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THE PROBABILITY OF RUIN IN AN AUTOREGRESSIVE MODEL

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1. Introduction.

The classical, discrete time model for ruin theory is based on a surplus process with independent increments. In [3], Gerber generalized portions of this theory by considering situations with dependent increments. A particular case, the first-order autoregressive model, is discussed in *Actuarial Mathematics* [1]. One of the restrictions inherent in these works is that the underlying random variables are bounded. (This is not explicitly stated in [1], but the boundedness is used in the proof of the main result as we will indicate.) Gerber asks if the boundedness condition can be relaxed, and leaves it as an open problem.

The restriction of boundedness may be reasonable from a practical point of view. However, we frequently use unbounded random variables, such as those with an exponential distribution, for modelling claims. Moreover, in teaching concepts of risk theory, the exponential distribution plays a key role in examples, due to its mathematical simplicity. Therefore, it is of significance to remove the boundedness criteria.

In this paper we focus on the first-order autoregressive case. A more general treatment appears in [4]. We show in section 4 below that the boundedness restriction can be removed from the basic result on the probability of ruin, [1, Theorem 12.3]. As a major tool, we use a certain concept of duality for distribution functions. This idea, which has some independent interest, is introduced in section 2. In section 5 we give some miscellaneous examples on estimating ruin probabilities.

2. A duality.

We first need some facts about the adjustment coefficient. These are essentially given in [1, p. 355], but for completeness and consistency of notation we will review this material here.

Given any random variable G, with distribution function F, define a function $\rho: [0,\infty) \to [0,\infty]$, by

$$\rho(r) = \int_{-\infty}^{\infty} e^{-rx} dF(x) - 1.$$

Let $\nu = \sup \{ r: \rho(r) < \infty \}$. (Since $\rho(0) = 0$, the set in question is not empty.) Suppose that G satisfies the following two conditions:

(i) $E(G) > 0$.

(ii) $\lim_{r\to\nu} \rho(r) = \infty$.

We require of course that

(iii) $\nu > 0$

in order that condition (ii) makes sense. Another necessary condition for (ii) to hold is that

(iv) $\Pr(G < 0)$ is positive,

since if $\Pr(G < 0)$ were equal to zero, $\rho(r)$ would be nonpositive. In certain cases, (iv) will imply (ii). For example, this holds if $\nu = \infty$, since (iv) implies that $\lim_{r\to\infty} \rho(r) = \infty$.

From (iii) and (i) we see that $\rho'(0)$ exists and is equal to $-E(G)$, which is negative. It is also clear that for r in $(0, \nu)$, $\rho''(r)$ exists and is positive. Together with (ii), this shows that ρ has a unique zero in the interval $(0, \nu)$. This point, known as the *adjustment coefficient*, will be denoted by R. It is clear from the above remarks that

$$\rho'(R) > 0. \tag{2.1}$$

Define a distribution function F^* by

$$F^*(t) = \int_{-\infty}^{t} e^{-Rx} dF(x).$$

From the definition of R we have a legitimate distribution function, that is, $F^*(\infty) = 1$. We will refer to this as the *dual distribution* of the original.

Throughout the paper we will use $*$ to refer to quantities, (such as expectations) calculated with respect to F^*.

From (2.1) we obtain the important fact that

$$E^*(G) = \int_{-\infty}^{\infty} x e^{-Rx} dF(x) = -\rho'(R) < 0. \tag{2.2}$$

Examples.

(a) G takes the value 1 with probability p, and -1 with probability q, where $p > q$. Then $e^{-R} = q/p$, so the dual distribution takes the value 1 with probability q and -1 with probability p.

(b) G is normal with mean μ and variance σ^2. Then

$$R = \frac{2\mu}{\sigma^2}.$$

(See [1, remark below formula 12.49]) The dual distribution then has the density function

$$\frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(\mu-x)^2}{2\sigma^2}\right] \exp\left[\frac{2\mu x}{\sigma^2}\right]$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(\mu+x)^2}{2\sigma^2}\right]$$

showing that it is normal with mean $-\mu$ and variance σ^2.

In both of the above cases, the dual distribution of G is distributed as $-G$. This is not always the case, and in fact this cannot hold when the range of G is not symmetric. Both distributions "take the same values". To put it more precisely, the probability measures are equivalent in the sense that each is absolutely continuous with respect to the other.

