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ASPECTS OF INTEREST RATE MODELS

ABSTRACT

Interest rate modelling is discussed, with special emphasis on the long and short rate model of Brennan and Schwartz (1979). Comment is made on an unexpected aspect of the solution of the resulting partial differential equation. In addition, an analysis is made of a related matter; the predictive power of the term structure of interest rates.

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1 Introduction

Many different types of interest rate models have been proposed. This paper concentrates on the models which use diffusion processes to model rates and proceed to set up a partial differential equation for the price of a unit discount bond. Such models were reviewed by Sharp(1990).

One of the earliest and most commonly used diffusion models is that of Brennan and Schwartz (1979). A derivation of this model is presented as Appendix A. The model uses diffusion processses for two rates, the instantaneous zero-term ("short") rate and the "long" yield on a consol (irredeemable) bond. The short rate is arrived to drift towards a value related to the current long rate.

In Section 2 of this paper is discussed the solution of the second order partial differential equation which results from the above model. It is demonstrated that although the analytic solution has not been determined, it can be shown to have a character which could be regarded as unrealistic.

In Section 3 is discussed the predictive power of the term structure. One might expect that if the yield curve is strongly upwards sloping, then Treasury bill rates would tend to increase. Indeed this behaviour would be in line with the form assumed for the drift of the short rate under the model of Brennan and Schwartz(1979).

The data available in 1979 was not inconsistent with such a drift. However it is shown that with the addition of data from the 1980's there is little evidence of any power of the term structure to predict future Treasury bill rates.

2 A Characteristic of the Brennan and Schwartz Model

It is useful to consider an interesting feature of the Brennan and Schwartz (1979) model which has hitherto not been mentioned in the literature despite its importance. The model is described in Appendix A.

Consider (A.26) for small values of u_l (e.g. $0.00 \leq u_l \leq 0.25$) and moderate values of u_r (e.g. $0.25 \leq u_r \leq 1$). Note that at the $u_l = 0$ boundary there is by (A.27) and (A.33) a discontinuity at $\tau = 0$. The numerical results confirm that $\partial b/\partial u_l$ reaches very high values as this discontinuity is propagated, and the dominant terms in (A.26) in the region considered are

$$-\frac{1}{n}\frac{\partial b}{\partial u_l}(1 - u_l)[1 - \frac{u_l}{u_r}] \doteq \frac{\partial b}{\partial \tau} \tag{2.01}$$

Then (2.01) is an advective equation by which a disturbance is propagated with speed given by the coefficient of $\partial b/\partial u_l$. Equations of this type are discussed by Vemuri and Karplus (1981, p. 159). The initial condition for (2.01) is just before bond maturity

$$b(u_r, u_l, 0+) \begin{cases} = 1 & 1 \geq u_\ell > 0 \\ = 0 & u_\ell = 0 \end{cases}$$

This shape is propagated into the $u_l > 0$ region as a Heaviside function. Neglecting the other terms in (A.26) or equivalently (A.23) one can show through consideration of the speed of propagation $\ell(\sigma_l^2 + \ell - r)$ that the time taken for the discontinuity of the Heaviside function to travel from infinity to the point r, l is

$$\tau_H(r, l) = \frac{1}{\sigma_l^2 - r} \ln \left(\frac{l + \sigma_l^2 - r}{l} \right) \tag{2.02}$$

In the limit $\sigma_l^2 - r \to 0$, $\tau_H(\sigma_l^2, \ell) = 1/\ell$. It is emphasized that this discontinuity is a genuine solution of (A.26) and is not a product of the numerical methods.

The propagation of the Heaviside function is actually modified by the terms found in (A.26) in addition to those found in (2.01). Nonetheless, the effect is seen in Appendix B which gives the price $P(r(20), \ell(20), 20)$ of a unit discount bond 20 years from maturity as produced by a hopscotch finite difference algorithm (Gourlay and McGee, 1977). The parameter values used are loosely based on those of Brennan and Schwartz (1979) and are intended as examples for demonstration purposes. Only a 9×9 subset of the 101×101 matrix of values is shown, but one can see the propagation of the very low bond values from the $\ell = \infty, r = 0$ corner. The values given by equation (2.02) are verified. At $\sigma_l = 0.0866, r = 0, \tau = 20$ the inversion of equation (2.02) gives $\ell = 0.0463$, and it can be seen that the disturbance has indeed reached approximately this point. It should be noted in verifying the position of the disturbance that the bond values are already reduced from 1 by the operation of the other terms in (A.26), so that the best estimate of the position of the disturbance is not where the bond value is one half.

A similar disturbance results from the $\partial b/\partial u_r$ term in (A.26), but its impact is not so great as that resulting from the $\partial b/\partial u_l$ term.

Thus this unusual behaviour of the solution leads to unusual values for the bond price. Considering a fixed time to maturity, eg 20 years. Then the bond price is close to zero over much of the plane of interest rate values, as in Appendix B. The boundary region over which the price rises to significant values is very small. This type of behaviour is not what one would expect to arise from a fully realistic model of interest rates.

3 Predictive Power of the Term Structure

One view of the term structure is based on variations of the "expectations hypothesis". Typically, forward rates are thought to be estimators of future short rates where the estimator has an upward bias because of investors' risk aversion. This view has been investigated eg by Fama (1984) who used U.S. interest rate data and regression techniques. He found that bond yields did give some information about movements of short rates up to about five months in the future.

