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OPTION BOUNDS IN DISCRETE TIME WITH TRANSACTION COSTS

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ABSTRACT

Option bounds are obtained in a discrete-time framework with transaction costs. The model represents an extension of the Cox-Ross-Rubinstein binomial option pricing model to cover the case of proportional transaction costs. The method proceeds by constructing the appropriate replicating portfolio at each trading interval. Numerical values of these bounds are presented for a range of parameter values.

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1. INTRODUCTION

The classic Black-Scholes option pricing model rests on perfect market assumptions. A replicating portfolio can be constructed consisting of a long position in the risky asset and a short position in bonds which is equal in value to the price of a call option. As time passes, the weights of this portfolio are rebalanced so that it replicates the pay-off of the option contract at maturity. Under perfect market assumptions this rebalancing is costless, but if we introduce transaction costs this is no longer the case. It is of interest to explore the implications for option pricing and option replication of the introduction of transaction costs. Our paper explores this issue in a discrete-time setting.

The first paper to relax the assumption of no transaction costs in the context of option pricing was by Gilster and Lee [1984]. These authors also incorporated differential borrowing and lending rates in their analysis. Their basic model was developed using a continuous time framework but to circumvent the problem of infinite transaction costs they used discrete time rebalancing in their hedge construction.

Leland [1985] also examined this problem in a continuous-time framework. He obtained a modified Black-Scholes formula and develops an alternative replicating strategy to incorporate the impact of transaction costs. The Black-Scholes formula is modified by increasing the variance by a factor which depends on the magnitude of the transaction costs. Under his alternative replicating strategy the pay-off on the portfolio at option expiration approximates the maturity value of the call option. Leland notes the difficulty of incorporating transaction costs in a continuous-time framework and he uses periodic portfolio revisions for his numerical simulations.

An alternative and more convenient approach is to embed the problem in a true discrete-time framework. Merton [1990] uses this approach to explore the impact of transaction costs on option prices in a two-period binomial model. He assumes that the upper bounds for the option value is equal to the value of a replicating portfc ɔ which has the same value at expiration as the option. The portfolio revisions at the intermediate trading date allow for transaction costs. Merton uses this approach to determine the production cost to a financial intermediary of manufacturing a call option and ɜxamines the relationship between the bid-ask spreads in the option market to the size of the transaction costs in the market for the underlying asset.

Our approach is similar to Merton [1990] but we extend the analysis to an arbitrary number of periods whereas Merton's model just involved two periods. We employ the discrete time framework of Cox, Ross, and Rubinstein [1979] and obtain upper and lower bounds for the European call price in this framework. The upper bound is obtained by finding the cost of replicating a long position in the option by a dynamic hedging strategy. Proportional transaction costs are incurred at each revision date upon either the purchase or the sale of shares of the risky asset. The lower bound is obtained, in a similar way, by finding the cost of replicating a short position in the option. If there were no transaction costs the (absolute) value of these two portfolios would be equal. The impact of transaction costs is to drive a wedge between these two values: the higher the transaction costs the wider the wedge.

These bounds stem from no-arbitrage arguments. If an investor can create a long synthetic call more cheaply than she can purchase a comparable call in the market, an arbitrage opportunity exists. Hence, the current value of the replicating portfolio (for the long call) provides an upper bound for the call price. In the same way, the (absolute) value of the replicating portfolio that precisely duplicates the maturity payoff to a short call position provides a lower bound for the call price.

We begin with a single period model and indicate how to obtain the upper bound in this case. This one period model is then extended to several periods and we develop a recursive procedure to obtain the replicating portfolio for the upper bound. The procedure for obtaining the lower bound is similar but not identical. The zero-transaction costs option values lie between the lower bound and the upper bound, as we would expect. Furthermore, as the transaction costs tend to zero, the bounds converge to the Cox-Ross-Rubinstein option prices. We are able to derive a compact, closed-form expression for the option upper bound when the number of trading intervals is large. In this case, the option upper bound can be approximated by the ordinary Black-Scholes formula with an adjusted variance. Our variance adjustment is similar to, but larger than, that derived by Leland [1985] in his paper on transaction costs.

Shen [1990] also employs a discrete-time framework to examine the impact of transaction costs on option prices and obtains upper and lower bounds. His conclusions are similar to ours but there are distinct differences between the two approaches. Our approach differs from Shen's in that we derive a Black-Scholes-type approximation for the value of the replicating portfolio with transaction costs. In addition, we focus on some technical differences between the construction of the upper and lower bounds. On the other hand, Shen covers several issues that we do not address. These include a discussion of two types of settlements (cash or stock), option pricing by a risk averse dealer, and optimization of the number of trading dates in a fixed time interval. Hence, the two papers complement one another.

Other authors have recently explored the impact of transaction costs on option prices in different settings. Hodges and Neuberger [1989] assume a continuous-time framework and derive bounds on option prices by assuming a particular utility function for the intermediary (or individual) creating the hedge. Figlewski [1989] uses simulation techniques to examine the impact of transaction costs on option prices and concludes that

> "transaction costs for the standard arbitrage trade, induce arbitrage bounds around the theoretical option values that are substantially wider than the bid-ask spreads observed in practice."

Biais and Hillion [1990] derive a model for the bid-ask spread in option prices using a market micro-structure approach.

The layout of the present paper is as follows. In Section 2 we derive the upper bound for the one-period case and indicate how to extend the approach to many periods. In Section 3 we derive an analytical expression for the option upper bound in the presence of transaction costs. Section 4 develops some convenient approximations and derives a closed form Black-Scholes-type expression for the upper bound. In Section 5 we demonstrate how the lower bound for the option price can be derived. We explore the numerical properties of these bounds in Section 6. Section 7 concludes the paper.

2. OPTION REPLICATION IN DISCRETE TIME WITH TRANSACTION COSTS

In this section we use no-arbitrage arguments to establish the cost of creating a long European call option by dynamic hedging when there are transaction costs. This furnishes an upper bound for the call price. In the two-period case our model reduces to that obtained by Merton [1990] when allowance is made for differences in notation and convention.[1] We use the multiplicative binomial lattice employed by Cox, Ross, and Rubinstein [1979] for the asset price

[1]Our notation differs from Merton's and we make different assumptions concerning the transaction costs incurred at the outset and in the final period.

		Su^2
	Su	
S		Sud
	Sd	
		Sd^2

where we assume that $u > R > d$, with R equal to one plus the one-period riskless rate.[2] We use a dynamic hedging strategy to replicate the pay-off to a European call option. The replicating portfolio will be constructed backwards from the maturity date, i.e., if we know the portfolio at the points Su and Sd in the above diagram we will construct the portfolio at the point S. In order to take the transaction costs into account it is not enough to know the value of the replicating portfolio at each node; we also have to know how much is invested in the risky asset and how much is borrowed. We will use the symbol Δ for the number of shares and B for the dollar investment in bonds. The following diagram gives the Δ's and B's at the different points in the previous diagram:

[2]We use essentially the same notation as Cox, Ross, and Rubinstein except that we use R for one plus the one-period riskless rate and they use r.

		(Δ_3, B_3)
	(Δ_1, B_1)	
(Δ, B)		(Δ_4, B_4)
	(Δ_2, B_2)	
		(Δ_5, B_5)

Table I provides values of the weights of the replicating portfolio in the case of a simple two-period model when there are no transaction costs. The current value of the portfolio is 17.687 and to avoid arbitrage this must also be the price of the European call.

Table I: Hedge Portfolio Weights- Long call, transaction costs zero. Parameters: current asset price = 100, strike price = 100, number of periods = 2, u = 1.25, d = 0.80, r = 1.07.

		(1,−100)
	(1,−93.458)	
(0.701,−52.406)		(0,0)
	(0,0)	
		(0,0)

Current value of the replicating portfolio = 100 (0.70093) − 52.406 = 17.687.

To introduce transaction costs, assume that proportional transaction costs are incurred when shares of the risky asset are traded.[3] Let k be the transaction costs measured as a fraction of the amount traded. We must select Δ and B so that the portfolio (Δ_1, B_1) can be bought if the up-state Su occurs and (Δ_2, B_2) can be bought if the down-state Sd occurs. This leads to the following two equations:

$$\Delta Su + BR = \Delta_1 Su + B_1 + k|\Delta - \Delta_1|Su \tag{1}$$

$$\Delta Sd + BR = \Delta_2 Sd + B_2 + k|\Delta - \Delta_2|Sd. \tag{2}$$

Equation (1) indicates that the value of the portfolio in the up-state is exactly enough to buy the appropriate replicating portfolio corresponding to this state and to cover the transaction costs incurred in the rebalancing. Equation (2) has a similar interpretation for the down-state. Since we don't know whether the risky asset has to be bought or sold, but in both cases transaction costs have to be paid, we use the absolute value of $\Delta - \Delta_1$ and $\Delta - \Delta_2$. Equations (1) and (2) are two nonlinear equations in Δ and B, and it is not obvious whether a solution exists, and, if a solution exists, whether it is unique. However, in the appendix we prove the following result.

[3]Proportional transaction costs on bonds can also be incorporated. However, the model becomes much more complicated without providing new insights.

