	0.941727
0.000010	0.993986	0.994022	0.994022
0.000001	0.999397	0.999401	0.999401

Remark. The reader may find tables in Keilson-Ross (1976) and Beekman-Fuelling (1977) useful. Those references provide tables for

$$P\{\max_{0 \leq t \leq Q} X(t) < A\sigma \mid X(0) = 0\}$$

for various values of Q , A , σ , and κ .

There are many papers and monographs concerned with Brownian motion approximations to solutions for boundary crossing problems. In some cases, these probabilistic works also relate to actuarial science, in particular collective risk theory. Two such references are the monograph Siegmund (1985), and the paper Asmussen (1984).

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