The method now described is based on nonparametric techniques. An advantage is that no assumption need be made, for example, about any change or lack of change over time of the level of the random fluctuations.

Appendix C presents monthly tender rates $t(i,j)$ of Canadian Treasury bills where i represents the calendar year and j the month. Appendix D presents monthly data $m(i,j)$ on yields to maturity of 1 to 3 year Canadian government bonds. A test was devised of the hypothesis that in each calendar year i the "spread" $m(i,1) - r(i,1)$ at the start of the calendar year has some power to predict the increase $r(i,12) - r(r,1)$ in Treasury bill rates during the year. The choice of calendar year periods is made for the sake of convenience.

Appendix E illustrates the operation of the test. Within the $N = 41$ year period 1949 - 1989, ranks $v_s(i)$, 1 to 41 are assigned to the January spreads. Ranks $v_c(i)$ are assigned separately to the eleven month increases in Treasury bill rates within the calendar year.

Then, denoting the beginning and ending calendar years by y_1 and y_N, a calculation is made of the statistic

$$D = \sum_{i=y_1}^{y_N} (v_s(i) - v_c(i))^2 \doteq \frac{1}{3}N(N+1)(2N+1) - 2\sum_{i=y}^{y_N} v_s(i)v_c(i) \tag{3.1}$$

where the equality is not exact in view of the possibility of tied ranks.

Under the null hypothesis of independence of the ranks, the expectation of D is given (Lehmann, 1975) by

$$E_{H_0}\ (D) = \frac{N^3 - N}{6}$$

and the standard deviation can be approximated by

$$(Var_{H_0}(D)) \doteq \left(\frac{N^2(N+1)^2(N-1)}{36}\right)^{1/2} \tag{3.2}$$

For the period I, 1949 - 1989 one finds the following:

$$\begin{aligned} D^I &= 11,416 \\ E_{H_0}(D^I) &= 11,480 \\ (Var_{H_0}(D))^{1/2} &\doteq 1,815.15 \end{aligned}$$

Thus the statistic D^I is only 0.0352 standard deviation from its expectation under the null hypothesis of independence of the spreads and the changes in Treasury bill rates. Thus there is no evidence of the power of the interest rate spread to predict Treasury bill movements over the period 1949-1989.

One might speculate that the predictive power may have been present in a stable period such as II: 1958 - 1974. The choice of upper limit could correspond to the oil price shock of 1974. However, the prime motivation for choice of this period was to repeat the test over a period where it appears from Appendix E ranks that the test may yield a positive result. Thus this unfairly chosen period could be regarded as giving an indication of the predictive power under the optimum circumstances. The results for 1958 - 1974 are:

$$D^{II} = 494$$

$$E_{H_0}(D^{II}) = 816$$

$$(Var_{H_0}(D^{II}))^{1/2} \doteq 204$$

Thus, this 1.58 σ result is very weak evidence of some predictive power over the period 1958 - 1974. In considering this result, the method of choice of period should be kept in mind.

4 Conclusion

The solution behaviour demonstrated in Section 2 must be considered a disadvantage of the Brennan and Schwartz(1979) model. The behaviour is related to the fact that model is not of the general equilibrium and arbitrage - free class described by Cox, Ingersoll and Ross (1985a and 1985b). In addition, the short rates drift towards the long rate under the Brennan and Schwartz (1979) model must be viewed somewhat skeptically in light of the results of Section 3. Nonetheless, the model continues to be one of the more practically useful and comprehensive description of the complicated behaviour of the term structure.

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APPENDIX A

Ito's Lemma

A generalized form of Ito's lemma is given (Malliaris and Brock, 1982, p. 85). Let $u(X(t), t)$: $[0, T] \times R^d \rightarrow R^k$ denote a continuous nonrandom function such that its partial derivatives $\partial u/\partial t, \partial u/\partial X_i (i = 1, 2, \cdots, d)$ and $\partial^2 u/\partial X_i \partial X_j$ $(i, j \leq d)$ are continuous. That is, u is now considered to be a k-vector and X a d-vector. Suppose that $X(t) = X(t, \omega) : [0, T] \times \Omega \rightarrow R^d$ is a process with stochastic differential

$$dX(t) = f(X(t), t)dt + \sigma(t)dz(t) \tag{A.01}$$

Suppose also that $\sigma(t) = \sigma(t, w) : [0, T] \times \Omega \rightarrow R^d \times R^m$ is a nonanticipating $(d \times m)$ matrix valued function and that $z(t) = z(t, w) : [0, T] \times \Omega \rightarrow R^m$ is a m-dimensional Wiener process. Let $Y(t) = u(t, X(t))$. Then $Y(t)$ has a differential on $[0, T]$ given by

$$dY(t) = \left\{ \frac{\partial u}{\partial t}(X(t), t) + \frac{\partial u}{\partial X}(X(t), t) f(X(t), t) \right.$$

$$\left. + \frac{1}{2} \sum_i \sum_j \frac{\partial^2 u}{\partial X_i \partial X_j} \right\} (X(t), t) \left[\sigma(t)\sigma'(t)\right]_{ij} dt$$

$$+ \frac{\partial u}{\partial X}(X(t), t)\sigma(X(t), t)dz(t)$$

$$= \left\{ \frac{\partial u}{\partial t} + \frac{\partial u}{\partial X} f + \frac{1}{2} tr \ [u_{XX} \ \sigma\sigma'] \right\} dt + \frac{\partial u}{\partial X} \sigma dz \tag{A.02}$$

where u_{XX} is the $(d \times d)$ matrix with i, jth element the k-vector $\partial^2 u / \partial X_i \partial X_j$.