Theorem 1. In the construction of a synthetic long European call option by dynamic hedging, equations (1) and (2) have a unique solution (Δ, B). Furthermore, for this solution the following inequality holds

$$\Delta_2 \leq \Delta \leq \Delta_1 . \tag{3}$$

This theorem enables us to rewrite equations (1) and (2) in the following form

$$\Delta Su + BR = \Delta_1 Su + B_1 + k(\Delta_1 - \Delta)Su \tag{4}$$

$$\Delta Sd + BR = \Delta_2 Sd + B_2 + k(\Delta - \Delta_2)Sd, \tag{5}$$

or equivalently,

$$\Delta S\bar{u} + BR = \Delta_1 S\bar{u} + B_1 \tag{6}$$

$$\Delta S\bar{d} + BR = \Delta_2 S\bar{d} + B_2, \tag{7}$$

where

$$\bar{u} = u(1 + k) \quad \text{and} \quad \bar{d} = d(1 - k). \tag{8}$$

Thus, the theorem permits us to reduce the nonlinear equations to linear ones. These can be readily solved. These equations form the basis of an iterative procedure which can be used to obtain the composition of the replicating portfolio at inception. Working backwards from the boundary we can compute the explicit portfolio weights at each node of the lattice. This procedure makes appropriate adjustment for transaction costs each time shares of the risky asset have to be traded to rebalance the portfolio. We use this procedure in Section 6 to compute numerical values of the upper bounds.

If we replace $\bar{u}$ by u and $\bar{d}$ by d in equations (6) and (7) we have the familiar equations for discrete-time option pricing without transaction costs. Hence, one might be tempted to calculate the current portfolio value with transaction costs by replacing u by $\bar{u}$ and d by $\bar{d}$ in the standard fo ula for the option price C [see Cox, Ross, Rubinstein formula (6)]. This would lead to

$$C = \frac{\sum_{j=0} \binom{n}{j} \bar{p}^j (1 - \bar{p})^{n-j} \max[0, \bar{u}^j \bar{d}^{n-j} S - K]}{R^n} \tag{9}$$

with

$$\bar{p} = \frac{R - \bar{d}}{\bar{u} - \bar{d}}.$$

This, however, is incorrect since the right-hand sides of equations (6) and (7) no longer represent the values of the call in the up-state and the down-state, as in the no-transaction cost case. The actual value of the call in the up-state is $\Delta_1 Su + B_1$ instead of $\Delta_1 S\bar{u} + B_1$ and similarly for the down-state.

Our equations (4) and (5) provide the basis for the recursive approach to determine the portfolio weights at intermediate trading dates in the presence of transaction costs. Apart from notational differences, they correspond to equation (14.2c) in Merton [1990]. We assume that the institution (or intermediary) creating the replicating portfolio does not have to buy the initial amount of the risky asset (Δ). Hence, we just take account of the additional transaction costs necessary to maintain the replicating portfolio. In our numerical examples we assume that the replicating portfolio at option expiration for an in-the-money call option consists of one unit of the risky asset and a short position in riskless bonds equal to the exercise price. Our conventions correspond to those employed by Leland [1985] rather than those used by Merton [1990].[4]

We can rework our earlier two-period example to illustrate the impact of transaction costs. Table II provides the portfolio weights required to create a long (synthetic) call option when $k = 0.01$. Note that the holdings of the risky asset are higher in Table II than in Table I. The current value of the replicating portfolio of 18.307 represents the upper bound.

[4]Merton assumes that the entity constructing the replicating portfolio pays the necessary transaction costs to establish the initial asset position and also that the asset holdings in the replicating portfolio are liquidated at expiration.

Table II: Hedge Portfolio Weights- Long call, transaction costs 0.01. Parameters: current asset price = 100, strike price = 100, number of periods = 2, u = 1.25, d = 0.80, r = 1.07.

		(1,−100)
	(0.983,−90.950)	
(0.705,−52.156)		(0,0)
	(0,0)	
		(0,0)

Current value of the replicating portfolio = 100 (0.70463) − 52.156 = 18.307.

3. THE REPLICATING PORTFOLIO AS A DISCOUNTED EXPECTATION

It is well-known that the value of a European call option without transaction costs can be expressed as a discounted expectation of the maturity value of the option, assuming that the risky asset price follows a certain risk-neutral binomial process. In this section we will derive an analogous expected value formulation for the value of the replicating portfolio with transaction costs. In the no-transaction costs case the European call price is given by the Cox-Ross-Rubinstein binomial model:

$$C = \frac{\left[\sum_{j=0} \binom{n}{j} p^j (1 - p)^{n-j} \max[0, u^j d^{n-j} S - K]\right]}{R^n}, \tag{10}$$

where

$$p = \frac{R - d}{u - d}.$$

Inside the brackets we have the expectation of $\max[0, u^j d^{n-j} S - K]$, i.e., the value of the call at maturity, if we assume that the call follows a multiplicative binomial process with the probability of the up-state equal to p and the down-state equal to $1 - p$. The factor R^n means that the future expectation is discounted for n periods at the riskfree rate. We can obtain an equivalent formulation for the option price in terms of an expectation as follows: Let X_1, ..., X_n be independent, identical, binomial variables with possible values $\log_e u$ and $\log_e d$ and probabilities p and $(1 - p)$, respectively. Let $Y = \Sigma X_i$. Then the term within brackets is the expectation of $\max[0, Se^Y - K]$ or equivalently:

$$C = \frac{\mathrm{Exp}\left[(Se^Y - K) I_{Se^Y \geq K}\right]}{R^n}, \tag{11}$$

where

$$I_{Se^Y \geq K}$$

is the indicator function which has value 1 if $Se^Y \geq K$ and 0 otherwise.

We can derive a similar expression for the option price upper bound when there are transaction costs. From (6), (7), and (8) it follows that for a two-period discrete model

$$C = \Delta S + B = \frac{\bar{p}[(1 + k)\Delta_1 Su + B_1] + (1 - \bar{p})[(1 - k)\Delta_2 Sd + B_2]}{R}, \qquad (12)$$

where C is the current value of the portfolio that exactly replicates the payoff to a long European call position (with transaction costs). This can be further simplified to

$$\begin{aligned} C &= \Delta S + B \\ &= [\bar{p}\,\bar{p}_u\{(1 + k)\Delta_3 Su^2 + B_3\} + \bar{p}(1 - \bar{p}_u)\{(1 - k)\Delta_4 Sud + B_4\} + \\ &\quad (1 - \bar{p})\bar{p}_d\{(1 + k)\Delta_4 Sud + B_4] + (1 - \bar{p})(1 - \bar{p}_d)\{(1 - k)\Delta_5 Sd^2 + B_5\}]/R^2, \end{aligned} \qquad (13)$$

where

$$\bar{p}_u = \frac{R(1 + k) - \bar{d}}{\bar{u} - \bar{d}} \quad \text{and} \quad \bar{p}_d = \frac{R(1 - k) - \bar{d}}{\bar{u} - \bar{d}}.$$

It immediately follows that

$$0 < \bar{p}_d < \bar{p}_u < 1.$$

From (13) we see that while the right-hand side can be interpreted as a discounted expectation, we have a process for which the probability for a particular state depends on whether the previous jump was upwards or downwards. After an up-jump the probability of another up-jump is $\bar{p}_u$ while just after a down-jump the probability of another up-jump is $\bar{p}_d$.

After a down-jump the probability of another down-jump is larger than in the case of a preceding up-jump. This process can be formalized as follows: Let $X_1, X_2, X_3, \ldots, X_n$ be a Markov process with two states and values $\log_e u$ and $\log_e d$. The transition matrix is given by:

$$\bar{P} = \begin{pmatrix} \bar{p}_u & \bar{p}_d \\ 1 - \bar{p}_u & 1 - \bar{p}_d \end{pmatrix}. \qquad (14)$$

The first column of $\bar{P}$ represents the probability distribution of X_{j+1} if $X_j = \log_e u$ and the second column represents the probability distribution if $X_j = \log_e d$. The starting distribution for X_1 is given by

$$\hat{p} = (\bar{p}, \ (1 - \bar{p}))^T$$

(^{T} means the transposed vector). The following theorem can be proved by an induction argument.

Theorem 2. The current cost of creating a synthetic long European call option in the presence of proportional transaction costs can be expressed as follows:

$$C = \frac{\mathrm{Exp}\left[\left((1 + \bar{X}_n k) S e^{Y} - K\right) I_{Se^{Y} - K}\right]}{R^n}, \tag{15}$$

where n is the number of periods to option expiration,

$$Y = \sum_{i=1}^{n} X_i, \text{ and } \bar{X}_n = 1 \text{ if } X_n = \log_e u \text{ and } \bar{X}_n = -1 \text{ if } X_n = \log_e d.$$

Apart from the $\bar{X}_n k$ factor, the portfolio value is the discounted expectation of the call value at maturity. However, in this case the expectation is based on a different stochastic process from that used in the no-transaction cost case. For a standard call option without transaction costs the binomial process can also be visualized as a Markov process, but one with identical columns. This reflects the fact that the distribution of X_{j+1} doesn't depend on X_j. Expression (15) also shows us that the cost of replicating a long call position with transaction costs is greater than the cost of replicating a call without transaction costs. Since after an up-jump the possibility of another up-jump is larger, there is a higher probability of a whole sequence of up-jumps leading to a higher probability for a high value of Y. The same holds for downward moves leading to a higher probability for low values of Y. Hence, the variance of Y is much larger than that of the Y for a call without transaction costs. The higher variance leads, in turn, to a higher price.

