Risk Theory with the Gamma Process

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Abstract

The aggregate claims process is modelled by a process with independent,

stationary and nonnegative increments. Such a process is either compound Poisson or

else a process with an infinite number of claims in each time interval, for example a

gamma process. It is shown how classical risk theory, and in particular ruin theory, can be

adapted to this model. A detailed analysis is given for the gamma process, for which

tabulated values of the probability of ruin are provided.

Keywords

1. Introduction

In classical collective risk theory, the aggregate claims process is assumed to be compound Poisson (Panjer and Willmot, 1984). Here we shall examine a more general model for the aggregate claims process: processes with independent, stationary and nonnegative increments. Such a process is either compound Poisson or else a process with an infinite number of claims in any time interval. The most prominent process with this intriguing property is the gamma process.

Since the process under consideration is either a compound Poisson process or a limit of compound Poisson processes, its properties can be derived from the basic properties of the compound Poisson process. The general results are derived in Section 2 (for the aggregate claims process) and Section 6 (for the probability of ruin). The gamma process is examined in detail in Sections 3, 4 and 5 (for the aggregate claims process) and Sections 7 and 8 (for the probability of ruin).

2. Processes with Independent, Stationary and Nonnegative Increments

Let $Q(x)$ be a nonnegative and nonincreasing function of x, $x > 0$, with the properties:

$$Q(x) \rightarrow 0 \text{ as } x \rightarrow \infty$$

and

$$\int_0^\infty Q(x)\, dx < \infty. \tag{2.1}$$

Condition (2.1) can also be written as

$$\int_0^\infty x\, [-dQ(x)] < \infty,$$

which, if $q(x) = -Q'(x)$ exists, becomes

$$\int_0^\infty x\, q(x)\, dx < \infty.$$

Such a function Q(x) defines an aggregate claims process $\{S(t)\}_{t \geq 0}$ in the following way.

For each x > 0, let N(t; x) denote the number of claims with an amount greater than x that occur before time t; let S(t; x) be the sum of these claims. We assume that $\{N(t; x)\}_{t \geq 0}$ is a Poisson process with parameter Q(x) and that $\{S(x; t)\}_{t \geq 0}$ is a compound Poisson process with Poisson parameter Q(x) and individual claim amount distribution

$$P(y; x) = \begin{cases} 0 & y \leq x \\ \dfrac{Q(x) - Q(y)}{Q(x)} & y > x \end{cases} \qquad (2.2)$$

The process {S(t)} is defined as the limit of the compound Poisson processes {S(t; x)} as x tends to 0.

We write

$$Q(0) = \lim_{x \to 0} Q(x).$$

We need to distinguish two cases: $Q(0) < \infty$, and $Q(0) = \infty$. In the first case, {S(t)} is a compound Poisson process with Poisson parameter Q(0) and individual claim amount distribution

$$P(y) = 1 - \frac{Q(y)}{Q(0)} , \qquad y \geq 0. \qquad (2.3)$$

This is the classical model for collective risk theory. Conversely, every compound Poisson process, given by Poisson parameter λ and individual claim amount distribution P(y), is of this type if we set

$$Q(y) = \lambda[1 - P(y)], \quad y \geq 0. \qquad (2.4)$$

In the second case, {S(t)} is the limit of compound Poisson processes, but is not a compound Poisson process itself, because the expected number of claims per unit time, Q(0), is infinite. Indeed, with probability one, the number of claims in any time interval is infinite. Nevertheless, S(t) is finite, as the majority of the claims are very small in some sense. In both cases, Q(y) is the expected number of claims per unit time with an amount exceeding y.

Since $\{S(t)\}$ is the limit of $\{S(t; x)\}$ as x tends to 0, we can use well-known results for

the compound Poisson process to obtain results for the process $\{S(t)\}$. For example, it

follows from

$$\begin{aligned} E[S(t; x)] &= t\,Q(x)\int_0^\infty [1 - P(y; x)]\,dy \\ &= t\,x\,Q(x) + t\int_x^\infty Q(y)\,dy \end{aligned}$$

that

$$E[S(t)] = t\int_0^\infty Q(y)\,dy = t\int_0^\infty y\,[-dQ(y)]. \tag{2.5}$$

To get the Laplace transform, we start with

$$\begin{aligned} E[e^{-z\,S(t; x)}] &= \exp\{t\,Q(x)[\int_x^\infty e^{-zy}\,dP(y; x) - 1]\} \\ &= \exp\{t\int_x^\infty [e^{-zy} - 1][-dQ(y)]\}. \end{aligned}$$

Letting $x \to 0$, we obtain

$$E[e^{-z\,S(t)}] = \exp\{t\int_0^\infty [e^{-zy} - 1][-dQ(y)]\}. \tag{2.6}$$

The process $\{S(t)\}$, defined by the function $Q(x)$, has independent, stationary and

nonnegative increments, and $E[S(t)] < \infty$. The converse is true in the following sense.