3. The classical case.

We will briefly review the standard, discrete time surplus model with independent increments, as outlined in [1, section 12.4]. (For the most part, we follow the notation used in [1]. However, we omit the ~ used to distinguish discrete time from continuous time, since we only discuss the former.) Let G_n denote the gain made in the n-th period. That is, G_n equals the excess of the periodic premium over the claims for the n-th period. Let U_n

denote the surplus at the end of n periods. Let u be the initial surplus. The model assumes that

$$U_n = u + G_1 + G_2 + \ldots + G_n,$$

where the G_i 's are independent and each distributed as some random variable G. Let

$$T = \min\{ n: U_n < 0 \}, \text{ the time of ruin,}$$

$$\psi(u) = \Pr(T < \infty), \text{ the probability of ruin.}$$

Then the main result is [1, Theorem 12.1] which states that

$$\psi(u) = \frac{e^{-Ru}}{E[e^{-RU_T} | T < \infty]} \qquad (3.1)$$

where R is the adjustment coefficient, calculated with respect to G, as outlined in section 2. The proof of this is in two stages. We must first show that the sequence e^{-RU_n} is a martingale. In other words, we must show that for all n,

$$E[e^{-RU_{n+1}} | U_n] = e^{-RU_n}. \qquad (3.2)$$

This is sufficient to obtain (3.1) with equality replaced by less than or equal to. To obtain the equality we must also show that

$$E[e^{-RU_n} | T > n)] \Pr(T > n) \rightarrow 0 \text{ as } n \rightarrow \infty. \qquad (3.3)$$

In [3], this is done by invoking the dominated convergence theorem from integration theory, which needs the boundedness of the underlying random variables to justify it. The same idea is used in a more general setting in chapter 9 of [2]. In [1], there is an alternate proof, based on the following result, which is of independent interest.

Proposition 1. Let X_n be a sequence of random variables, with means μ_n and standard deviations σ_n satisfying the following:

(i) $\mu_n \rightarrow \infty$, as $n \rightarrow \infty$.

(ii) $\frac{\sigma_n}{\mu_n} \rightarrow 0$, as $n \rightarrow \infty$.

(iii) There exists $m < \infty$, such that

$$\Pr(X_n \leq m) = 0, \text{ for all } n.$$

Let g be any nonincreasing function defined on the real line such that $\lim_{t\rightarrow\infty} g(t) = 0$. (Necessarily, $g \geq 0$.)

Then,

$$E[g(X_n)] \rightarrow 0.$$

Proof. Let F_n denote the distribution function of X_n. Then,

$$0 \leq E[g(X_n)] = \int_m^{\mu_n/2} g(t)dF_n(t) + \int_{\mu_n/2}^{\infty} g(t)dF_n(t)$$

$$\leq g(m)\Pr\left(X_n \leq \frac{\mu_n}{2}\right) + g\left(\frac{\mu_n}{2}\right). \qquad (3.4)$$

By Chebychev's inequality, $\Pr(X_n \leq \mu_n/2) \leq 4\sigma_n^2/\mu_n^2$. Taking limits on the right hand side completes the proof.

This is essentially the derivation of (3.3) given in [1]. That proof is given in the particular case when $g(t) = e^{-Rt}$, and X_n is U_n conditioned by the fact that ruin has not yet occurred at time n. Hence, U_n is nonnegative and we can take $m = 0$. Note that the existence of the lower bound is needed to ensure that the first term in (3.4) will approach zero. The following counterexample shows that the proposition can fail without hypothesis (iii).

Example. Let

$$X_n = \begin{cases} -n \text{ with probability } 2^{-n} \\ n \text{ with probability } 1-2^{-n} \end{cases}$$

and let

$$g(t) = 2^{-t}.$$

Then, $\mu_n = n(1-2^{1-n}) \to \infty$. Moreover, $\mu_n/n \to 1$, so that

$$\frac{\sigma_n^2}{\mu_n^2} = \frac{n^2-\mu_n^2}{\mu_n^2} \to 1-1 = 0.$$

However, $E[2^{-X_n}]$ is clearly greater than $2^n \times 2^{-n} = 1$.