Partial Equilibrium Development

There are in the literature several interest rate models where the price of a pure discount bond is assumed to depend on one or two state variables which are expressed in terms of Wiener processes. In this Appendix the models are summarized in a general framework corresponding to that of Buser, Hendershott and Sanders (1988) and Hull and White (1988). The common thread linking the models is that the use of Ito's lemma and an arbitrage argument lead to a partial differential equation for the bond price. In a few cases a closed form solution can be found while in other cases a numerical solution is necessary.

We are interested in the price of a pure discount (zero coupon) bond which matures at \$1 at time τ years hence (so $d\tau = -dt$). It is assumed that this bond is one of at least $n+1$ traded securities $B_j, j =, 2, \cdots n+1$ the prices of which are functions of n state variables $X_i, i = 1, 2, \cdots, n$. The state variables are assumed to follow the joint diffusion process

$$dX_i = \beta_i dt + \eta_i dz_i \qquad i = 1, 2, \cdots, n \tag{A.03}$$

The drift and instantaneous variance rates β_i and η_i are functions of time and of all the $X_i, i = 1, 2, \cdots n$. The dz_i are standard Wiener processes such that $E(dz_i) = 0, E(dz_i^2) = dt$

and $E(dz_i dz_j) = \rho_{ij} dt$ where ρ_{ij} is the instantaneous correlation between the ith and jth Wiener processes.

Now we apply Ito's lemma (A.02) to the price $B_j(X_1, \cdots, X_n, \tau)$ of the jth security and find

$$\frac{dB_j}{B_j} = \mu_j dt + \sum_{i=1}^{n} s_{ij}\, dz_i \qquad j = 1, 2, \cdots n+1 \tag{A.04}$$

where the μ_j and s_{ij} are functions given by

$$\mu_j = \frac{1}{B_j}\left[\sum_{i=1}^{n} \frac{\partial B_j}{\partial X_i}\,\beta_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n} \frac{\partial^2 B_j}{\partial X_i \partial X_k}\,\eta_i \eta_k \rho_{ik} - \frac{\partial B_j}{\partial \tau}\right] \quad j = 1, 2, \cdots n+1 \tag{A.05}$$

$$s_{ij} = \frac{1}{B_j}\left[\frac{\partial B_j}{\partial X_i}\eta_i\right], \qquad j = 1, 2, \cdots n+1 \tag{A.06}$$

In the above the μ_j is the instantaneous expected rate of return on security B_j and s_{ij} is the portion of the instantaneous standard deviation of B_j which is produced by its dependence on X_i.

Now an arbitrage argument can be developed. Since there are $n+1$ securities and n Wiener processes a self financing (no cash injection required) portfolio can be formed which will be instantaneously riskless; that is its instantaneous rate of return can be predicted with certainty. Denoting the portfolio value by I and the nominal quantity held of security B_j by y_j we have

$$I = \sum_{j=1}^{n+1} y_j B_j \tag{A.07}$$

Consider the instantaneous change in I,

$$dI = \sum_{j=1}^{n+1} y_j\, dB_j + \sum_{j=1}^{n+1} B_j\, dy_j$$

$$= \sum_{j=1}^{n+1} y_j dB_j \tag{A.08}$$

where the second term in the expression for dI is zero because of the self-financing nature of the portfolio (Ingersoll, 1981). Now using (A.04) in (A.08) we have

$$dI = \sum_{j=1}^{n+1} y_j \mu_j B_j dt + \sum_{j=1}^{n+1} \sum_{i=1}^{n} y_j \ s_{ij} \ B_j \ dz_i \tag{A.09}$$

We can choose the y_j to produce an instantaneously riskless portfolio by setting equal to zero the second term in (A.09). Then our riskless portfolio must by arbitrage arguments have a return equal to the riskless rate of interest r, often treated as being well approximated by the overnight inter-bank rate or the rate on Treasury bills. Thus $dI = Irdt$ and hence from (A.07) and (A.09) we have the set of equations

$$\sum_{j=1}^{n+1} y_j \ B_j(\mu_j - r) = 0 \tag{A.10}$$

$$\sum_{j=1}^{n+1} y_j \ B_j \ s_{ij} = 0 \quad i = 1, 2, \cdots, n \tag{A.11}$$

By a well known result in linear algebra, this set of $n+1$ homogeneous equations for $n+1$ unknowns y_j (or equivalently $y_j \ B_j$) has a non-zero solution iff for a set $\lambda_1, \lambda_2, \cdots, \lambda_n$ of variables which depend only on the state variables and time we have

$$\mu_j - r = \sum_{i=1}^{n} \lambda_i \ s_{ij} \qquad j = 1, 2, \cdots, \ n+1 \tag{A.12}$$

Then (A.12) is an asset pricing model where the λ_i express the extra return demanded by investors to compensate for the volatility of the ith state variable X_i. For most choices of state variables (e.g. interest rates) the s_{ij} from (A.06) will be negative, hence the λ_i will be negative. λ_i is often referred to as the (negative of) the market price of risk.