Every process $\{X(t)\}$ with these properties is of the form

$$X(t) = S(t) + bt,$$

where $\{S(t)\}$ is a process of the type presented above and b is a nonnegative constant.

This is a consequence of the connection between processes with independent and

stationary increments and infinitely divisible distributions, and the characterization of

infinitely divisible distributions with nonnegative support (Feller, 1971, p. 450, Theorem 2;

p. 571, formula (4.7)).

3. The Gamma Process

Assume that the function $Q(x)$ is differentiable and that $-Q'(x)$ is

$$q(x) = \frac{a}{x}e^{-bx}, \quad x > 0, \tag{3.1}$$

where a and b are positive constants. Let $\{S(t)\}$ be the associated aggregate claims process. In a time interval of length t, the expected number of claims with an amount exceeding x is

$$tQ(x) = at\int_x^\infty \frac{e^{-by}}{y}dy.$$

Since $Q(0) = \infty$, there is an infinite number of claims in each time interval. By (2.5) the expected aggregate claims in a time interval of length t are

$$E[S(t)] = t\int_0^\infty y\,q(y)\,dy = at\int_0^\infty e^{-by}dy = \frac{at}{b}. \tag{3.2}$$

To obtain the distribution of S(t), we compute its Laplace transform by (2.6):

$$\begin{aligned}
E[e^{-z\,S(t)}] &= \exp\{t\int_0^\infty [e^{-zy} - 1]q(y)dy\} \\
&= \exp\{at\int_0^\infty \frac{e^{-(z+b)y} - e^{-by}}{y}dy\} \\
&= \left(\frac{b}{z+b}\right)^{at}.
\end{aligned} \tag{3.3}$$

To verify the last step, consider the function

$$\phi(z) = \int_0^\infty \frac{e^{-(z+b)y} - e^{-by}}{y}dy;$$

observe that $\phi(0) = 0$ and $\phi'(z) = -(z+b)^{-1}$. Formula (3.3) shows that the distribution of S(t) is gamma, with shape parameter $\alpha_t = at$ and scale parameter $\beta_t = b$. Hence the process $\{S(t)\}$ is called a *gamma process*.

A gamma process with $a = b = 1$ is called a *standardized gamma process*. For an

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arbitrary gamma process with parameters a and b, we may set $t^* = at$ and $S^*(t^*) = bS(t)$.

It follows from (3.3) that

$$E(e^{-zS^*(t^*)}) = \left(\frac{1}{z+1}\right)^{t^*}.$$ (3.4)

Thus the transformed process $\{S^*(t^*)\}$ is a standardized gamma process.

In Section 2, the aggregate claims process $\{S(t)\}$ is obtained as a limit of the compound Poisson processes $\{S(t; x)\}$ as x tends to 0. Here, we have a specific $Q(x)$ and we now show that the gamma process $\{S(t)\}$ is the limit of a certain family of compound Poisson processes, which are not $\{S(t; x)\}$.

Let a and b be the positive constants in (3.1). Consider the family of compound Poisson processes with Poisson parameter λ, $\lambda > 0$, and individual claim amount density function

$$\underset{\lambda}{p}(x) = \frac{b^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-bx}, \qquad x > 0,$$ (3.5)

where

$$\alpha = a/\lambda.$$

In view of (2.4), we examine

$$\lambda \underset{\lambda}{p}(x) = \frac{ab^{\alpha} x^{\alpha-1} e^{-bx}}{\Gamma(\alpha+1)}.$$

For $\lambda \to \infty$ ($\alpha \to 0$) we obtain the right-hand side of (3.1). Thus the gamma process with parameters a and b is indeed the limit of this family of compound Poisson processes.

