ACTUARIAL RESEARCH CLEARING HOUSE
1993 VOL. 2

AN EXPLICIT FORMULA FOR THE PROBABILITY OF RUIN

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ABSTRACT

It was pointed out in [2](P.363) that for certain families of claim amount distributions, the ruin probability $\psi(u)$ can be expressed as

$$\sum_{i=1}^{n} C_i e^{-r_i u}. \qquad (1)$$

So far only relationships among C_i's and r_i's have been found such as in [3](P.22) and [4](P.484-5). For diexponential claim amount distributions (defined in [1], P.237), C_i's and r_i's will be expressed here explicitly in terms of parameters in p(x).

Let

$$\{ S(t) = X_1 + X_2 + \dots + X_{N(t)} \mid t \geq 0 \}$$

be a compound Poisson process with parameter λ and let

$$p(x) = pae^{-ax} + qbe^{-bx}, \quad x \geq 0,$$

with $p + q = 1$, be the probability density function of the identically and independently distributed random variables X_i's. For $\theta > 0$, let $c = (1 + \theta)\lambda p_1$. For $u \geq 0$, let $U(t) = u + ct - S(t)$ and let $\psi(u) = \Pr(T(u) < \infty)$, where $T = \min_{t>0} \{ t : U(t) < 0 \}$. The following formula will be derived

$$\int_0^{\infty} e^{ur} [-\psi'(u)]du = \frac{\theta}{1+\theta} \cdot \frac{ab(pb + qa - r)}{D(r)}, \qquad (2)$$

where

$$D(r) = (1+\theta)(pb+qa)r^2 - [\theta ab+(1+\theta)(pb^2+qa^2)]r + \theta ab(pb+qa). \qquad (3)$$

Proof of (2).

Since $p_1 = p/a + q/b$ and since $M_X(r) = pa/(r-a) + qb/(r-b)$, the formula (12.6.9) in [2] can be rewritten as

$$\int_0^{\infty} e^{ur}[-\psi'(u)]du = \frac{\theta}{1+\theta} \cdot \frac{1}{\dfrac{(1+\theta)(p/a + q/b)r}{pa/(a-r) + qb/(b-r) - 1} - 1}$$

$$= \frac{\theta}{1+\theta} \cdot \frac{1}{\dfrac{(1+o)(p/a + q/b)}{p/(a-r) + q/(b-r)} - 1}$$

$$= \frac{\theta}{1+\theta} \cdot \frac{ab(pb + qa - r)}{D(r)},$$

where

$$D(r) = (1 + \theta)(pb + qa)(a - r)(b - r) - ab(pb + qa - r),$$

which can be rewritten as (3).

The zeros of D(r),

$$r = \frac{\theta ab + (1 + \theta)(pb^2 + qa^2) \pm d(\theta)}{2(1 + \theta)(pb + qa)},$$

with

$$d(\theta) = \{[\theta ab + (1+\theta)(pb^2 + qa^2)]^2 - 4\theta(1+\theta)ab(pb+qa)^2\}^{1/2},$$

can be used in (2) to derive an explicit expression for $\psi(u)$.

Two special cases will be considered.

Case 1. Let $\theta = q/(1 + p)$ and $b = 2a$.

Then $1 + \theta = 2/(1 + p)$, $pb + qa = (1 + p)a$, and

$pb^2 + qa^2 = (3p + 1)a^2$. It follows that $d(\theta) = 4a^2 p^{1/2}$ and that

the zeros of $D(r)$ are $r = a(1 \pm p^{1/2})$. Hence (2) becomes

$$\int_0^\infty e^{ur}[-\psi'(u)]du = \left(\frac{1 - p^{1/2}}{2}\right)^2 \frac{a(1 + p^{1/2})}{a(1 + p^{1/2}) - r} + \left(\frac{1 + p^{1/2}}{2}\right)^2 \frac{a(1 - p^{1/2})}{a(1 - p^{1/2}) - r},$$

which implies

$$\psi(u) = \left(\frac{1 - p^{1/2}}{2}\right)^2 e^{-a(1+p^{1/2})u} + \left(\frac{1 + p^{1/2}}{2}\right)^2 e^{-a(1-p^{1/2})u}.$$

For example, $\psi(u)$ of Exercise 12.17 in [2] can be obtained as follows.

For $\lambda = 3$, $c = 1$ and

$$p(x) = (1/3)e^{-3x} + (16/3)e^{-6x}, \quad x > 0,$$

$p = 1/9$, $a = 3$, $b = 6 = 2a$, $p_1 = 5/27$ and $\theta = 4/5 = q/(1 + p)$.

Hence

$$\psi(u) = \left(\frac{1 - 1/3}{2}\right)^2 e^{-3(1 + 1/3)u} + \left(\frac{1 + 1/3}{2}\right)^2 e^{-3(1 - 1/3)u}$$

$$= (1/9)e^{-4u} + (4/9)e^{-2u}.$$

Case 2. Let $\theta = (b - a)/(b + a)$ and $p = q = 1/2$.

Then $d(\theta)$, denoted by d, can be found to be $[(b - a)^2 + a^2]^{1/2}$.

It follows that

$$\int_0^\infty e^{ur}[-\psi'(u)]du = \frac{a(b - a)(d + a)}{2bd(b - d)} \; \frac{(b - d)/2}{(b - d)/2 - r}$$

$$+ \frac{a(b - a)(d - a)}{2bd(b + d)} \; \frac{(b + d)/2}{(b + d)/2 - r}.$$

Since $b^2 - d^2 = 2a(b - a)$, the following formula can be obtained.

$$\psi(u) = \frac{(b+d)(d+a)}{4bd} e^{-u(b-d)/2} + \frac{(b-d)(d-a)}{4bd} e^{-u(b+d)/2}.$$

$\psi(u)$ of Example 12.10 in [2] can be obtained as follows.

For $\theta = 2/5$ and

$$p(x) = (3/2)e^{-3x} + (7/2)e^{-7x}, \quad x > 0,$$

$p = q = 1/2$, $a = 3$, $b = 7$ and $\theta = (b - a)/(b + a)$. Since

$$d = [(7 - 3)^2 + 3^2]^{1/2} = 5,$$

$$\psi(u) = \frac{(7+5)(5+3)}{4\times7\times5} e^{-u(7-5)/2} + \frac{(7-5)(5-3)}{4\times7\times5} e^{-u(7+5)/2}$$

$$= (24/35)e^{-u} + (1/35)e^{-6u}.$$

The author would like to thank Dr. Elias Shiu for generously sharing the related references.

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