By choosing one particular security B out of the $B_1, B_2, \cdots, B_{n+1}$ and substituting (A.05) and (A.06) in (A.12) we derive the important equation

$$\frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 B}{\partial X_i \partial X_k} \eta_i \, \eta_k \, \rho_{ik} + \sum_{i=1}^{n} \frac{\partial B}{\partial X_i} \, (\beta_i - \lambda_i \, \eta_i) - \frac{\partial B}{\partial \tau} - rB = 0 \qquad (A.13)$$

This equation (A.13) comprises a partial differential equation for the bond price. It was not derived from general equilibrium arguments, and the functional forms of the prices of risk λ_i are left unspecified. This is in contrast with the Cox, Ingersoll and Ross (1985a and 1985b) general equilibrium model which effectively specifies relationships between the λ_i, β_i and η_i.

Partial Equilibrium Model

Special cases of the pricing model (A.13) have been derived by several sets of authors.

Brennan and Schwartz (1979) point out that a single parameter model, such as one based on the spot rate of interest, will be unable to reproduce observed yield curves. The riskless or "spot" rate is the limit as the term tends to zero of the yield on a risk-free bond. It is observed that at different dates this rate may be identical while the rest of the yield curve differs. This is a result of such factors as differences in investors' expectations about future changes in the inflation rate. If it is anticipated that the inflation rate will increase in the future, then long rates will exceed short rates by a greater margin than would otherwise be the case. Brennan and Schwartz develop a model in which the riskless and long rates follow a joint stochastic process. Thus account can be taken of information about the future course of the riskless rate which is contained in the current value of the long rate. A disadvantage of the model is that two exogenous variables are required, the riskless and long rates, rather than only the riskless rate.

Brennan and Schwartz again use the riskless rate r as being one of their variables, while the other state variable is the long rate l. Then r and l follow the process corresponding to (A.03)

$$dr = \beta_r \; dt + \eta_r \; dz_r \tag{A.14}$$

$$dl = \beta_l \; dt + \eta_l \; dz_l \tag{A.15}$$

where the forms of β_r, β_l, η_r and η_l are yet to be specified and in general the two Wiener processes are correlated. They assume that a consol bond (infinite term bond) of value $V(l)$ exists which pays a continuous coupon of \$1 per annum, so that l is defined by

$$V(l) = \frac{1}{l}. \tag{A.16}$$

Now considering equation (A.13) for the consol bond we notice that $\partial V/\partial r = 0$ and that, in view of the infinite maturity, $\partial V/\partial \tau = 0$. Substituting $\partial V/\partial l = -1/l^2$ and $\partial^2 V/\partial l^2 = 2/l^3$, and adding a term 1 to the left hand side of (A.13) to allow for the coupon payment per annum, we find

$$\lambda_l(r, l, t) = -\frac{\eta_l}{l} + (\beta_l - l^2 + rl)/\eta_l. \tag{A.17}$$

Thus we are left with only one utility dependent function $\lambda_r(r, l, t)$.

Substitution of (A.17) into (A.13) gives the partial differential equation for a bond of arbitrary maturity

$$\frac{1}{2}\frac{\partial^2 B}{\partial r^2}\eta_r^2 + \frac{\partial^2 B}{\partial r \partial l}\rho_{rl}\,\eta_r\,\eta_l + \frac{1}{2}\frac{\partial^2 B}{\partial l^2}\eta_l^2 + \frac{\partial B}{\partial r}(\beta_r - \lambda_r\,\eta_r)$$
$$+\frac{\partial B}{\partial l}\left(\frac{\eta_l^2}{l} + l^2 - rl\right) - \frac{\partial B}{\partial \tau} - Br = 0. \qquad (A.18)$$

It will be noted that this important partial differential equation contains neither λ_l nor β_l, the drift parameter for the long rate l. In general, $B, \eta_r, \eta_l, \rho_{rl}, \beta_r$ and λ_r are functions of r, l and t.

Equation (A.18) has no known analytic closed form solution, and Brennan and Schwartz use numerical methods in order to solve it.

Specialization of the Brennan and Schwartz Model

The Brennan and Schwartz two state variable model (A.18) is probably the model which has seen the widest use because the number of parameters is less than that of most two state variable models. Since the price of a security, the consol bond, can be expressed in terms of one of the state variables, the long rate, the number of parameters is reduced as shown in the derivation of (A.18). In this Appendix, which owes much to Brennan and Schwartz (1979), the specialization of (A.18) and the transformation of the semi-infinite ranges of the short and long rate state variables are described. Then the boundary conditions are derived.

Equation (A.18) was specialized by Brennan and Schwartz (1979). They assumed that

$$\eta_r(r,l,t) = r\sigma_r \tag{A.19}$$

and

$$\eta_l(r,l,t) = l\sigma_l \tag{A.20}$$

which ensures for $\beta_r \geq 0$ and $\beta_l \geq 0$ that r and l cannot become negative, and corresponds with a view that interest rate fluctuations will be on a proportionate basis. They assumed that the short rate r drifts towards the long term rate l modified by an offset parameter p, so

$$d\ ln\ r = \alpha\,[\ln\ l - \ln\ p - \ln\ r]\,dt + \sigma_r dz_r, \qquad r > 0 \tag{A.21}$$

and hence by Ito's lemma

$$\beta_r(r,l,t) = r\left[\alpha\ \ln(l/pr) + \frac{1}{2}\sigma_r^2\right]. \tag{A.22}$$