4. Parameter Estimation for the Gamma Process

Let $\{S(t)\}$ be a gamma process with (at time $t = 0$) unknown parameters a and b. We claim that, if we can observe the process for a time interval of (arbitrarily short) length h, $h > 0$, the value of a can be obtained as a limit: For $0 < x < 1$, we define the random variable

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$$A(x) = -\frac{N(h; x)}{h \ln(x)};$$ (4.1)

then

$$\lim_{x \to 0} A(x) = a.$$ (4.2)

(We remark that a similar situation exists for the diffusion process with *a priori* unknown but constant infinitesmal drift μ and variance σ^2: If the sample path for an arbitrarily small time interval is known, σ^2 can be calculated.)

To prove (4.2), we write (4.1) as

$$A(x) = \frac{\int_x^{\infty} \frac{e^{-by}}{y} dy}{\int_x^1 \frac{dy}{y}} \cdot \frac{N(h; x)}{ah \int_x^{\infty} \frac{e^{-by}}{y} dy} \cdot a.$$

Applying L'Hôpital's rule, we see that the first ratio tends to 1 as x tends to 0. The second ratio is $N(h; x)/[hQ(x)]$; by the strong law of large numbers it converges to 1 (with probability one) as x tends to 0.

In the following we assume that the value of a is known, but that b is unknown. If the aggregate claims process has been observed to time t, S(t) is a sufficient statistic, i.e., any additional information about the sample path is irrelevant for the estimation of b (DeGroot, 1975, p. 304, #5). To illustrate this, let us treat the unknown b as a random variable Θ with prior probability density function $u(\theta)$, $\theta > 0$. Then the posterior density of Θ at time t, given the value of S(t), is

$$u(\theta; t) = \frac{\theta^{at} e^{-\theta S(t)} u(\theta)}{\int_0^{\infty} r^{at} e^{-r S(t)} u(r) dr}.$$

Let us now assume that $u(\theta)$ is gamma, say,

$$u(\theta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta \theta}, \quad \theta > 0,$$

with $\beta > 0$ and $\alpha > 1$. Then the posterior density is also gamma, with parameters

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$$\alpha_t = \alpha + at$$

and

$$\beta_t = \beta + S(t).$$

At time $t = 0$, the expected aggregate claims per unit time are

$$E(\frac{a}{\Theta}) = a\int_0^{\infty}\frac{u(\theta)}{\theta}d\theta = a\frac{\beta}{\alpha - 1}.$$

Hence, with $S(t)$ known, the conditional expectation of the aggregate claims per unit time

is

$$a\frac{\beta_t}{\alpha_t - 1} = a\frac{\beta + S(t)}{\alpha + at - 1}$$

$$= (1 - Z_t)a\frac{\beta}{\alpha - 1} + Z_t\frac{S(t)}{t}, \tag{4.3}$$

where $Z_t = at/(at + \alpha - 1)$. Formula (4.3) corresponds to the well-known result for exact

credibility in the gamma/gamma model.

5. Simulation of the Gamma Process

We can simulate a compound Poisson process by simulating the times and

amounts of the claims. This straightforward approach is not applicable to the gamma

process, since there are infinitely many claims in each time interval. We now present a

method for simulating the gamma process.

Let $\{S(t)\}$ be the gamma process with parameters a and b. To simulate a sample

path, we use the following result. For time $\tau > 0$, the conditional distribution of the ratio

$S(\tau/2)/S(\tau)$, given $S(\tau)$, is symmetric beta with parameter $a\tau/2$ (DeGroot, 1975, p. 244, #5).

Thus, if we want to simulate a sample path for $S(t)$, $0 \leq t \leq T$, we can proceed as follows.

First, we simulate a value for $S(T)$, whose distribution is gamma with shape parameter aT

and scale parameter b. Then we obtain $S(T/2)$ by simulating a value for $S(T/2)/S(T)$,

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which has a symmetric beta distribution with parameter aT/2. Next, we obtain S(T/4) and S(3T/4) by simulating the values of S(T/4)/S(T/2) and [S(3T/4) − S(T/2)]/[S(T) − S(T/2)], respectively, each of which has a symmetric beta distribution with parameter aT/4. Similarly, we can generate the values of S(T/8), S(3T/8), S(5T/8), S(7T/8), and so on.