It is convenient to define the constants and matrices

$$\theta = \frac{Rk}{\bar{u} - \bar{d}}, \quad A = \begin{pmatrix} 1 \\ -1 \end{pmatrix}(1, \ -1), \quad \text{and} \quad P = \begin{pmatrix} \bar{p} \\ 1 - \bar{p} \end{pmatrix}(1, \ 1). \tag{16}$$

We have the following matrix identity:

$$\bar{P} = \begin{pmatrix} \bar{p} + \theta & \bar{p} - \theta \\ 1 - \bar{p} - \theta & 1 - \bar{p} + \theta \end{pmatrix} = P + \theta A. \tag{17}$$

We see that the difference between our stochastic Markov process and a process with independent increments is given by the matrix θA. With no transaction costs, i.e., $k = 0$, this matrix is also zero and we are back to the standard case.

4. AN APPROXIMATION FOR THE OPTION PRICE UPPER BOUND WITH TRANSACTION COSTS WHEN THE NUMBER OF PERIODS IS LARGE

For numerical computations of the option price upper bound with transaction costs, it is convenient to use the recursive formulae given by equations (6) and (7) and work backwards through the lattice to obtain the explicit portfolio weights at each node. Equation (15) is less useful for practical computations, but it does have certain other advantages. In particular it can be used as a springboard to develop an accurate and very convenient closed-form approximation to the option price upper bound in discrete time when there are transaction costs. In this section we sketch the derivation of the approximation. It turns out that the approximation formula corresponds to the Black-Scholes formula with an adjusted variance.

To develop this value for large n, we first have to define a binomial tree. We will use the standard binomial tree as in Cox and Rubinstein [1985] with parameters[5] u, d, and R given by

$$u = e^{\sigma\sqrt{h}}, \quad d = e^{-\sigma\sqrt{h}}, \quad R = e^{rh} \tag{18}$$

where $h = T/n$, σ is the volatility of the risky asset, and r is the riskless continuous interest rate. We assume that the time to maturity of the option, T, is exactly one year.

In deriving the approximation formula it is necessary to make some approximations. However, we will show in the next section that the approximation formula is very accurate for the parameter values that are likely to be of interest.

If we consider the Markov process described in the previous section, we have the following result.

Lemma 1. The variance of the random variable Y of Theorem 2 has the following behaviour for large n and small k

[5]Clearly the parameters u, d, and R depend on n. We suppress this dependence for convenience.

$$\mathrm{Var}(Y) = \sigma^2\left(1 + \frac{2k}{\sigma}\sqrt{n}\right) + O\left(\frac{1}{\sqrt{n}}\right). \tag{19}$$

This result is proved in the appendix.

It is easy to show using the same method as in the proof of Lemma 1 that p (and also $\bar{p}$) tends to 0.5 as n becomes large. In particular, this implies that without transaction costs the probability of an up-jump tends to one-half for large n. When there are positive transaction costs the probability of an up-jump depends on whether the previous jump was an up-jump or a down-jump. If the previous jump was an up-jump the probability of an up-jump is $\bar{p} + \theta$. As n gets large $\bar{p}$ tends to 0.5 while for the range of parameter values we consider that θ is a positive number. For finite n, θ is less than one-half. This means that with transaction costs, if the process has an up-jump, the probability of moving to the up-state in the next period is greater than the probability of moving to the down-state.[6] The opposite holds in the case of a down-jump. If the process has just had a down-jump the probability of the subsequent jump being downwards is greater than one-half. Because of this property the process has a tendency to push more of the probability weight out towards the tails of the lattice.

The next stage of the approximation is to replace (15) by

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[6]Some specimen numerical values may help illustrate this point. We use parameter values in line with those used for our numerical computations in Section 5. Assume that $k = 0.005$, $\sigma = 0.2$, n (the number of periods) = 250, and the riskless rate = 10% p.a. With these values $p = 0.5119$, $\bar{p} = 0.5007$, $\theta = 0.1417$, $\bar{p}_u = 0.6424$, and $1 - \bar{p}_d = 0.6410$. If the process has just had an up-jump, the probability of the next jump being an up-jump is 0.6424. However, after a down-jump the probability of the next jump being a down-jump is 0.6410.

$$C = \frac{\mathrm{Exp}\left[(Se^{Y} - K)I_{Se^{Y} \geq K}\right]}{R^{n}}. \tag{20}$$

Hence, we omit the factor $(1 + \tilde{X}_n k)$. This omission does not mean that we neglect the impact of transaction costs. The influence of transaction costs arises mainly from the previous $(n - 1)$ realizations of the stochastic process rather than from this last factor. Furthermore, the factor $(1 + \tilde{X}_n k)$ does not differ much from 1 provided that k is small.[7] In expression (15) the impact of this factor is negligible compared with the influence of the underlying stochastic process on the call price.

The next step is to establish that the asset price process is risk-neutral under the new Markov process. To do this we compute an expression for the expected value of Y in Lemma 2. It turns out that it is convenient to have the expression for the variance of Y from Lemma 1 available in performing this derivation. That is why we derived the expression for the variance of Y first.

Lemma 2. The expected value of the random variable Y of Theorem 2 has the following behaviour for large n and small k:

[7]This restriction is important to ensure that equation (20) gives an accurate approximation for the option upper bound. As a practical matter k will be small: of the order of 1% or less. The approximation works very well for such values of k.

$$E(Y) = r - \frac{1}{2}[\mathrm{Var}(Y)] + O\left(\frac{1}{\sqrt{n}}\right), \tag{21}$$

where Var(Y) is given by equation (19). This lemma is proved in the appendix.

For large n the distribution of the random variable Y tends[8] to a normal distribution with mean and variance given by Lemmas 1 and 2. Hence, the distribution of the asset price tends to the corresponding lognormal distribution. We can use the standard Black-Scholes methodology to compute expression (20). This leads to the following theorem.

Theorem 3. For large n and small k the initial value of the hedge portfolio under a dynamic portfolio strategy that replicates a long call option at the maturity date and is self-financing inclusive of transaction costs, is equal to the Black-Scholes value but with modified variance given by

$$\sigma^2\left(1 + \frac{2k\sqrt{n}}{\sigma\sqrt{T}}\right), \tag{22}$$

where T is the time to option maturity.

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[8]See Billingsley [1979] Example 25.5 and Theorem 27.5 for the justification of using this limit.

Theorem 3 provides a very convenient method to compute the upper bound. As noted above we will illustrate the accuracy of the approximation formula in Section 6. We can compare our formula with that of Leland. In Leland's approach the dynamic portfolio strategy is not self-financing since he uses a continuous model with discrete revision times. Leland also derives a Black-Scholes-type formula with modified variance. The two expressions for the variance are very similar, but where Leland has a factor of $\sqrt{(2/\pi)}$ we have unity. Since $\sqrt{(2/\pi)} \approx 0.8$, our model leads to higher option values than Leland's. This is to be expected since our discrete model involves no residual hedging errors.

In the analysis thus far we have concentrated on dynamic hedging strategies that exactly replicate the payoff to a long position in a European call option. The current value of the replicating portfolio represents the cost of creating a synthetic option with the same terminal value to an economic agent facing proportional transaction costs **k**. As such, it provides an upper bound for the option price. We now examine the lower bound.

5. LOWER BOUNDS FOR THE OPTION PRICE

In this section we explore the derivation of a lower bound for the option price in a discrete-time model when there are proportional transaction costs.[9] To obtain the lower bound we compute the cost of creating a self-financing replicating portfolio which has exactly the same value at

[9] We are grateful to Fischer Black for suggesting we examine this issue.

expiration as a short position in a European call. The dynamic hedging strategy takes account of the transaction costs incurred at each trading date. The intuition corresponds closely with that behind the derivation of the upper bound. However, there are some important technical differences.

First consider a one-period model. Note that if we are replicating a short position in a call option the replicating portfolio at expiration will consist of a short position in the asset plus a long position in the risky asset (or else zero shares of each security). If the call is in the money at expiration the value the replicating portfolio at expiration will be negative. The replicating portfolio at the start of the period also involves a short position in the risky security.

We assume the same notation as in Section 2 and derive some arbitrage bounds that will be useful in the sequel. Assume an investor purchases one share of the risky asset at the start of the period and sells it at the end of the period. The initial amount required is $S(1+k)$ and if the up-state occurs the net proceeds upon sale are $Su(1-k)$. If the initial amount were invested in the riskless asset the proceeds would be $SR(1+k)$. Since the maximum return from the risky strategy must exceed the riskless return we have

$$u(1 - k) > R(1 + k). \tag{23}$$

In the same way if we consider the short sale of the risky asset we obtain:

$$R(1 + k) > d(1 + k). \tag{24}$$

From these two equations we note that

$$u(1 - k) > d(1 + k). \tag{25}$$

Now the recursive equations for the replicating portfolio which has a payoff equal to the short position in the call option are exactly the same equations as (1) and (2). One difference is that the sign of the holdings in the risky asset is now negative (or zero) on the boundary. Corresponding to Theorem 1 we have

Theorem 4. In the construction of a synthetic short call position by dynamic hedging there is a unique solution to equations (1) and (2) provided that equation (25) is satisfied.

The proof is given in the Appendix. It is instructive to compare this result with Theorem 1 for the long call position. There are some important differences. First, note that we require equation (25) to hold for Theorem 4 to be valid. No such requirement was needed to prove Theorem 1. Hence, we would expect Theorem 1 to be valid for a wider range of parameter values than the present theorem. We will find this to be the case in Section 6. Second, in the present case the number of shares of the risky asset at successive nodes on the expiration boundary satisfy

$$\Delta_{j+1} \geq \Delta_j,$$

where Δ_j is the number of shares at node j and Δ_{j+1} is the number of shares at the node (j + 1) just below it. The asset price at node j is higher than the asset price at node (j + 1). It is not necessarily the case that the number of shares to be held one period earlier at the node from which both j and (j + 1) can be accessed lies in the interval $[\Delta_j, \Delta_{j+1}]$. This contrasts with the situation in Theorem 1 where the number of shares held at a given node always lay in the interval spanned by the number of shares held at the two adjacent nodes one period later.