Then substituting (A.19), (A.20) and (A.22) in (A.18) Brennan and Schwartz find

$$\frac{1}{2}\,\frac{\partial^2 B}{\partial r^2}\,r^2\sigma_r^2 + \frac{\partial^2 B}{\partial r \partial l}\,rl\ \rho_{rl}\ \sigma_r\ \sigma_l + \frac{1}{2}\,\frac{\partial^2 B}{\partial l^2}\,l^2\sigma_l^2 + \frac{\partial B}{\partial r}\,r\left[\alpha \ln\left(\frac{l}{pr}\right) + \frac{1}{2}\ \sigma_r^2 - \lambda_r\sigma_r\right]$$

$$+\frac{\partial B}{\partial l}l\left[\sigma_l^2 + l - r\right] - \frac{\partial B}{\partial \tau} - Br = 0. \tag{A.23}$$

In order to handle numerically the possibilities $l \to \infty$ and $r \to \infty$, the transformations are made

$$u_r = \frac{1}{1+nr} \tag{A.24}$$

$$u_l = \frac{1}{1+nl} \tag{A.25}$$

and $b(u_r, u_l, \tau)$ is substituted for $B(r, l, \tau)$. Then (A.23) becomes, with the addition of a term in q, the coupon per unit face value,

$$\begin{aligned}\frac{1}{2}\frac{\partial^2 b}{\partial u_r^2}\ u_r^2\ (1-u_r)^2\sigma_r^2 + \frac{\partial^2 b}{\partial u_r \partial u_l}\ u_r u_l (1-u_r)(1-u_l)\rho_{rl}\sigma_r\sigma_l + \frac{1}{2}\frac{\partial^2 b}{\partial u_l^2}\ u_l^2(1-u_l)^2\sigma_l^2 \\ +\frac{\partial b}{\partial u_r}\ u_r(1-u_r)\ \left[\sigma_r^2\left(\frac{1}{2}-u_r\right) - \alpha \ln\left[\frac{u_r(1-u_l)}{pu_l(1-u_r)}\right] + \lambda_r\sigma_r\right] \\ +\frac{\partial b}{\partial u_l}\ u_l(1-u_l)\left[-u_l\sigma_l^2 - \frac{1}{nu_l}(1-u_l) + \frac{1}{nu_r}(1-u_r)\right] - \frac{\partial b}{\partial \tau} - b\frac{(1-u_r)}{nu_r} + q = 0.\end{aligned} \tag{A.26}$$

It is required to solve (A.26) numerically.

The boundary conditions on (A.26) are important, and are derived by Brennan and Schwartz (1979). The condition at maturity of the bond $b(u_r, u_l, \tau)$ is that it be worth the b_0 at which it is redeemed:

$$b(u_r, u_l, 0) = b_0. \tag{A.27}$$

By multiplying (A.26) by nu_r and letting $u_r \to 0$ and $u_l \to 0$ the boundary condition is obtained

$$b(0,0,\tau) = 0, \qquad \tau > 0 \tag{A.28}$$

which is intuitively reasonable in view of the discounting of a future maturity value at an infinite interest rate.

Another boundary condition results from multiplying (A.26) by nu_r and letting $u_r \to 0$ to obtain

$$\frac{\partial b}{\partial u_l}\ (0, u_l, \tau)u_l(1-u_l) - b(0, u_l, \tau) = 0. \tag{A.29}$$

Solving (A.29) gives

$$b(0,u_l,\tau) = b(0,u_l^0,\tau)\ \exp\left[\int_{u_l^0}^{u_l} \frac{du_l'}{u_l'(1-u_l')}\right]$$
$$= b(0,u_l^0,\tau)\ \frac{u_l(1-u_l^0)}{u_l^0(1-u_l)} \qquad (A.30)$$

where u_l^0 is some value of u_l chosen as an origin. We always have a finite non-negative bond value since the value cannot exceed the sum of the total future coupons and the maturity value of the bond. Thus consideration of (A.30) for the case $u_l = 1$ leads to the conclusion that

$$b(0,u_l,\tau) = 0, \qquad \tau > 0, \qquad 0 \le u_l \le 1. \qquad (A.31)$$

By dividing (A.26) by $\ln u_l$ and letting $u_l \to 0$ one finds

$$\alpha u_r(1-u_r)\ \frac{\partial b}{\partial u_r}(u_r,0,\tau) = 0. \qquad (A.32)$$

Thus, in view of (A.28), another boundary condition is

$$b(u_r,0,\tau) = 0, \qquad \tau > 0, \qquad 0 \le u_r \le 1. \qquad (A.33)$$

In other words, (A.31) and (A.33) state that if either the short or long interest rates r and l are infinite, then the bond value is zero.

If one sets $u_r = 1$ and $u_l = 1$ in (A.27) one finds

$$\frac{\partial b}{\partial \tau}\ (1,1,\tau) = 0 \qquad (A.34)$$

and hence from (A.27) one has the boundary condition

$$b(1,1,\tau) = b_0 + \tau q, \qquad \tau \ge 0. \qquad (A.35)$$

This is not surprising since by (A.22) and (A.19) a zero short rate r will remain at zero.