4. The Autoregressive model of order 1.

We begin by reviewing the model described in [1, p. 357]. Let Y denote the underlying claim random variable. Let W_n denote the loss in year n. We assume that

$$W_n = Y_n + a W_{n-1}$$

where $-1 < a < 1$, and the Y_n 's are independent and each distributed as some random variable Y.

We will later use the fact that if Y is bounded, the same is true for W_n. A more precise formulation is as follows.

Proposition 2. Suppose that

$$b \le Y \le d.$$

Then there exists m and M, such that for all n,

$$m \le W_n \le M.$$

Moreover, m and M can be defined in terms of b, d and w as follows.

If $a \ge 0$,

$$m = \min\left(\frac{b}{1-a}, w\right), \quad M = \max\left(\frac{d}{1-a}, w\right)$$

If $a < 0$, let

$$m_1 = \frac{b+ad}{1-a^2}, \quad M_1 = \frac{d+ab}{1-a^2}.$$

Then:

(i) If $m_1 \le w \le M_1$, then $m = m_1$ and $M = M_1$.

(ii) If $w \le m_1$, then $m = w$ and $M = \frac{w-b}{a}$.

(iii) If $M_1 \le w$, then $m = \frac{w-d}{a}$ and $M = w$.

Proof. We will prove the more complicated case of *negative a*. (The proof for nonnegative a is similar). We use induction on n, noting that the conclusion is trivial when the index is 0. Assume it is true for some index n.

Suppose that (i) holds. Then,

$$W_{n+1} = Y_{n+1} + aW_n \le d + am_1 = M_1.$$

Similarly,

$$W_{n+1} \geq b + aM_1 = m_1.$$

Suppose that (ii) holds. Then

$$w \leq m_1 \text{ implies that } d \leq \frac{w(1-a^2)-b}{a},$$

and so

$$W_{n+1} \leq d + aw \leq \frac{w-b}{a},$$

and

$$W_{n+1} \geq b + a\frac{w-b}{a} = w.$$

Case (iii) follows similarly, completing the proof.

We now return to the description of the model. Let c denote the periodic premium, let u denote the initial surplus and let w denote W_0, the initial value of W. The surplus at the end of n periods is given by

$$U_n = u + nc - (W_1 + W_2 + \dots + W_n).$$

We define T as in section 3. The main quantity of interest is

$$\psi(u,w) = \Pr(T < \infty),$$

the probability that ruin occurs, given initial values of u and w. Let

$$\hat{G}_n = c - \frac{1}{1-a} Y_{n+1}$$

so that each $\hat{G}_n$ is distributed as the the random variable

$$\hat{G} = c - \frac{1}{1-a} Y$$

We will assume that c satisfies

$$\frac{1}{1-a} \sup (Y) > c > \frac{1}{1-a} E(Y)$$

This implies that conditions (i) and (iv) of section 2 hold for $\hat{G}$. We assume in addition that Y is such that condition (ii) of section 2 holds. It follows that the adjustment coefficient R, with respect to $\hat{G}$, exists.

We can motivate the definition of $\hat{G}$ as follows. A unit of claim in one period will result in a units of claim the following period, and that in turn will result in a^2 units the period after that and so on. In the limiting case, each unit of claim will eventually cause 1/(1-a) units of loss. Moreover, from the initial surplus of u we should set aside w/(1-a) to cover the future losses occasioned by our initial value of W_0. It is natural therefore to consider an associated independent increment model, with claims multiplied by 1/(1-a), an initial surplus of u - w/(1-a), and the same periodic premium of c. For such a model, the periodic gain is distributed as $\hat{G}$.

Let $\hat{U}_n$ be the surplus at the end of n years in this associated model. This definition agrees with that defined in [1, formula 12.4.20] by

$$\hat{U}_n = U_n - \frac{a}{1-a} W_n. \qquad (4.1)$$

We can verify this by induction. From our definition of the initial surplus, (4.1) holds trivially for n = 0. Assuming its validity for index n,

$$\hat{U}_{n+1} = \hat{U}_n + \hat{G}_{n+1} = \hat{U}_n + c - \frac{1}{1-a} Y_{n+1}$$

$$= U_n - \frac{a}{1-a} W_n + c - \frac{1}{1-a} Y_{n+1}$$

$$= U_n + c - \frac{1}{1-a} [W_{n+1}]$$

$$= U_n + c - W_{n+1} - \frac{a}{1-a} W_{n+1}$$

$$= U_{n+1} - \frac{a}{1-a} W_{n+1}$$

In the case that *a* is nonnegative we can immediately say something about the probability of ruin. From (4.1) we know that

