ACTUARIAL RESEARCH CLEARING HOUSE
1993 VOL. 2

Geometric Realization of Convolution of Uniform Distributions

by Hung-Ping Tsao

ABSTRACT

Let X_i, $1 \leqslant i \leqslant n$, have a uniform distribution F_i on $[0, a_i]$. Assume that F_i's are mutually independent. We shall determine the distribution function of $S = \sum_{i=1}^{n} X_i$ by way of geometric realization of convolution. We shall adopt notation used in Actuarial Mathematics written by Newton L. Bowers, Jr., et al.

For each i, $1 \leqslant i \leqslant n$, consider the sum of i distinct a_k's. There are exactly $\binom{n}{i}$ such sums; denote them by a_{ij}, $1 \leqslant j \leqslant \binom{n}{i}$. Note that not all a_{ij}'s are distinct. Let $b_1, b_2, b_3, \ldots, b_m$ (with $m \leqslant 2^n - 1$) be the distinct values of a_{ij}'s in ascending order. Then the n-th convolution of F_i's can be expressed as

$$F_S(s) = \begin{cases} 0 & s < 0 \\ \dfrac{1}{(n!)\prod_{i=1}^{n} a_i}\left\{s^n + \sum_{i=1}^{n-1}\left[(-1)^i \sum_{a_{ij} < b_k} (s - a_{ij})^n\right]\right\} & b_{k-1} \leqslant s < b_k \quad (1 < k < m) \\ 1 & s \geqslant b_m \end{cases}$$

We shall only verify this result for $n = 2$ and 3. As for the case where $n > 3$, the same idea can be applied with imagination on the n-dimensional box.

We shall first look at the case where $n = 2$. Note that $a_{11} = \mathrm{Min}(a_1, a_2)$, $a_{12} = \mathrm{Max}(a_1, a_2)$ and $a_{21} = a_1 + a_2$. Therefore, $b_1 = a_{11}$, $b_2 = a_{12}$ and $b_3 = a_{21}$. For simplicity, we assume that $a_1 < a_2$. From Figure 1, we can calculate $F_S(s)$ geometrically.

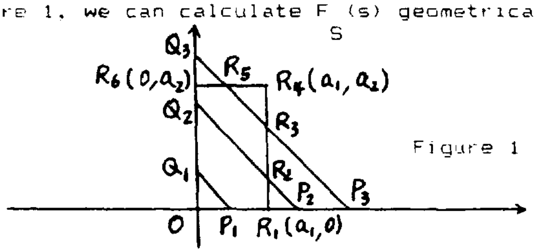

Figure 1

We shall use A(region) to denote the area of the region in question. Let $s \in [0, b_1)$ and let OP_1Q_1 be the isosceles right triangle with s being the length of each leg. Then

$$F_S(s) = \frac{1}{a_1 a_2} A(OP_1Q_1) = \frac{1}{2 a_1 a_2} s^2 .$$

Likewise, we can see on $[b_1, b_2)$ that

$$F_S(s) = \frac{1}{a_1 a_2} [A(OP_2Q_2) - A(R_1P_2R_2)] = \frac{1}{2 a_1 a_2} [s^2 - (s - a_1)^2]$$

and on $[b_2, b_3)$ that

$$F_S(s) = \frac{1}{a_1 a_2} [A(OP_3Q_3) - A(R_1P_3R_3) - A(R_6R_5Q_3)]$$

$$= \frac{1}{2 a_1 a_2} [s^2 - (s - a_1)^2 - (s - a_2)^2]$$

$$= \frac{1}{a_1 a_2} [A(OR_1R_4R_6) - A(R_3R_4R_5)] = 1 - \frac{1}{2 a_1 a_2} (a_1 + a_2 - s)^2$$

Now, let's look at the case where $n=3$. For simplicity, we assume that $b_1 = a_1$, $b_2 = a_2$, $b_3 = a_3$, $b_4 = a_1 + a_2$, $b_5 = a_1 + a_3$, $b_6 = a_2 + a_3$ and $b_7 = a_1 + a_2 + a_3$. In Figure 2, we illustrate the case where s $[b_2, b_3)$.

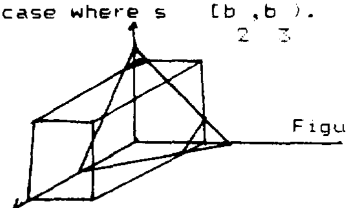

Figure 2

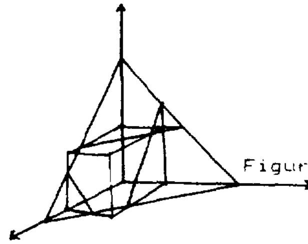

Figure 3

As can be seen in Figure 2, $a_1 a_2 a_3 F_S(s)$ equals the volume of the largest isosceles right tetrahedral minus the sum of the volumes of the smaller isosceles right tetrahedrals outside of the rectangular box. Hence on $[b_2, b_3)$, i.e. $[a_2, a_3)$, we have

$$F_S(s) = \frac{1}{a_1 a_2 a_3}\left[\frac{1}{6} s^3 - \frac{1}{6}(s-a_1)^3 - \frac{1}{6}(s-a_2)^3 \right]$$

$$= \frac{1}{6 a_1 a_2 a_3}\left[s^3 - (s-a_1)^3 - (s-a_2)^3 \right]$$

Likewise, we can see from Figure 3 that

$$F_S(s) = \frac{1}{6 a_1 a_2 a_3}\left[s^3 - \sum_{i=1}^{3} (s-a_i)^3 + (s-a_1-a_2)^3 \right]$$

on $[b_4, b_5)$, i.e. $[a_1 + a_2, a_1 + a_3)$.

The other cases are similar. On $[a_2 + a_3, a_1 + a_2 + a_3)$ we can also obtain

$$F_S(s) = 1 - \frac{1}{6 a_1 a_2 a_3}(a_1 + a_2 + a_3 - s)^3$$