For the case $u_r = 1$, terms involving differentiation by u_r drop out and (A.26) becomes

$$\frac{1}{2}\frac{\partial^2 b}{\partial u_l^2}(1, u_l, \tau)u_l^2(1-u_l)^2\sigma_l^2 + \frac{\partial b}{\partial u_l}(1, u_l, \tau)u_l(1-u_l)\left[-u_l\sigma_l^2 - \frac{1}{nu_l}(1-u_l)\right]$$

$$+q - \frac{\partial b}{\partial \tau}(1, u_l, \tau) = 0. \qquad (A.36)$$

Thus $b(1, u_l, \tau)$ is the solution of (A.36) subject to the boundary conditions (A.27), (A.33) and (A.35).

For $u_l \to 1$, (7.2.08) is dominated by the $\ln(1-u_l)$ term so

$$\alpha u_r(1-u_r)\frac{\partial b}{\partial u_r}(u_r, 1, \tau) = 0. \qquad (A.37)$$

Thus solving (A.37) subject to (A.35), the final boundary condition is

$$b(u_r, 1, \tau) = b_0 + \tau q, \qquad \tau \geq 0, \quad 0 u_r \leq 1. \qquad (A.38)$$

APPENDIX B - UNIT DISCOUNT BOND 20 YEARS FROM MATURITY

r values ··· l values	0.	0.02500	0.03750	0.05833	0.07917	0.10000	0.1313	0.2250	∞
0.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	0.00000
0.02500	0.98565	0.57777	0.51902	0.44900	0.39772	0.35744	0.31005	0.21807	0.00000
0.03750	0.79889	0.50031	0.44710	0.38348	0.33743	0.30171	0.26023	0.18117	0.00000
0.05833	0.24756	0.32148	0.31281	0.28626	0.25789	0.23274	0.20178	0.14091	0.00000
0.07917	0.04087	0.12586	0.14925	0.16953	0.17325	0.16784	0.15363	0.11262	0.00000
0.10000	0.00555	0.03413	0.04985	0.07389	0.09161	0.10192	0.10624	0.08945	0.00000
0.13125	0.00029	0.00370	0.00703	0.01494	0.02472	0.03480	0.04755	0.05903	0.00000
0.22500	0.00000	0.00001	0.00002	0.00009	0.00028	0.00066	0.00172	0.00865	0.00000
∞	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Diffusion Parameters	Grid Parameters
$\sigma_r = 0.2550$	$g = 0.01$ (spacing of u_r)
$\sigma_l = 0.0866$	$h = 0.01$ (spacing of u_l)
$\rho_r = 0.3747$	$n = 40$
$\alpha = 0.0701$	
$\lambda = 0.0355$	
$p = 1.06173$	

APPENDIX C

CANADIAN TREASURY BILL RATES 1946 - 1989

	JAN	FEB	MAR	APR	MAY	JUN	JUL	AUG	SEP	OCT	NOV	DEC
1949	0.41	0.41	0.44	0.49	0.50	0.51	0.51	0.51	0.51	0.51	0.51	0.51
1950	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.53	0.61	0.62	0.62	0.63
1951	0.63	0.70	0.75	0.75	0.75	0.75	0.77	0.78	0.86	0.92	0.92	0.90
1952	0.89	0.90	0.93	0.97	1.01	1.06	1.11	1.10	1.12	1.19	1.21	1.30
1953	1.35	1.46	1.51	1.53	1.57	1.69	1.75	1.81	1.91	1.93	1.90	1.88
1954	1.85	1.75	1.62	1.58	1.60	1.57	1.38	1.32	1.21	1.18	1.17	1.36
1955	0.99	0.90	1.13	1.23	1.24	1.36	1.43	1.60	1.77	2.07	2.33	2.59
1956	2.58	2.50	2.61	2.83	2.84	2.63	2.53	2.95	3.06	3.30	3.39	3.61
1957	3.70	3.76	3.71	3.72	3.77	3.80	3.81	3.97	3.94	3.84	3.66	3.65
1958	3.54	2.99	2.44	1.67	1.56	1.75	1.31	1.29	2.02	2.48	3.00	3.46
1959	3.34	3.70	4.16	4.52	4.98	5.15	5.23	5.82	5.73	5.14	4.87	5.02
1960	4.81	4.69	3.87	3.40	2.87	2.87	3.13	2.66	1.91	2.65	3.42	3.61
1961	3.20	3.05	3.21	3.30	3.19	2.76	2.61	2.48	2.42	2.53	2.42	2.82
1962	3.08	3.08	3.12	3.08	3.36	4.48	5.47	5.15	5.03	4.54	3.88	3.88
1963	3.82	3.68	3.63	3.58	3.33	3.23	3.39	3.60	3.69	3.57	3.64	3.71
1964	3.76	3.81	3.88	3.75	3.66	3.56	3.60	3.80	3.81	3.70	3.73	3.85
1965	3.78	3.72	3.71	3.66	3.84	3.95	4.00	4.08	4.11	4.14	4.17	4.45
1966	4.61	4.68	4.87	5.09	5.10	5.06	5.07	5.08	5.03	5.13	5.19	5.07
1967	4.83	4.62	4.26	4.00	4.12	4.32	4.27	4.33	4.50	4.91	5.15	5.73
1968	5.94	6.57	6.90	6.91	6.96	6.75	6.26	5.81	5.62	5.64	5.62	5.96
1969	6.36	6.31	6.62	6.66	6.75	7.03	7.46	7.65	7.75	7.68	7.71	7.78
1970	7.80	7.70	7.32	6.81	6.51	5.90	5.79	5.66	5.44	5.25	4.76	4.47
1971	4.59	4.51	3.30	3.05	3.06	3.15	3.58	3.88	3.93	3.79	3.31	3.25
1972	3.29	3.48	3.51	3.65	3.68	3.58	3.48	3.47	3.57	3.57	3.61	3.66
1973	3.79	3.92	4.29	4.73	5.08	5.40	5.65	6.03	6.41	6.51	6.46	6.38
1974	6.28	6.11	6.28	7.13	8.24	8.68	8.92	9.09	9.03	8.60	7.73	7.32
1975	6.65	6.34	6.29	6.54	6.90	6.96	7.29	7.72	8.37	8.31	8.44	8.56
1976	8.59	8.70	9.04	8.97	8.94	8.99	9.02	9.12	8.97	9.07	8.88	8.41
1977	8.08	7.67	7.61	7.55	7.26	7.07	7.12	7.16	7.09	7.19	7.25	7.18
1978	7.14	7.24	7.62	8.18	8.13	8.24	8.43	8.77	9.02	9.52	10.29	10.43
1979	10.80	10.78	10.90	10.84	10.84	10.82	10.91	11.32	11.57	12.86	13.61	13.63
1980	13.54	13.56	14.35	15.64	12.54	11.15	10.10	10.21	10.63	11.57	12.87	16.31
1981	16.77	16.87	16.64	16.92	18.61	18.83	19.27	20.85	19.70	18.19	15.87	14.81
1982	14.46	14.54	14.88	15.07	15.08	16.11	15.69	14.41	13.15	11.54	10.72	10.25
1983	9.53	9.39	9.21	9.21	9.12	9.24	9.24	9.34	9.26	9.22	9.34	9.74
1984	9.73	9.77	10.32	10.56	11.27	11.74	12.81	12.21	12.08	11.74	10.79	10.13
1985	9.52	10.56	11.08	9.92	9.56	9.35	9.14	9.01	8.95	8.58	8.72	9.08
1986	10.02	11.55	10.49	9.14	8.33	8.60	8.29	8.33	8.32	8.32	8.27	8.21
1987	7.70	7.32	7.00	7.52	8.05	8.30	8.53	8.95	9.22	8.72	8.24	8.45
1988	8.42	8.31	8.43	8.75	8.88	9.20	9.27	9.62	10.26	10.29	10.60	10.94
1989	11.15	11.42	11.99	12.29	12.20	12.06	12.07	12.15	12.20	12.20	12.23	12.21