However, we can still compute the numerical values of the risky asset and bond holdings at each node to replicate the maturity payoff to a short call option. This is again accomplished by solving equations (1) and (2) recursivley. There are two changes with the earlier case. First, the boundary values will be exactly the negative of those for the long call position. Second, we have to be more careful at each step of the iteration to ensure we have obtained the correct holdings of the risky asset at each node in the correct region. We know from Theorem 4 that there is a unique solution but we cannot guarantee that it is in the region spanned by the holdings at the two adjacent next-period nodes. It is easy adapt our numerical algorithm to ensure that we have the correct solution in the appropriate region.

As noted earlier the current value of the replicating portfolio that generates the synthetic short call position provides (the negative of) a lower bound for the call option price. We provide some numerical estimates of the lower bounds obtained in this way in the next section. We also know that the value of a European call cannot be less than the difference between the current asset price and the present value of the strike price (or zero if this expression is negative). We can use this lower bound for those parameter values for which the approach suggested here breaks down.

It is instructive to illustrate the procedure involved using our earlier two-period example. Table III provides the portfolio weights at each node when we replicate the payoff to a short call position. These weights were obtained by finding the (unique) solution to equations (1) and (2) at each node of the lattice. The current portfolio value of −17.031 represents (the negative) of the option's lower bound. The weights of the risky asset holdings in Table III illustrate the second remark made after the proof of Theorem 4.

Table III: Hedge Portfolio Weights- Short call, transaction costs 0.01. Parameters: current asset price = 100, strike price = 100, number of periods = 2, u = 1.25, d = 0.80, r = 1.07.

		(−1,100)
	(−1.018,96.054)	
(−0.696,52.524)		(0,0)
	(0,0)	
		(0,0)

Current value of the replicating portfolio = 100 (− 0.69555) + 52.524 = − 17.031

In the case of the upper bound we were able to develop a compact expression for the long call and ultimately a Black-Scholes-type approximation to its value. In the present case the nature of the result in Theorem 4 suggests that we cannot write down an algebraic expression for the value of the lower bound corresponding to equation (15). The problem arises because the number of shares of the risky asset, Δ, sometimes lies within the interval $[\Delta_1,\Delta_2]$ and sometimes outside this interval. The portfolio weights in Table III illustrate this behaviour. In the topmost triangle, -1.018 lies outside the interval $[-1,0]$, whereas the asset portfolio weights for the other nodes lie within their appropriate intervals.

If we could assume that Δ always lay within the interval $[\Delta_1,\Delta_2]$ then equations (1) and (2) for the lower bound would be structurally the same as equations (6) and (7) for the upper bound except that $-\mathbf{k}$ replaces $\mathbf{k}$. Were this the case, we could develop the analogue of (15) by replacing $\mathbf{k}$ with $-\mathbf{k}$. This in turn would justify a Black-Scholes-type approximation for the lower bound with the variance given by equation (22) with $-\mathbf{k}$ replacing $\mathbf{k}$. This procedure of replacing $\mathbf{k}$ by $-\mathbf{k}$ and using the corresponding formula for the upper bound produces answers that are often quite close[10] to the accurate lower bounds even though the procedure lacks a rigorous justification.

[10] We investigated the accuracy of the lower bound results obtained by this procedure by comparing them with the accurate values obtained from solving (1) and (2) recursivley. The agreement was generally very good. To conserve space we do not report the detailed results here.

6. NUMERICAL CALCULATIONS

In this section we compute option price bounds for a range of parameter values.[11] In addition to illustrating the comparative statics.[12] These computations form a basis for comparison with the approximation we introduced earlier. For all our simulations we take the current price of the risky asset to be 100, the time to option expiry one year, and the (effective)[13] interest rate to be 10% p.a. For our base case assumptions the standard deviation of the return on the risky asset is 20% p.a. We examine the impact on the option bounds of variations in the strike price and of variations in the transaction costs. The zero transaction cost case corresponds to the Cox-Ross-Rubinstein case and is used as a benchmark.

Our approach is to present the results for the upper bound case first. ´ ıble IV provides the option upper bounds for a range of transaction cost and strike price assumptions. The first panel, corresponding to **k** = 0, contains the zero-transaction cost benchmark prices. The impact of transaction costs on the upper bound is most transparent if we subtract these reference prices from the option upper bounds. These differences are tabulated in Table IVa. As we would expect the influence of the transaction costs increases with the frequency of trading and also with the magnitude of the costs. As the strike price increases, the size of the upper bound spread

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[11] Our parameter values correspond to those assumed by Leland.

[12] Many of the comparative statics for the upper bound case were also observ Leland(1985).

[13] This means that one dollar invested for one year at the riskless rate accumulates to $1.10.

increases, reaches a maximum, and then decreases. The upper bound spread is highest when the current asset price is equal to the discounted strike price. This corresponds to the case when the option's time value is highest. In a discrete-time model we can see the intuition behind this result. Consider a call which is very deep in the money so that there is no chance[14] it will mature out of the money. The dynamic replicating portfolio in this case is certain to consist of a long position in the underlying asset and a short position in the discounted strike price. If this portfolio is maintained throughout the lattice there will be no transactions required. At the other extreme, consider an option which is so far out of the money that there is zero chance that it will end up in the money. In this case the option value at expiration will be zero so that the hedge portfolio is degenerate consisting of no risky asset and no bonds. To maintain such a portfolio throughout the lattice costs nothing and so transaction costs have no impact on the call's price (of zero). As the strike price movc away from either of these extremes the importance of transaction costs increases, reaching a maximum when the option's time value attains its maximum.

Tables V and Va examine the impact of the asset return variance on the call price upper bound when there are transaction costs. We know from the benchmark case that as the variance increases, the option price increases. Table Va shows that the upper bound spread generated by the inclusion of transaction costs also increases with the asset return variance.

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[14]This can occur in a true discrete-time binomial model. In the standard continuous-time model with lognormal asset returns there is always some chance that the call will not end up in the money.

In Table VI we compare the option price upper bounds produced by our exact discrete-time model with those of Leland for corresponding parameter values. In our model where the trading strategy is self-financing, the transaction costs exceed those of Leland's model.[15] As the magnitude of the transaction costs increases, the difference between the prices generated by the two models increases.

In Table VII we compare the upper bounds generated by our exact discrete model to the continuous-time approximation given in Theorem 3. We see that the approximation formula is very accurate and that the accuracy increases as the number of trading intervals increases. Recall that our maintained assumption is that the true asset return process follows a multiplicative binomial process as in Cox-Ross-Rubinstein. This table shows the Black-Scholes-type formula from Theorem 3 yields very close approximations to the true upper bounds.

The comparative statics for the lower bound case are very similar to those for the upper bound case. Table VIII provides lower bound values for the same set of parameters as Table IV. The lower bound spread increases with the frequency of trading and also with the size of the transaction costs. However, for certain parameter combinations the necessary conditions for the validity of Theorem 4 are violated and we cannot use our recursive procedure to compute the lower bound. These combinations are denoted with an asterisk in Table VIII. They correspond

[15]Recall that Leland used a discrete set of revision intervals to approximate a continuous-time model and that even if the stock follows the assumed (continuous) process there will be a hedging error at option maturity. A similar hedging error arises in the no-transaction cost case if the true stock return process has a continuous (lognormal) distribution and we approximate it by a multiplicative binomial process. The first panel of Table VII illustrates this point.

to situations where both k and n have higher values. For each of these combinations the inequality in equation (25) is violated. In these cases we have used the theoretical[16] lower bound values.

The final table, Table IX, presents combined information on the magnitude of the spread between the upper and lower bounds as a percentage of the no-transaction costs benchmark price. An example will illustrate how the figures in this table were obtained. Suppose that the strike price equals 100, the number of revisions is 52, and the transaction cost parameter $k = 0.00125$. For these parameter values the upper bound is 13.256 (from the second panel of Table IV) and the lower bound is 12.637 (from Table VIII). The benchmark option price in this case ($k = 0$) is 12.953. Hence, the lower bound-upper bound interval measured in terms of differences from the benchmark price is [−0.316,0.303]. We can express this range in terms of percen ge deviations from the benchmark price as [−2.44,2.34]. This is the format we use in Table IX. Table IX illustrates that these deviations are not quite symmetrical. These (percentage) spreads increase with increases in the strike price even though the dollar amount of the spreads increases and then decreases as the strike price increases.

[16]The theoretical bound is the maximum of zero and the difference between the current asset price and the present value of the strike price.

7. CONCLUSIONS

This paper derived a procedure for computing option price bounds in a discrete-time model when there are proportional transaction costs. The upper bound corresponds to the current value of a portfolio which exactly replicates the payoff to a long call. The corresponding lower bound can be established by finding the cost of replicating a short call position. We explored the numerical values of these bounds and concluded that the impact of transaction costs can be substantial especially if the number of revision times is large.

While our analysis just dealt with European call options it can be extended to cover European put options. We could derive the corresponding recursive equations for the put case and develop an algorithm for the multi-period case. It is more convenient to derive the put values from put-call parity.