We have simulated the standardized Gamma process for T = 5.11 and T = 10.23. The resulting sample paths are shown in Figures 1 and 2.

Figure 1

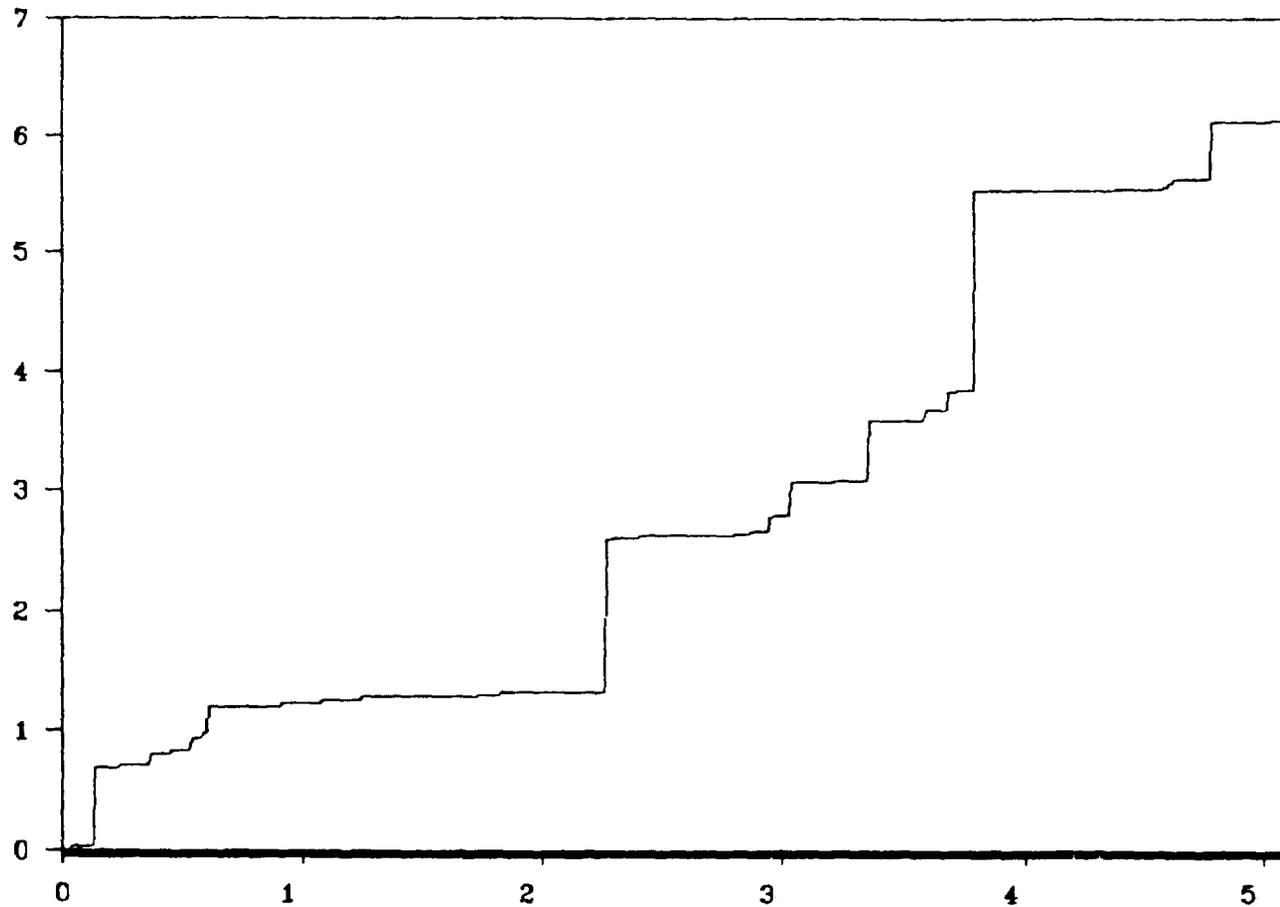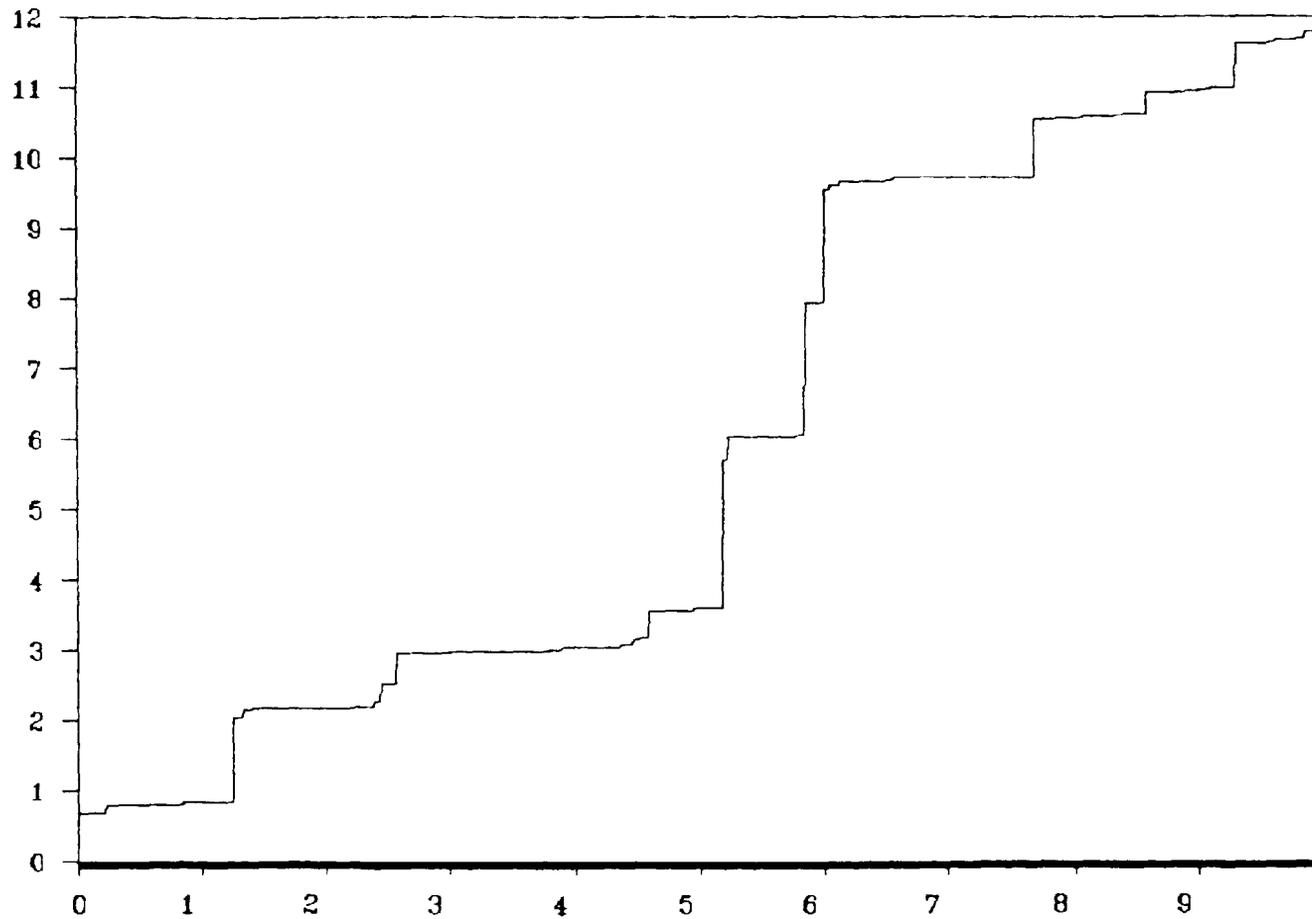Figure 2