Note: The source for this data is Statistics Canada CANSIM series B14001: 91 day Treasury bill tender rate (monthly average).

APPENDIX D

CANADIAN 1 - 3 YEAR BOND RATES 1949 - 1989

	JAN	FEB	MAR	APR	MAY	JUN	JUL	AUG	SEP	OCT	NOV	DEC
1949	1.63	1.62	1.69	1.68	1.68	1.67	1.68	1.60	1.58	1.61	1.61	1.70
1950	1.71	1.71	1.73	1.73	1.72	1.75	1.75	1.75	1.80	1.83	2.04	2.13
1951	2.18	2.23	2.72	2.52	2.38	2.48	2.48	2.45	2.43	2.39	2.39	2.33
1952	2.47	2.43	2.53	2.54	2.55	2.72	2.89	3.01	3.03	3.17	3.22	3.19
1953	3.04	3.09	3.12	3.12	3.13	3.06	3.29	3.42	3.36	3.35	3.26	3.26
1954	3.02	2.75	2.72	2.17	2.19	2.01	2.01	1.91	1.85	1.92	1.85	1.79
1955	1.55	1.65	1.59	1.72	1.84	1.91	2.21	2.32	2.48	2.60	3.17	3.25
1956	3.01	3.01	3.10	3.56	3.36	3.06	3.40	3.80	3.90	4.09	4.39	4.54
1957	4.77	4.18	4.20	4.27	4.70	4.79	4.71	4.84	4.87	4.46	3.92	3.84
1958	3.63	3.55	3.18	3.00	2.80	3.14	2.37	2.69	3.09	3.35	4.00	4.52
1959	4.32	4.66	4.72	4.95	5.06	5.21	5.33	5.54	5.75	5.04	4.78	4.96
1960	4.89	4.81	4.21	4.14	4.30	4.06	3.69	2.98	3.07	3.50	3.92	3.99
1961	3.78	3.59	3.84	4.00	4.20	3.58	3.42	3.22	3.57	3.26	3.24	3.39
1962	3.50	3.40	3.20	3.45	3.91	5.49	5.63	5.37	5.12	4.22	3.99	4.12
1963	4.14	4.29	4.38	4.38	3.97	3.81	4.26	4.45	4.22	4.12	4.22	4.28
1964	4.38	4.33	4.49	4.50	4.30	4.33	4.45	4.55	4.40	4.44	4.49	4.21
1965	4.01	4.31	4.10	4.09	4.19	4.29	4.49	4.75	4.86	5.01	5.03	5.11
1966	4.99	5.19	5.27	5.20	5.17	5.16	5.44	5.91	5.49	5.54	5.74	5.43
1967	4.92	5.05	4.35	4.47	4.92	5.34	5.40	5.49	5.80	5.79	5.80	6.16
1968	6.35	6.51	6.69	6.58	6.71	6.63	6.17	5.87	5.94	6.16	6.13	6.71
1969	6.71	6.82	7.00	7.22	7.54	7.53	7.77	7.69	7.86	7.73	7.94	8.07
1970	7.95	7.66	7.09	6.83	6.78	6.52	6.44	6.52	6.47	6.36	5.37	4.89
1971	5.05	5.05	4.77	4.88	4.97	5.31	5.51	5.33	5.26	4.41	4.21	4.42
1972	4.76	5.18	5.51	5.73	5.96	5.86	5.87	5.97	5.85	5.66	5.03	5.15
1973	5.48	5.45	5.77	6.24	7.15	6.94	7.09	7.27	6.94	6.61	6.57	6.92
1974	6.75	6.58	7.55	8.83	8.93	9.29	9.18	9.30	8.87	7.47	6.98	6.66
1975	5.91	6.22	6.56	7.23	7.09	7.35	7.90	8.37	8.76	8.21	8.48	8.36
1976	8.13	8.36	8.63	8.46	8.25	8.40	8.44	8.45	8.30	8.35	8.01	7.50
1977	7.57	7.42	7.46	7.56	7.33	7.31	7.37	7.36	7.43	7.52	7.60	7.59
1978	7.70	7.82	8.28	8.58	8.58	8.60	8.63	8.63	8.77	9.48	10.07	10.14
1979	10.08	10.07	10.10	9.76	9.76	9.87	10.06	10.89	11.17	12.78	12.41	12.24
1980	12.79	13.62	14.27	12.35	10.85	10.48	11.11	11.98	12.69	13.11	13.08	12.95
1981	13.06	13.66	14.04	15.78	16.22	16.19	18.77	18.82	18.94	17.33	13.57	15.22
1982	15.95	15.03	15.43	[illegible]	14.71	16.50	15.69	13.53	12.75	11.57	10.80	10.24
1983	10.28	10.23	10.18	[illegible]	9.75	10.08	10.38	10.86	10.10	9.88	10.03	10.39
1984	10.23	10.74	11.50	11.76	12.92	12.89	13.02	12.39	12.04	11.44	[illegible]	10.44
1985	10.27	11.69	11.14	10.59	10.16	10.02	10.06	9.79	9.88	9.44	9.26	9.10
1986	9.88	9.66	9.36	8.82	8.98	8.83	9.07	9.02	9.10	8.99	8.70	8.63
1987	7.85	8.01	7.71	9.23	9.33	9.11	9.73	9.88	10.61	9.46	9.65	9.69
1988	9.04	8.83	8.99	9.32	9.46	9.40	9.67	10.38	10.17	9.80	10.36	10.58
1989	10.58	11.30	11.60	11.01	10.61	10.19	10.05	10.63	10.52	10.24	10.99	10.84