We also demonstrated that the upper bound can be expressed as a discounted expectation under a new Markov process. This lead to an approximation for the upper bound in terms of a modified Black-Scholes formula. This modification involves increasing the variance as described in Theorem 3. The accuracy of the approximation increase with the number of trading intervals. We noted some interesting asymmetries between the properties of the upper bound and the lower bound.

Our approach assumes that the frequency of transactions is specified exogenously. To derive the bounds we have assumed within our discrete-time model that the replication is exact and that there will be no hedging errors at maturity. To ensure this, trading occurs at each trading date. Risk-averse economic agents will be willing to tolerate less than perfect hedging for a reduction in transaction costs. This leads to the possibility of determining the transaction frequency endogenously and some progress in this direction has been made in recent papers.[17]

[17]For example, Hodges and Neuberger [1989] and Shen [1990] consider this problem in an option context while Dumas and Luciano [1989] examine optimal portfolio revision policies in the presence of transaction costs.

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Table IV: European call option upper bounds. Option upper bounds computed in discrete-time setting using recursive procedure based on equations (6) and (7). Parameters: asset price = 100, standard deviation = 20% p.a., time to expiry = 1 year, interest rate = 10% p.a. effective.

OPTION PRICE UPPER BOUNDS

	Number of revision times			
	6	13	52	250
Strike Price				
	k = 0%			
80	27.703	27.701	27.665	27.675
90	19.821	19.740	19.667	19.674
100	12.655	13.093	12.953	12.984
110	8.129	8.026	7.972	7.965
120	4.216	4.427	4.548	4.551
	k = 0.125%			
80	27.735	27.747	27.753	27.876
90	19.894	19.842	19.865	20.103
100	12.770	13.248	13.256	13.630
110	8.254	8.205	8.324	8.715
120	4.329	4.595	4.882	5.269
	k = 0.5%			
80	27.837	27.894	28.047	28.574
90	20.113	20.149	20.453	21.346
100	13.106	13.699	14.111	15.339
110	8.618	8.721	9.300	10.649
120	4.663	5.084	5.820	7.161
	k = 2%			
80	28.297	28.563	29.409	31.568
90	20.983	21.346	22.643	25.524
100	14.358	15.333	16.966	20.413
110	9.965	10.555	12.469	16.192
120	5.926	6.859	8.950	12.750

Table IVa: Difference between option price upper bounds and zero transaction cost prices for same parameter values as Table IV. Option upper bounds with transaction costs computed in discrete-time setting using recursive procedure based on equations (6) and (7). Parameters: asset price = 100, standard deviation = 20% p.a., time to expiry = 1 year, interest rate = 10% p.a. effective.

UPPER BOUND SPREAD

	Number of revision times			
	6	13	52	250
Strike Price				
	k = 0.125%			
80	0.032	0.046	0.089	0.201
90	0.073	0.102	0.198	0.429
100	0.115	0.155	0.304	0.646
110	0.124	0.179	0.352	0.750
120	0.113	0.168	0.334	0.718
	k = 0.5%			
80	0.134	0.193	0.383	0.900
90	0.292	0.408	0.786	1.672
100	0.452	0.606	1.158	2.354
110	0.489	0.695	1.328	2.684
120	0.447	0.657	1.272	2.610
	k = 2%			
80	0.594	0.862	1.744	3.893
90	1.162	1.606	2.976	5.849
100	1.703	2.240	4.013	7.429
110	1.836	2.529	4.497	8.227
120	1.710	2.432	4.402	8.199

Table V: European call option upper bounds with transaction costs for different volatility assumptions. Option upper bounds with transaction costs computed in discrete-time setting using recursive procedure based on equations (6) and (7). Parameters: asset price = 100, standard deviation = 10%, 20%, 30% p.a., time to expiry = 1 year, interest rate = 10% p.a. effective.

	OPTION PRICE UPPER BOUNDS			
	Number of revision times			
	6	13	52	250
Standard deviation in %				
	k = 0%			
10	9.772	9.960	9.936	9.954
20	12.655	13.093	12.953	12.984
30	16.009	16.677	16.435	16.480
	k = 0.125%			
10	9.852	10.073	10.155	10.424
20	12.770	13.248	13.256	13.630
30	16.131	16.841	16.757	17.169
	k = 0.5%			
10	10.089	10.406	10.785	11.700
20	13.106	13.699	14.111	15.339
30	16.490	17.320	17.677	19.034
	k = 2%			
10	11.005	11.646	12.927	15.475
20	14.358	15.333	16.966	20.413
30	17.844	19.087	20.839	24.788

Table Va: Difference between option price upper bounds and zero-transaction cost prices for same parameter values as Table V. Option upper bounds with transaction costs computed in discrete-time setting using recursive procedure based on equations (6) and (7). Parameters: asset price = 100, standard deviation = 10%, 20%, 30% p.a., time to expiry = 1 year, interest rate = 10% p.a. effective.

	UPPER BOUND SPREADS			
	Number of revision times			
	6	13	52	250
Standard deviation in %				
	k = 0.125%			
10	0.079	0.113	0.218	0.470
20	0.115	0.155	0.304	0.646
30	0.122	0.164	0.323	0.689
	k = 0.5%			
10	0.317	0.446	0.849	1.746
20	0.452	0.606	1.158	2.354
30	0.481	0.643	1.242	2.554
	k = 2%			
10	1.233	1.685	2.991	5.522
20	1.703	2.240	4.013	7.429
30	1.835	2.411	4.405	8.308

Table VI: Comparison of option upper bounds generated by our discrete-time model and those obtained from Leland's model. Option upper bounds computed in our model using recursive procedure based on equations (6) and (7). Parameters: asset price = 100, standard deviation = 20% p.a., time to expiry = 1 year, interest rate = 10% p.a. effective. L = Leland, BV = Ours.

	Number of revision times					
	6L	6BV		52L	52BV	
Strike Price			Absolute Difference			Absolute Difference
	k = 0%					
80	27.675	27.703	0.028	27.675	27.665	0.010
90	19.675	19.821	0.147	19.675	19.667	0.008
100	12.993	12.655	0.338	12.993	12.953	0.040
110	7.966	8.129	0.164	7.966	7.972	0.006
120	4.555	4.216	0.339	4.555	4.548	0.007
	k = 0.125%					
80	27.698	27.735	0.037	27.745	27.753	0.008
90	19.728	19.894	0.167	19.831	19.865	0.034
100	13.075	12.770	0.305	13.232	13.256	0.024
110	8.062	8.254	0.192	8.246	8.324	0.077
120	4.647	4.329	0.318	4.822	4.882	0.060
	k = 0.5%					
80	27.771	27.837	0.066	27.974	28.047	0.073
90	19.887	20.113	0.227	20.296	20.453	0.157
100	13.317	13.106	0.210	13.915	14.111	0.196
110	8.344	8.618	0.274	9.035	9.300	0.265
120	4.916	4.663	0.254	5.582	5.820	0.238
	k = 2%					
80	28.091	28.297	0.206	29.026	29.409	0.383
90	20.515	20.983	0.468	22.052	22.643	0.592
100	14.225	14.358	0.133	16.253	16.966	0.714
110	9.388	9.965	0.577	11.659	12.469	0.809
120	5.926	5.926	0.000	8.172	8.950	0.778

Table VII: *Comparison of accurate option upper bounds based on our discrete-time model with the Black-Scholes type approximation used in Theorem 3 of this paper.* Accurate upper bounds with transaction costs computed using recursive procedure based on equations (6) and (7). Parameters: asset price = 100, standard deviation = 20% p.a., time to expiry = 1 year, interest rate = 10% p.a. effective. BV = Our discrete model, A = Approximation value (Theorem 3).

	Number of revision times								
Strike Price	6BV	6A	Absolute Difference	52BV	52A	Absolute Difference	250BV	250A	Absolute Difference
	k = 0%								
80	27.703	27.675	0.028	27.665	27.675	0.010	27.675	27.675	0.000
90	19.821	19.675	0.147	19.667	19.675	0.008	19.674	19.675	0.000
100	12.655	12.993	0.338	12.953	12.993	0.040	12.984	12.993	0.008
110	8.129	7.966	0.164	7.972	7.966	0.006	7.965	7.966	0.000
120	4.216	4.555	0.339	4.548	4.555	0.007	4.551	4.555	0.004
	k = 0.125%								
80	27.735	27.705	0.031	27.753	27.764	0.010	27.876	27.876	0.000
90	19.894	19.741	0.153	19.865	19.870	0.006	20.103	20.102	0.001
100	12.770	13.096	0.326	13.256	13.292	0.036	13.630	13.636	0.006
110	8.254	8.086	0.167	8.324	8.316	0.008	8.715	8.714	0.001
120	4.329	4.670	0.341	4.882	4.889	0.007	5.269	5.272	0.004
	k = 0.5%								
80	27.837	27.797	0.040	28.047	28.056	0.009	28.574	28.572	0.002
90	20.113	19.940	0.173	20.453	20.451	0.002	21.346	21.342	0.004
100	13.106	13.397	0.291	14.111	14.135	0.023	15.339	15.340	0.001
110	8.618	8.438	0.180	9.300	9.286	0.014	10.649	10.645	0.004
120	4.663	5.006	0.343	5.820	5.826	0.006	7.161	7.162	0.002
	k = 2%								
80	28.297	28.207	0.090	29.409	29.398	0.011	31.568	31.549	0.019
90	20.983	20.724	0.260	22.643	22.603	0.040	25.524	25.498	0.025
100	14.358	14.513	0.155	16.966	16.941	0.025	20.413	20.389	0.024
110	9.965	9.715	0.250	12.469	12.418	0.050	16.192	16.166	0.026
120	5.926	6.246	0.320	8.950	8.933	0.017	12.750	12.733	0.017

Table VIII: European call option lower bounds. Option lower bounds computed in discrete-time setting using recursive procedure based on equations (1) and (2). Parameters: asset price = 100, standard deviation = 20% p.a., time to expiry = 1 year, interest rate = 10% p.a. effective.