6. Ruin Theory

Let {S(t)} be the aggregate claims process introduced in Section 2. In this section we present some ruin probability results for this process. In the next section, we specialize to the case that {S(t)} is a gamma process.

Let the surplus of an insurance company at time t, $t \geq 0$, be

$$U(t) = u + ct - S(t). \tag{6.1}$$

Here u is a nonnegative number denoting the initial surplus and c is the rate at which the premiums are received. The relative security loading θ is defined by the equation

$$c = (1 + \theta) E[S(1)] = (1 + \theta) \int_0^\infty Q(x)\, dx. \tag{6.2}$$

We assume that $\theta > 0$. Let $\psi(u)$ denote the probability of ultimate ruin, i.e., the probability that the surplus becomes negative at some future time.

In view of formula (2.4), results for this model can be obtained via those for the compound Poisson model with the following recipe. We start with a formula for the case of the compound Poisson process with Poisson parameter λ and individual claim amount distribution P(y). Then we substitute Q(y) for $\lambda[1 - P(y)]$ (or q(y) for $\lambda p(y)$ if the derivatives exist) to obtain the corresponding formula for the more general model.

For example, in the compound Poisson model the probability of ruin satisfies the following defective renewal equation [e.g., Bowers *et al* (1986, p. 373, #12.11)]:

$$c\psi(u) = \lambda \int_0^u \psi(u - y)[1 - P(y)]dy + \lambda \int_u^\infty [1 - P(y)]dy, \quad u \geq 0.$$

Substituting Q(y) for $\lambda[1 - P(y)]$, we get

$$c\psi(u) = \int_0^u \psi(u - y) Q(y) dy + \int_u^\infty Q(y)\, dy, \quad u \geq 0. \tag{6.3}$$

For $u = 0$, this gives

$$\psi(0) = \frac{1}{c}\int_0^\infty Q(y)\,dy = \frac{1}{1+\theta}. \tag{6.4}$$

Let us now consider the maximal loss random variable

$$L = \max_{t \geq 0} \{S(t) - ct\}. \tag{6.5}$$

It is of interest since $1 - \psi(u)$ is its distribution function. In the compound Poisson model, it is well known (Bowers *et al*, 1986, Section 12.6) that L has a compound geometric distribution:

$$L = L_1 + L_2 + \dots + L_N. \tag{6.6}$$

Here $N, L_1, L_2, \dots$ are independent random variables, the L_i's are identically distributed with the probability density

$$h(x) = \frac{1 - P(x)}{\int_0^\infty [1 - P(y)]\,dy}, \qquad x > 0, \tag{6.7}$$

and N has a geometric distribution defined by

$$\Pr(N = n) = \frac{\theta}{1+\theta}\left(\frac{\theta}{1+\theta}\right)^n, \qquad n = 0, 1, 2, \dots. \tag{6.8}$$

If we multiply both numerator and denominator of (6.7) by λ, we see that (6.6) is valid for the general model, with

$$h(x) = \frac{Q(x)}{\int_0^\infty Q(y)\,dy}, \qquad x > 0. \tag{6.9}$$

These formulas can be used to determine numerical lower and upper bounds for the ruin probability; see Method 1 in Dufresne and Gerber (1989).

For the next result we assume that $p(x) = P'(x)$ and $q(x) = -Q'(x)$ exist. Let T denote the time of ruin. Put $X = U(T-)$, the surplus immediately before ruin, and $Y = |U(T)|$, the deficit at the time of ruin. We assume that $u = 0$. Given that ruin occurs, the joint probability density of X and Y in the compound Poisson case is

$$h(x, y) = \frac{p(x + y)}{\int_0^\infty [1 - P(s)]\,ds}, \qquad x > 0, y > 0 \qquad (6.10)$$

(Dufresne and Gerber, 1988). Thus, in the general model, the joint density of X and Y is

$$h(x, y) = \frac{q(x + y)}{\int_0^\infty Q(s)\,ds}, \qquad x > 0, y > 0. \qquad (6.11)$$

We note that both (6.10) and (6.11) are symmetric in x and y. The probability density of Z = X + Y (the amount of the claim that causes ruin) is

$$g(z) = \int_0^z h(x, z - x)\,dx = \frac{zq(z)}{\int_0^\infty Q(s)\,ds}, \qquad z > 0. \qquad (6.12)$$

The conditional probability density of X given Z = z (and u = 0) is

$$\frac{h(x, z - x)}{g(z)} = \frac{1}{z}, \qquad 0 < x < z.$$

This is the somewhat surprising result that the conditional distribution of X (given Z = z) is uniform between 0 and z.

We wish to remark that, if Q(0) = ∞, the notion of an individual claim amount distribution of the process {S(t)} *per se* does not make sense. However, the conditional claim amount distribution, given certain information, may still exist. For example, (2.2) is the distribution of an individual claim amount given that it exceeds x. Likewise, g(z) is the probability density function of the amount of the claim that causes ruin.