Note: The source for this data is Statistics Canada CANSIM series B14009: Government of Canada bond yield average, 1 to 3 years.

APPENDIX E

TEST OF PREDICTIVE POWER 1949 - 1989

Year i	Spread $m(i,1) - r(i,1)$	Rank $v_s(i,)$	Bill Rate Change $r(i,12) - r(i,1)$	Rank $v_c(i)$
1949	1.220	35	0.100	17
1950	1.200	34	0.120	18
1951	1.550	38	0.270	20
1952	1.580	39	0.410	23
1953	1.690	40	0.530	25
1954	1.170	33	-0.490	8
1955	0.560	25	1.600	34
1956	0.430	20	1.030	30
1957	1.070	32	-0.050	14
1958	0.090	10	-0.080	13
1959	0.980	31	1.680	35
1960	0.080	9	-1.200	6
1961	0.580	26	-0.380	10
1962	0.420	19	0.800	28
1963	0.320	15	-0.110	12
1964	0.620	27	0.090	16
1965	0.230	14	0.670	26
1966	0.380	17	0.460	24
1967	0.090	11	0.900	29
1968	0.410	18	0.020	15
1969	0.350	16	1.420	33
1970	0.150	12	-3.330	2
1971	0.460	21	-1.340	5
1972	1.470	36	0.370	21
1973	1.690	41	2.590	38
1974	0.470	22	1.040	31
1975	-0.740	3	1.930	36
1976	-0.460	7	-1.180	11
1977	-0.510	6	-0.900	7
1978	0.560	24	3.290	41
1979	-0.720	4	2.830	40
1980	-0.750	2	2.770	39
1981	-3.710	1	-1.960	3
1982	1.490	37	-4.210	1
1983	0.750	29.5	0.210	19
1984	0.500	23	0.400	22
1985	0.750	29.5	-0.440	9
1986	-0.140	8	-1.810	4
1987	0.150	13	0.750	27
1988	0.620	28	2.520	37
1989	-0.570	5	1.060	32

APPENDIX F

TEST OF PREDICTIVE POWER 1958 - 1974

Year i	Spread $m(i,1) - r(i,1)$	Rank $v_s(i)$	Bill Rate Change $r(i,12) - r(i,1)$	Rank $v_c(i)$
1958	0.090	2	-0.080	6
1959	0.980	15	1.680	16
1960	0.080	1	-1.200	3
1961	0.580	13	-0.380	4
1962	0.420	10	0.800	12
1963	0.320	6	-0.110	5
1964	0.620	14	0.090	8
1965	0.230	5	0.670	11
1966	0.380	8	0.460	10
1967	0.090	3	0.900	13
1968	0.410	9	0.020	7
1969	0.350	7	1.420	15
1970	0.150	4	-3.330	1
1971	0.460	11	-1.340	2
1972	1.470	16	0.370	9
1973	1.690	17	2.590	17
1974	0.470	12	1.040	14