$$\hat{U}_T \leq U_T \tag{4.2}$$

Note that we do *not* assume that claims are positive, so we cannot say that $\hat{U}_n \leq U_n$ for all n. It is true for n equal to T, since the loss in the year of ruin is necessarily positive. We see therefore that there is less chance of ruin in the autoregressive model than there is in the associated independent increment model. This is intuitively clear, since the ruining claim of the associated model will not necessarily ruin us in the autoregressive case, where a portion of this loss will only appear in future periods. Let $\hat{T}$ denote the time of ruin in the associated model. It follows from (3.1) that *for nonnegative a*

$$\psi(u,w) \leq \frac{e^{-R(u-\frac{a}{1-a}w)}}{E[e^{-R\hat{U}_{\hat{T}}} | \hat{T} < \infty]} \leq e^{-R(u-\frac{a}{1-a}w)} \tag{4.3}$$

The second inequality was obtained in [1, Corollary to Theorem 12.3]. The above derivation is somewhat easier, as it does not require that we first prove this theorem.

Theorem 12.3 of [1] states that we can replace the first inequality sign by an equality, if we also replace $\hat{T}$ by T in the denominator. That is

$$\psi(u,w) = \frac{e^{-R(u-\frac{a}{1-a}w)}}{E[e^{-R\hat{U}_T} \mid T<\infty]}. \tag{4.4}$$

This is intuitively logical when a is nonnegative We would then expect that

$$\hat{U}_T \leq \hat{U}_{\hat{T}}.$$

Indeed, we have already noted that time T will occur after time $\hat{T}$. Hence, at time T, we will have already encountered ruin in the associated model and may have accumulated a large deficit in this model. This suggests that the second term in (4.3) should be larger than the right hand side of (4.4).

We have motivated (4.4) by considering nonnegative a, but it is in fact true for all values of a.

To prove (4.4), we must verify (3.2) and (3.3) with U replaced by $\hat{U}$. There is no difficulty with (3.2) and the derivation of this goes through as in the classical case with $a = 0$. The problem in deriving (3.3) is that unlike U_n, $\hat{U}_n$ need *not* be nonnegative at a time prior to ruin. We do know from (4.1) that $\hat{U}_n \geq -aW_n/(1-a)$. Proposition 2 then shows that if Y is bounded, $\hat{U}_n$ is uniformly bounded below, and we can apply Proposition 1. However, if Y is not bounded, we need a different method. One problem is that the existing proofs of (3.3) do not use the full definition of R, but simply treat it as any positive constant. By a more subtle employment of the properties of the adjustment coefficient, we are able to handle the general case.

General proof of (4.4).

We must show

$$E[\, e^{-R\hat{U}_n} \mid T > n \,)\ ]\, Pr(\, T > n) \quad \rightarrow \quad 0 \text{ as } n \rightarrow \infty. \qquad (4.5)$$

We consider now the dual distribution for $\hat{G}$, as defined in section 2. This induces, in a natural way, a new distribution on the sample space consisting of all sequences of independent observations of $\hat{G}$. From the definition of this dual distribution, statement (4.5) is equivalent to

$$Pr^*(\, T > n) \quad \rightarrow \quad 0 \text{ as } n \rightarrow \infty. \qquad (4.6)$$

It is a straightforward exercise to see that

$$U_n = (1-a)\hat{G}_n + (1-a^2)\hat{G}_{n-1} + \ldots + (1-a^n)\hat{G}_1 + (1-a^n)(c-w)+u. \qquad (4.7)$$

(See [1, formula 12.4.16] for example.) Let

$$E^*(\hat{G}) = \alpha, \quad Var^*(\hat{G}) = \sigma^2.$$

From (4.7),

$$E^*(U_n) = \alpha\Big[\, n - \frac{a-a^{n+1}}{1-a}\,\Big] + (c-w)(1-a^n) + u \qquad (4.8)$$

We know from (2.2) that $\alpha < 0$, and (4.8) shows that

$$E^*(U_n) < 0 \text{ for sufficiently large } n. \qquad (4.9)$$

Using the independence of the $\hat{G}_n$'s and the fact that $|1-a^n|$ is less than or equal to 2, we see from (4.7) that

$$Var^*(U_n) \leq 4n\sigma^2. \qquad (4.10)$$

If ruin has not yet occurred, the surplus must be nonnegative. Hence,

$$P^*(T > n) \le P(U_n \ge 0) \le P(U_n > -\frac{1}{2} E^*(U_n))$$

$$\le \frac{16n\sigma^2}{[E^*(U_n)^2]},$$

which by (4.8), approaches 0 as n approaches infinity. (Note that we use (4.9) for the second inequality and (4.10) together with Chebychev's inequality for the third.) We have now established (4.5), completing the proof.