We now turn to Lundberg's asymptotic formula. The adjustment coefficient R is defined as the positive value of r for which

$$E[e^{r[S(t) - ct]}] = 1 \qquad (6.13)$$

or

$$E[e^{r\,S(t)}] = e^{rct}. \qquad (6.14)$$

(Note that the left-hand side of (6.14) is (2.6) with r = −z and that some regularity conditions have to be imposed on Q(y) to guarantee the existence of R.) Lundberg's

famous asymptotic formula states that

$$\psi(u) \sim C e^{-Ru} \qquad \text{for } u \to \infty. \tag{6.15}$$

In the compound Poisson case,

$$C = \frac{\theta \lambda \int_0^{\infty} y \, dP(y)}{\lambda \int_0^{\infty} y e^{Ry} dP(y) - c} \tag{6.16}$$

(Seal, formula (4.64)), which is translated as

$$C = \frac{-\theta \int_0^{\infty} y \, dQ(y)}{-\int_0^{\infty} y e^{Ry} dQ(y) - c}. \tag{6.17}$$

7. Ruin Theory for the Gamma Process

We now consider the special case that $\{S(t)\}$ is a gamma process. As we pointed out in Section 3, any gamma process can be transformed into a standardized gamma process. Thus we assume that, for $x > 0$,

$$q(x) = \frac{e^{-x}}{x} \tag{7.1}$$

or

$$Q(x) = \int_x^{\infty} \frac{e^{-y}}{y} dy. \tag{7.2}$$

In Abramowitz and Stegun (1964, p.227), the exponential integral (7.2) is denoted as $E_1(x)$.

Since

$$\int_0^{\infty} Q(x) \, dx = \int_0^{\infty} x \, q(x) \, dx = \int_0^{\infty} e^{-x} dx = 1,$$

formula (6.2) becomes

$$1 + \theta = c. \tag{7.3}$$

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By (6.9) the common probability density function of the random variables $\{L_i\}$ is

$$h(x) = Q(x) = E_1(x), \quad x > 0, \tag{7.4}$$

and their distribution function is

$$H(x) = \int_0^x h(y) \, dy = 1 - e^{-x} + xE_1(x), \quad x \geq 0. \tag{7.5}$$

From (6.11) and (6.12) we obtain

$$h(x, y) = \frac{e^{-(x + y)}}{x + y} \tag{7.6}$$

and

$$g(z) = e^{-z}, \tag{7.7}$$

respectively. Formula (7.7) is especially interesting, as it says that (if $u = 0$) the amount of the claim that causes ruin is exponentially distributed.

Substituting (3.4) and (7.3) in (6.14) yields the equation

$$\frac{1}{1 - r} = e^{r(1 + \theta)}. \tag{7.8}$$

The adjustment coefficient R is the positive root of (7.8). It follows from (6.17) and (7.3) that the asymptotic constant C in Lundberg's formula is

$$C = \frac{\theta}{\dfrac{1}{1 - R} - (1 + \theta)} = \frac{\theta(1 - R)}{R - \theta(1 - R)}. \tag{7.9}$$

Remark: As pointed out in Section 3, the gamma process is the limit of a certain family of compound Poisson processes, each with a gamma claim amount distribution. For these Willmot (1988) has given an elegant method to evaluate the probability of ruin.

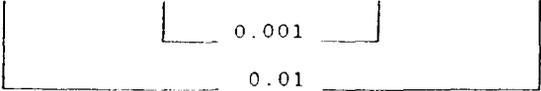## 8. The Probability of Ruin for the Gamma Process

As in the last section we assume that the aggregate claims process is the standardized gamma process. Since (7.5) gives an explicit expression for $H(x)$, we can apply the method of lower and upper bounds to calculate the probability of ruin (Dufresne and Gerber, 1989). We have calculated lower and upper bounds for $\psi(u)$ for different

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values of the initial surplus u (0, 1, 2, ... , 20) and the relative security loading θ (0.1, 0.2, 0.3, ... , 1.0), for intervals of discretisation with length 0.01 and 0.001. For θ = 0.5 these bounds are displayed in Table 1. Thus the exact value of the probability of ruin is known with sufficient accuracy (4 decimals). Table 2 shows these values.

