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Derivatives of Decreasing Life Annuity

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This note is motivated by Exercise 5.35 on page 156 of *Actuarial Mathematics*:

Prove that

$$\frac{\partial}{\partial n}(\bar{D}\bar{a})_{x:\overline{n}|} = \bar{a}_{x:\overline{n}|}. \tag{1}$$

Perhaps some would find the derivation below amusing.

Define

$$x_+ = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}.$$

Then

$$(\bar{D}\bar{a})_{x:\overline{n}|} = \int_0^\infty (n-t)_+ \, {}_tE_x \, dt,$$

and

$$\frac{\partial}{\partial n}(\bar{D}\bar{a})_{x:\overline{n}|} = \int_0^\infty \frac{\partial}{\partial n}(n-t)_+ \, {}_tE_x \, dt. \tag{2}$$

Let $H(\cdot)$ denote the Heaviside function,

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}.$$

Then, for $x \neq 0$,

$$\frac{d}{dx} x_+ = H(x).$$

Thus (2) becomes

$$\frac{\partial}{\partial n}(\bar{D}\bar{a})_{x:\overline{n}|} = \int_0^\infty H(n-t)\, {}_tE_x \, dt$$

$$= \int_0^n {}_tE_x \, dt$$

$$= \bar{a}_{x:\overline{n}|}.$$

It is also interesting to note that, because the derivative of the Heaviside function is the Dirac delta function,

$$\frac{d}{dx} H(x) = \delta(x),$$

we have

$$\frac{\partial^2}{\partial n^2}(\bar{D}\bar{a})_{x:\overline{n}|} = \int_0^\infty \frac{\partial^2}{\partial n^2}(n-t)_+ \, {}_tE_x \, dt$$

$$= \int_0^\infty \delta(n-t)\, {}_tE_x \, dt$$

$$= {}_nE_x.$$