OPTION PRICE LOWER BOUNDS

	Number of revision times			
	6	13	52	250
Strike Price				
k = 0%				
80	27.703	27.701	27.665	27.675
90	19.821	19.740	19.667	19.674
100	12.655	13.093	12.953	12.984
110	8.129	8.026	7.972	7.965
120	4.216	4.427	4.548	4.551
k = 0.125%				
80	27.671	27.656	27.582	27.502
90	19.749	19.638	19.469	19.246
100	12.538	12.935	12.637	12.286
110	8.003	7.843	7.604	7.136
120	4.102	4.256	4.202	3.773
k = 0.5%				
80	27.582	27.534	27.383	27.273
90	19.531	19.333	18.889	18.221
100	12.168	12.445	11.597	9.684
110	7.614	7.269	6.374	3.647
120	3.754	3.726	3.077	0.879
k = 2%				
80	27.327	27.276	27.273*	27.273*
90	18.697	18.281	18.182*	18.182*
100	10.323	10.115	9.091*	9.091*
110	5.845	4.311	0.0*	0.0*
120	2.266	1.266	0.0*	0.0*

*signifies that inequality (25) is violated for these parameter values.

Table IX: Upper and lower bounds expressed as a percentage deviation from the benchmark no-transaction costs case. Parameters: asset price = 100, standard deviation = 20% p.a., time to expiry = 1 year, interest rate = 10% p.a. effective.

	Number of revision times		
	13	52	250
Strike Price			
	k = 0.125%		
80	[-0.16, 0.17]	[-0.30, 0.32]	[-0.62, 0.73]
90	[-0.52, 0.52]	[-1.01, 1.01]	[-2.18, 2.18]
100	[-1.21, 1.19]	[-2.44, 2.34]	[-5.38, 4.97]
110	[-2.28, 2.23]	[-4.62, 4.41]	[-10.41, 9.41]
120	[-3.86, 3.80]	[-7.60, 7.34]	[-17.10, 15.77]
	k = 0.5%		
80	[-0.61, 0.70]	[-1.02, 1.38]	[-1.45, 3.25]
90	[-2.06, 2.07]	[-3.95, 4.00]	[-7.39, 8.50]
100	[-4.95, 4.63]	[-10.47, 8.94]	[-25.42, 18.13]
110	[-9.43, 8.66]	[-20.04, 16.66]	[-54.21, 33.70]
120	[-15.84, 14.85]	[-32.34, 27.98]	[-80.68, 57.34]

APPENDIX

PROOF OF THEOREM 1

We prove Theorem 1 by backward induction. By induction we may assume that $\Delta_4 \leq \Delta_1 \leq \Delta_3$ and $\Delta_5 \leq \Delta_2 \leq \Delta_4$. Thus, $\Delta_2 \leq \Delta_1$. Subtracting (2) from (1), transferring everything to the right-hand side, and introducing the function $f(\Delta)$ we get

$$f(\Delta) = \Delta S(u - d) - \Delta_1 Su + \Delta_2 Sd - B_1 + B_2 - k|\Delta - \Delta_1|Su + k|\Delta - \Delta_2|Sd = 0. \quad (A1)$$

The function $f(\Delta)$ is continuous and piecewise linear, i.e., it is a linear function on $(-\infty, \Delta_2)$, (Δ_2, Δ_1), and (Δ_1, ∞) with constant derivatives on each interval with values $[(1 + k)u - (1 + k)d]S$, $[(1 + k)u - (1 - k)d]S$, and $[(1 - k)u - (1 - k)d]S$, respectively. Since all of these numbers are strictly positive, $f(\Delta)$ is a monotonically increasing piecewise linear function. Hence, it has a unique zero. This proves the first claim of Theorem 1. For the second part it is enough to show that

$$f(\Delta_2) \leq 0 \quad \text{and} \quad f(\Delta_1) \geq 0 \quad (A2)$$

since this implies that $\Delta \in [\Delta_2, \Delta_1]$. Now

$$f(\Delta_2) = (\Delta_2 - \Delta_1)Su(1 + k) - B_1 + B_2 \quad (A3)$$

$$f(\Delta_1) = (\Delta_2 - \Delta_1)Sd(1 - k) - B_1 + B_2. \tag{A4}$$

Define

$$g(t) = (\Delta_2 - \Delta_1)St - B_1 + B_2. \tag{A5}$$

g is a decreasing linear function of t. Since by induction $\Delta_4 \leq \Delta_1 \leq \Delta_3$, we know that one of the equations from which Δ_1 has been deduced reads as follows:

$$\Delta_1 Sud + B_1 R = \Delta_4 Sud + B_4 + k(\Delta_1 - \Delta_4)Sud. \tag{A6}$$

Similarly, since $\Delta_5 \leq \Delta_2 \leq \Delta_4$ we have:

$$\Delta_2 Sdu + B_2 R = \Delta_4 Sdu + B_4 + k(\Delta_4 - \Delta_2)Sdu. \tag{A7}$$

Subtracting (A6) from (A7) and dividing by R gives

$$\frac{(\Delta_2 - \Delta_1)Sdu}{R} + B_2 - B_1 = \frac{k[(\Delta_4 - \Delta_2) - (\Delta_1 - \Delta_4)]Sdu}{R}. \tag{A8}$$

In case $\Delta_1 = \Delta_2$, we must have that $\Delta_4 = \Delta_1 = \Delta_2$ and hence $B_2 = B_1$. Thus, $\Delta = \Delta_1$ and $B = B_1/R$ is the unique solution of (1) and (2) which means that $\Delta_1 = \Delta = \Delta_2$. Hence, we may assume from now on that $\Delta_1 > \Delta_2$.

First, we remark that $|(\Delta_4 - \Delta_2) - (\Delta_1 - \Delta_4)| \leq \Delta_1 - \Delta_2$. We consider the two cases $(\Delta_4 - \Delta_2) - (\Delta_1 - \Delta_4) \geq 0$ and $(\Delta_4 - \Delta_2) - (\Delta_1 - \Delta_4) \leq 0$. In the first case we know that $g(du/r) \geq 0$. Hence,

$$f(\Delta_1) = g(d(1 - k)) \geq g(du/R) \geq 0.$$

Further

$$\begin{aligned} f(\Delta_2) &= g(u(1 + k)) \\ &\leq g\left(\frac{du(1 + k)}{R}\right) \\ &= g\left(\frac{du}{R}\right) + \frac{k(\Delta_2 - \Delta_1)Sdu}{R} \\ &= \frac{k[(\Delta_4 - \Delta_2) - (\Delta_1 - \Delta_4)]Sdu}{R} + \frac{k(\Delta_2 - \Delta_1)Sdu}{R} \\ &\leq 0. \end{aligned}$$

Thus we have proved (A2) in this case. If $(\Delta_4 - \Delta_2) - (\Delta_1 - \Delta_4) \leq 0$ the proof goes similarly.

To start the induction we consider the option at maturity. At maturity there are two possible portfolios: $\Delta = 1$ and $B = -K$, if the asset price is above the exercise price and $\Delta = 0$ and $B = 0$, if the asset price is below the exercise price. Hence, at maturity we always have $\Delta_1 \geq \Delta_2$ in the notation of this appendix. One period before maturity there are three different cases. First, $\Delta_1 = \Delta_2 = 1$ in which case $\Delta = \Delta_1$ and $B = -K/R$ is the unique solution, which indeed has $\Delta_2 \leq \Delta \leq \Delta_1$. Second, $\Delta_1 = \Delta_2 = 0$, in which case $\Delta = 0$ and $B = 0$ is the unique solution which indeed has $\Delta_2 \leq \Delta \leq \Delta_1$. Finally, $\Delta_1 = 1$, $\Delta_2 = 0$. In this case the unique solution is

$$\Delta = \frac{(S\bar{u} - K)}{(S\bar{u} - Sd)}.$$

Hence, $\Delta_2 = 0 < \Delta < 1 = \Delta_1$. This completes the first steps of the induction proof.