5. Estimating the probability of ruin.

One should note that (4.4) does *not* give an explicit formula for the probability of ruin. There is a circularity involved, since the time of ruin T appears on both sides of the formula. However, one can often use the formula to estimate ruin probabilities. We will illustrate a few examples of this. We consider two cases, depending on the sign of the coefficient a.

Case 1: $a \ge 0$.

It is not hard to verify that

$$\hat{U}_n = \frac{1}{1-a} U_n - \frac{a}{1-a} U_{n-1} - \frac{a}{1-a} c \,. \qquad (5.1)$$

We always have $U_T < 0$ and $U_{T-1} \ge 0$. With $a > 0$, it follows from (5.1) that,

$$\hat{U}_T \le -\frac{a}{1-a} c.$$

(This idea, due to Gerber, was used in a more general setting in [3, section 4]). From (4.4) we obtain

$$\psi(u,w) \leq e^{-R[\,u + \frac{a}{1-a}(c-w)]} \tag{5.2}$$

which improves the estimate given in (4.3).

If Y is bounded above, then we can obtain a lower bound for $\psi(u,w)$. Let M be as in Proposition 2. Then

$$U_T = U_{T-1} + c - W_T \geq c - M \tag{5.3}$$

and

$$\hat{U}_T = U_T - \frac{a}{1-a} W_T \geq c - \frac{1}{1-a} M \tag{5.4}$$

so that

$$\psi(u,w) \geq e^{-R(\,u + \frac{1}{1-a}[M-aw] - c\,)} . \tag{5.5}$$

Case 2: $a < 0$.

As the second term on the right hand side of (5.1) is no longer negative for $n = T$, we cannot obtain as good an upper bound as in the case of nonnegative a. Assume Y is bounded and let M and m be as in Proposition 2. Then, from (5.3) and (4.1)

$$\frac{-a}{1-a}M \geq \hat{U}_T \geq c - M - \frac{a}{1-a} m,$$

and we obtain

$$e^{-R[u + \frac{a}{1-a}(M-w)]} \geq \psi(u,w) \geq e^{-R[u+M-c+\frac{a}{1-a}(m-w)]}$$

Example.

For a simple numerical example, suppose that

$$Y = \begin{cases} 1 & \text{with probability } .6 \\ 2 & \text{with probability } .4 \end{cases}$$

$a = 12,\ c = 3,\ u = 0$, and $w = 0$.

Then

$$\hat{G} = \begin{cases} 1 & \text{with probability } .6 \\ -1 & \text{with probability } .4 \end{cases}$$

and it is easy to calculate that

$$e^{-R} = 2/3.$$

Using the second inequality of (4.3) does not help here. We just get an upper bound of 1. Using the first inequality of (4.3), we get

$$\psi(0,0) \leq \frac{2}{3},$$

since we know that $\hat{U}_T$ is necessarily equal to -1.

This is can be improved by (5.2), which gives

$$\psi(0,0) \leq \frac{8}{27}.$$

For the lower bound, we calculate that $M = 4$, and see from (5.5) that

$$\psi(0,0) \geq \frac{31}{243}.$$

References.

[1] Bowers, N.L., Gerber, H.U., Hickman, J.C., Jones, D.A., Nesbitt C.J. *Actuarial Mathematics*. Itasca,IL: Society of Actuaries, 1986.

[2] Gerber, H.U. *An Introduction to Mathematical Risk Theory*, Huebner Foundation Monograph No. 8. Homewood IL: Richard D. Irwin, 1979.

[3] Gerber, H.U. "Ruin theory in the linear model", *Insurance: Mathematics and Economics* 1 (1982), 177-184.

[4] Promislow, S.D. "The probability of ruin in a process with dependent decrements", Preprint (1990).