PROOF OF LEMMA 1

To calculate the variance of Y we first introduce the vector $v^T = (\log_e u, \log_e d)$. It follows for properties of transition matrices like $\bar{P}$ that:

$$EX_i = v^T \bar{P}^{i-1} \hat{p} \tag{A9}$$

$$EX_i X_{i+j} = v^T \bar{P}^j \begin{pmatrix} \log_e u & 0 \\ 0 & \log_e d \end{pmatrix} \bar{P}^{i-1} \hat{p}. \tag{A10}$$

Hence,

$$\begin{aligned} Cov(X_i, X_{i+j}) &= EX_i X_{i+j} - EX_i EX_{i+j} \\ &= v^T \bar{P}^j \left[\begin{pmatrix} \log_e u & 0 \\ 0 & \log_e d \end{pmatrix} - \bar{P}^{i-1} \hat{p} v^T \right] \bar{P}^{i-1} \hat{p}. \end{aligned} \tag{A11}$$

Since

$$(1, \quad 1)\left[\begin{pmatrix}\log_e u & 0 \\ 0 & \log_e d\end{pmatrix} - \bar{P}^{i-1}\hat{p}v^T\right] = 0 \tag{A12}$$

and PA = 0 we can reduce (A11) to

$$\begin{aligned} v^T(\theta A)^j\left[\begin{pmatrix}\log_e u & 0 \\ 0 & \log_e d\end{pmatrix} - \bar{P}^{i-1}\hat{p}v^T\right]\bar{P}^{i-1}\hat{p} = \\ \theta^j 2^{j-1}(\log_e u - \log_e d)(1, \quad -1)\left[\begin{pmatrix}\log_e u & 0 \\ 0 & \log_e d\end{pmatrix} - \bar{P}^{i-1}\hat{p}v^T\right]\bar{P}^{i-1}\hat{p}. \end{aligned} \tag{A13}$$

If we denote

$$\bar{P}^{i-1}\hat{p} = (p_i, \quad 1 - p_i)^T$$

we find that

$$\mathrm{Cov}(X_i, X_{i+j}) = p_i(1 - p_i)\theta^j 2^j(\log_e u - \log_e d)^2. \tag{A14}$$

To calculate $\mathrm{Var}(Y) = \mathrm{Var}(\Sigma X_i)$ we simply have to add all the covariances, i.e., $\Sigma\mathrm{Var}(X_i) + 2\Sigma\Sigma\mathrm{Cov}(X_i, X_{i+j})$. We thus find as the total variance

$$(\log_e u - \log_e d)^2 \left[\sum_{i=1}^{n} p_i(1 - p_i)\left(2\sum_{j=0}^{n-i} (2\theta)^j - 1 \right) \right] = (\log_e u - \log_e d)^2 \left[\sum_{i=1}^{n} p_i(1 - p_i)\left(2\left[\frac{1 - (2\theta)^{n-i+1}}{1 - 2\theta} \right] - 1 \right) \right]. \tag{A15}$$

Furthermore,

$$\begin{aligned} \tilde{P}^{i-1}\hat{p} &= (P + A\theta)^{i-1}\hat{p} \\ &= \sum_{j=0}^{i-1} (\theta A)^j P^{i-1-j}\hat{p} \\ &= \sum_{j=1}^{i-1} (\theta A)^j \hat{p} + \hat{p} \\ &= \sum_{j=1}^{i-1} \theta^j 2^{j-1}(2\bar{p} - 1)\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \hat{p} \\ &= \theta\left(\frac{1 - (2\theta)^{i-1}}{1 - 2\theta} \right)(2\bar{p} - 1)\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \hat{p}, \end{aligned} \tag{A16}$$

where we have used that

$$PA = 0 \quad \text{and} \quad P\hat{p} = \hat{p}.$$

Hence,

$$p_i(1 - p_i) = \left(\bar{p} + \theta\left(\frac{1 - (2\theta)^{i-1}}{1 - 2\theta}\right)(2\bar{p} - 1)\right) * \left(1 - \bar{p} - \theta\left(\frac{1 - (2\theta)^{i-1}}{1 - 2\theta}\right)(2\bar{p} - 1)\right). \tag{A17}$$

Substituting (A17) in (A15) and simplifying the resulting expression leads to[18]

$$\begin{aligned}(\log_e u - \log_e d)^2 \Bigg\{ n\Bigg[\left(\frac{\bar{p} - \theta}{1 - 2\theta}\right)\left(\frac{1 - \bar{p} - \theta}{1 - 2\theta}\right)\left(\frac{1 + 2\theta}{1 - 2\theta}\right) - \frac{2(2\bar{p} - 1)^2\theta(2\theta)^n}{(1 - 2\theta)^3}\Bigg] \\ + \left(\frac{\bar{p} - \theta}{1 - 2\theta}\right)\left(\frac{1 - \bar{p} - \theta}{1 - 2\theta}\right)\left(\frac{-4\theta}{1 - 2\theta}\right)\left(\frac{1 - (2\theta)^n}{1 - 2\theta}\right) \\ + \frac{(2\bar{p} - 1)^2}{(1 - 2\theta)^4}\left[\theta + \theta^2 - \theta(2\theta)^n - \theta^2(2\theta)^{2n}\right]\Bigg\}.\end{aligned} \tag{A18}$$

To prove Lemma 1 we remark that

$$(\log_e u - \log_e d)^2 = \frac{4\sigma^2}{n}. \tag{A19}$$

The next step is to examine each of the terms in expression (A18) to determine their dependence on n. At this stage we do not require assumptions concerning the magnitude of k. (Later on we assume k to be small.) In expression (A18) the major contribution will arise from the first term in square brackets:

---

[18]A detailed derivation is available from the authors.

$$\frac{\bar{p} - \theta}{1 - 2\theta} = \frac{e^{r/n} - e^{-\sigma/\sqrt{n}}(1 - k) - e^{r/n}k}{e^{\sigma/\sqrt{n}}(1 + k) - e^{-\sigma/\sqrt{n}}(1 - k) - 2e^{r/n}k}$$

$$= \frac{(1 - k)[1 + r/n + O(1/n^2)] - (1 - k)[1 - \sigma/\sqrt{n} + \sigma^2/2n + O(1/n\sqrt{n})]}{(1 + k)[1 + \sigma/\sqrt{n} + \sigma^2/2n + O(1/n\sqrt{n})] - (1 - k)[1 - \sigma/\sqrt{n} + \sigma^2/2n + O(1/n\sqrt{n})] - 2k[1 + r/n + O(1/n^2)]}$$

$$= \frac{(1 - k)\sigma/\sqrt{n} + (1 - k)(r - \sigma^2/2)/n + O(1/n\sqrt{n})}{2\sigma/\sqrt{n} - 2k(r - \sigma^2/2)/n + O(1/n\sqrt{n})}$$

$$= \left(\frac{1 - k}{2}\right)\left\{\frac{1 + (r - \sigma^2/2)/\sigma\sqrt{n} + O(1/n)}{1 - k(r - \sigma^2/2)/\sigma\sqrt{n} + O(1/n)}\right\}$$

$$= \left(\frac{1 - k}{2}\right)\left\{1 + \frac{(1 + k)(r - \sigma^2)/2}{\sigma\sqrt{n}} + O\left(\frac{1}{n}\right)\right\}, \qquad \text{(A20)}$$

which, if less precision is required, can be written as

$$\left(\frac{1 - k}{2}\right)\left\{1 + O\left(\frac{1}{\sqrt{n}}\right)\right\}.$$

$$\frac{1 - \bar{p} - \theta}{1 - 2\theta} = 1 - \frac{\bar{p} - \theta}{1 - 2\theta}$$

$$= 1 - \left(\frac{1 - k}{2}\right)\left\{1 + \frac{(1 + k)(r - \sigma^2/2)}{\sigma\sqrt{n}} + O\left(\frac{1}{n}\right)\right\} \qquad \text{(A21)}$$

$$= \frac{1 + k}{2}\left\{1 - \frac{(1 - k)(r - \sigma^2/2)}{\sigma\sqrt{n}} + O\left(\frac{1}{n}\right)\right\}.$$

or, again depending on the required precision,

$$\left(\frac{1 + k}{2}\right)\left\{1 + O\left(\frac{1}{\sqrt{n}}\right)\right\}.$$

$$\begin{aligned}\frac{1 + 2\theta}{1 - 2\theta} &= \frac{(1 + k)e^{\sigma/\sqrt{n}} - (1 - k)e^{-\sigma/\sqrt{n}} + 2ke^{r/n}}{(1 + k)e^{\sigma/\sqrt{n}} - (1 - k)e^{-\sigma/\sqrt{n}} - 2ke^{r/n}} \\ &= \frac{4k + 2\sigma/\sqrt{n} + O(1/n)}{2\sigma/\sqrt{n} + 2k(\sigma^2/2 - r)/n + O(1/n\sqrt{n})} \\ &= \frac{2k\sqrt{n}/\sigma + 1 + O(1/\sqrt{n})}{1 + k(\sigma^2/2 - r)/\sigma\sqrt{n} + O(1/n)} \qquad \text{(A22)} \\ &= \frac{2k\sqrt{n}}{\sigma} + 1 - \frac{2k^2(\sigma^2/2 - r)}{\sigma^2} + O\left(\frac{1}{\sqrt{n}}\right) \\ &= \sqrt{n}\left\{\frac{2k}{\sigma} + \frac{1 + 2k^2(r - \sigma^2/2)/\sigma^2)}{\sqrt{n}} + O\left(\frac{1}{n}\right)\right\},\end{aligned}$$

or, again depending on the required precision,

$$\sqrt{n}\left\{\frac{2k}{\sigma} + O\left(\frac{1}{\sqrt{n}}\right)\right\}.$$

Finally,

$$
\begin{aligned}
\frac{\theta}{1 - 2\theta} &= \frac{1}{4}\left(\frac{1 + 2\theta}{1 - 2\theta} - 1\right) \\
&= \frac{1}{4}\left[\sqrt{n}\left(\frac{2k}{\sigma} + O\left(\frac{1}{\sqrt{n}}\right)\right) - 1\right] \\
&= \sqrt{n}\left[\frac{k}{2\sigma} + O\left(\frac{1}{\sqrt{n}}\right)\right],
\end{aligned} \tag{A23}
$$

$$
\begin{aligned}
\frac{1}{1 - 2\theta} &= \frac{1 + 2\theta}{1 - 2\theta} - \frac{2\theta}{1 - 2\theta} \\
&= \sqrt{n}\left[\frac{k}{\sigma} + O\left(\frac{1}{\sqrt{n}}\right)\right],
\end{aligned} \tag{A24}
$$

$$
\begin{aligned}
\frac{2\bar{p} - 1}{1 - 2\theta} &= \left(\frac{\bar{p} - \theta}{1 - 2\theta}\right) - \left(\frac{1 - \bar{p} - \theta}{1 - 2\theta}\right) \\
&= -k + \frac{(1 + k)(1 - k)(r - \sigma^2/2)}{\sigma\sqrt{n}} + O\left(\frac{1}{\sqrt{n}}\right) \\
&= -k + O\left(\frac{1}{\sqrt{n}}\right),
\end{aligned} \tag{A25}
$$

depending on the required precision. Also,

$$
|(2\theta)^n| = \left|\frac{2^n k^n e^r}{e^{\sigma\sqrt{n}}\{(1 + k) - (1 - k)e^{-2\sigma/\sqrt{n}}\}^n}\right|. \tag{A26}
$$

The expression on the right-hand side of (A26) can be shown to be less than

$$e^{r}e^{-[\sigma + \log_e(1 + \alpha)]\sqrt{n}},$$

where

$$\alpha = \frac{(1 - k)\sigma(1 - \sigma)}{k}.$$

From this last inequality we see that we can skip all terms with a factor of $(2\theta)^n$ in expression (A18) if we want to calculate the limit behaviour of (A18) for $n \to \infty$. Substituting (A20)-(A25) in what remains of (A18) and using alternatives in (A20)-(A22) depending on the required precision we derive

$$\frac{4\sigma^2}{n}\left\{n\left(\frac{1 - k^2}{4}\right)\sqrt{n}\left(\frac{2k}{\sigma} + \frac{1}{\sqrt{n}} + \frac{6k^2(r - \sigma^2/2)}{\sigma^2\sqrt{n}} + O\left(\frac{1}{n}\right)\right) + \right.$$
$$\left.(\sqrt{n})^2\left\{-\left(\frac{1 - k^2}{4}\right)\frac{2k^2}{\sigma^2} + O\left(\frac{1}{\sqrt{n}}\right)\right\} + (\sqrt{n})^2\left\{\frac{3k^4}{4\sigma^2} + O\left(\frac{1}{\sqrt{n}}\right)\right\}\right\}$$
$$= \sigma^2\sqrt{n}(1 - k^2)\left(\frac{2k}{\sigma} + \left[\frac{1 + 6k^2(r - \sigma^2/2)/\sigma^2 - 2k^2/\sigma^2 + 3k^4/(1 - k^2)\sigma^2}{\sqrt{n}}\right] + O\left(\frac{1}{n}\right)\right).$$

If k is small the expression simplifies to

$$\sigma^2\left(1 + \frac{2k\sqrt{n}}{\sigma}\right).$$

This completes the proof of Lemma 1.

PROOF OF LEMMA 2

We use the same type of approach here as was used in establishing Lemma 1. From equation (A9) and the definition of E(Y) we have

$$E(Y) = \sum_{i=1}^{n} v^T \bar{P}^{i-1} \hat{p}$$

$$= \sum_{i=1}^{n} v^T \left[\theta \left(\frac{1 - (2\theta)^{i-1}}{1 - 2\theta} \right) (2\bar{p} - 1) \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \hat{p} \right]$$

$$= (\log_e u - \log_e d) \sum_{i=1}^{n} \theta \left(\frac{1 - (2\theta)^{i-1}}{1 - 2\theta} \right) (2\bar{p} - 1) + \sum_{i=1}^{n} [\bar{p} \log_e u + (1 - \bar{p}) \log_e d]$$

$$= (\log_e u - \log_e d) \frac{n\theta(2\bar{p} - 1)}{1 - 2\theta} - (\log_e u - \log_e d) \left(\frac{\theta(2\bar{p} - 1)}{1 - 2\theta} \right) \left(\frac{1 - (2\theta)^n}{1 - 2\theta} \right)$$

$$+ n\bar{p} \log_e u + n(1 - \bar{p}) \log_e d$$

$$= 2\sigma\sqrt{n}\,\theta \left(\frac{2\bar{p} - 1}{1 - 2\theta} \right) - \frac{2\sigma}{\sqrt{n}} \left(\frac{\theta}{1 - 2\theta} \right) \left(\frac{2\bar{p} - 1}{1 - 2\theta} \right) + 2\sigma\sqrt{n}\,\bar{p} - \sigma\sqrt{n}$$

$$= 2\sigma\sqrt{n}\,\theta \left(\frac{2\bar{p} - 1}{1 - 2\theta} \right) - \frac{2\sigma}{\sqrt{n}} \left(\frac{\theta}{1 - 2\theta} \right) \left(\frac{2\bar{p} - 1}{1 - 2\theta} \right) + \sigma\sqrt{n} \left(\frac{2\bar{p} - 1}{1 - 2\theta} \right) (1 - 2\theta)$$

$$= \left(\frac{2\bar{p} - 1}{1 - 2\theta} \right) \sigma\sqrt{n} - \frac{2\sigma}{\sqrt{n}} \left(\frac{\theta}{1 - 2\theta} \right) \left(\frac{2\bar{p} - 1}{1 - 2\theta} \right)$$

$$= \left(\frac{2\bar{p} - 1}{1 - 2\theta} \right) \left\{ \sigma\sqrt{n} - \frac{2\sigma}{\sqrt{n}} \left(\frac{\theta}{1 - 2\theta} \right) \right\}$$

$$= \left[-k + \frac{(1 - k^2)(r - \sigma^2/2)}{\sigma\sqrt{n}} + O\left(\frac{1}{n}\right)\right]\left[\sigma\sqrt{n} - k + O\left(\frac{1}{\sqrt{n}}\right)\right]$$

$$= -k\sigma\sqrt{n} + k^2 + (1 - k^2)\left(r - \frac{\sigma^2}{2}\right) + O\left(\frac{1}{\sqrt{n}}\right)$$

$$= r - \frac{\sigma^2}{2}\left[1 + \frac{2k\sqrt{n}}{\sigma}\right] + k^2\left(1 - r + \frac{\sigma^2}{2}\right) + O\left(\frac{1}{\sqrt{n}}\right)$$

$$= r - \frac{1}{2}\mathrm{Var}(Y) + k^2\left(1 - r + \frac{\sigma^2}{2}\right) + O\left(\frac{1}{\sqrt{n}}\right),$$

where we have used the results of Lemma 1. When k is small we have

$$E(Y) = r - \frac{1}{2}\mathrm{Var}(Y).$$

This completes the proof of Lemma 2.

PROOF OF THEOREM 4

First we show that if a solution exists it is unique. Then we establish existence.

Recall equations (1) and (2):

$$\Delta Su + BR = \Delta_1 Su + B_1 + k|\Delta - \Delta_1|Su \quad (1)$$

$$\Delta Sd + BR = \Delta_2 Sd + B_2 + k|\Delta - \Delta_2|Sd. \quad (2)$$

Subtracting (2) from (1) and transferring everything to the right-hand side,

$$f(\Delta) = \Delta S(u - d) - \Delta_1 Su + \Delta_2 Sd - B_1 + B_2 - k|\Delta - \Delta_1|Su + k|\Delta - \Delta_2|Sd$$
$$= 0.$$

There are two possible cases: $\Delta_1 \leq \Delta_2$ and $\Delta_1 > \Delta_2$.

When $\Delta_1 \leq \Delta_2$, $f(\Delta)$ is a linear function on $(-\infty,\Delta_1)$, (Δ_1,Δ_2), and (Δ_2,∞), with constant derivatives on each interval with values

$$(1 + k)S(u - d), \quad [(1 - k)u - (1 + k)d]S, \quad \text{and} \quad (1 - k)(u - d)S,$$

respectively. If $(1 - k)u \geq (1 + k)d$, all of these derivatives are positive, and $f(\Delta)$ is an increasing function.

When $\Delta_1 > \Delta_2$, $f(\Delta)$ is a linear function on $(-\infty,\Delta_2)$, (Δ_2,Δ_1), and (Δ_1,∞), with constant derivatives on each interval with values

$$(1 + k)S(u - d), \quad [(1 + k)u - (1 - k)d]S, \quad \text{and} \quad (1 - k)(u - d)S,$$

respectively. All of these derivatives are positive, and therefore, $f(\Delta)$ is an increasing function. Thus, $f(\Delta)$ is a monotonically increasing function. Hence, if $f(\Delta)$ has a zero, it has a unique zero, i.e., the system has a unique solution.

We now prove that a solution to equations (1) and (2) always exists.

Select Δ_{01} such that

$$\Delta_{01} < \frac{(B_1 - B_2) + (1 + k)S(\Delta_1 u - \Delta_2 d)}{(1 + k)S(u - d)} \quad \text{and} \quad \Delta_{01} \le \min\{\Delta_1, \Delta_2\}.$$

Select Δ_{02} such that

$$\Delta_{02} > \frac{(B_1 - B_2) + (1 - k)S(\Delta_1 u - \Delta_2 d)}{(1 - k)S(u - d)} \quad \text{and} \quad \Delta_{02} \ge \max\{\Delta_1, \Delta_2\}.$$

It is easy to show that $f(\Delta_{01}) < 0$ and $f(\Delta_{02}) > 0$. Since $f(\Delta)$ is a continuous function, there exists a Δ, ($\Delta_{01} < \Delta < \Delta_{02}$), such that $f(\Delta) = 0$. This completes the proof of Theorem 4.