Table 1

Lower and Upper Bounds for the Probability of Ruin

θ = 0.5

u	Lower Bounds		Upper Bounds	
0	0.666667	0.666667	0.666667	0.666667
1	0.321352	0.322741	0.323055	0.324488
2	0.175016	0.176268	0.176550	0.177839
3	0.096653	0.097604	0.097819	0.098798
4	0.053619	0.054288	0.054439	0.055129
5	0.029801	0.030250	0.030352	0.030817
6	0.016577	0.016870	0.016936	0.017240
7	0.009225	0.009412	0.009454	0.009649
8	0.005135	0.005252	0.005279	0.005401
9	0.002858	0.002931	0.002948	0.003024
10	0.001591	0.001636	0.001646	0.001693
11	0.000886	0.000913	0.000919	0.000948
12	0.000493	0.000510	0.000513	0.000531
13	0.000275	0.000284	0.000287	0.000297
14	0.000153	0.000159	0.000160	0.000166
15	0.000085	0.000089	0.000089	0.000093
16	0.000047	0.000049	0.000050	0.000052
17	0.000026	0.000028	0.000028	0.000029
18	0.000015	0.000015	0.000016	0.000016
19	0.000008	0.000009	0.000009	0.000009
20	0.000005	0.000005	0.000005	0.000005

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                        |_____ 0.001 _____|
              |_____ 0.01 _____|
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length of the interval of discretisation

Table 2

The Probability of Ruin for the Standardized Gamma process
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Relative Security Loading   Theta

u	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0.9091	0.8333	0.7692	0.7143	0.6667	0.6250	0.5882	0.5556	0.5263	0.5000
1	0.7395	0.5736	0.4613	0.3816	0.3229	0.2782	0.2434	0.2155	0.1929	0.1743
2	0.6184	0.4165	0.2990	0.2253	0.1764	0.1424	0.1178	0.0994	0.0854	0.0743
3	0.5182	0.3038	0.1952	0.1344	0.0977	0.0741	0.0582	0.0470	0.0388	0.0327
4	0.4345	0.2219	0.1277	0.0805	0.0544	0.0388	0.0289	0.0224	0.0178	0.0145
5	0.3643	0.1621	0.0836	0.0482	0.0303	0.0204	0.0144	0.0107	0.0082	0.0065
6	0.3054	0.1185	0.0548	0.0289	0.0169	0.0107	0.0072	0.0051	0.0038	0.0029
7	0.2561	0.0866	0.0359	0.0173	0.0094	0.0056	0.0036	0.0025	0.0018	0.0013
8	0.2148	0.0632	0.0235	0.0104	0.0053	0.0030	0.0018	0.0012	0.0008	0.0006
9	0.1801	0.0462	0.0154	0.0062	0.0029	0.0016	0.0009	0.0006	0.0004	0.0003
10	0.1510	0.0338	0.0101	0.0037	0.0016	0.0008	0.0005	0.0003	0.0002	0.0001
11	0.1266	0.0247	0.0066	0.0022	0.0009	0.0004	0.0002	0.0001	0.0001	0.0001
12	0.1062	0.0180	0.0043	0.0013	0.0005	0.0002	0.0001	0.0001		
13	0.0890	0.0132	0.0028	0.0008	0.0003	0.0001	0.0001			
14	0.0746	0.0096	0.0019	0.0005	0.0002	0.0001				
15	0.0626	0.0070	0.0012	0.0003	0.0001					
16	0.0525	0.0051	0.0008	0.0002						
17	0.0440	0.0038	0.0005	0.0001						
18	0.0369	0.0027	0.0003	0.0001						
19	0.0309	0.0020	0.0002							
20	0.0259	0.0015	0.0001							

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Illustration: Assume that the annual aggregate claims have an expectation $\mu = 100,000$ and a standard deviation $\sigma = 20,000$. The initial reserve is 48,000 and the annual premium (net of expenses) is 120,000. What is the probability of ultimate ruin?

Solution: We assume that the premiums are received continuously and the aggregate claims process can be modelled by a gamma process with parameters a and b. Then $a/b = \mu = 100,000$ and $a/b^2 = \sigma^2 = (20,000)^2$. It follows that $b = \mu/\sigma^2 = 1/4,000$. In order to use Table 2 (which is for the standardized gamma process), we have to transform the initial reserve to $u = 48,000 \times b = 12$. The relative security loading $\theta = 0.2$ does not change. Looking up Table 2, we obtain the probability of ruin $\psi(12) = 0.018$.

References

Abramowitz, M. and Stegun, I.A. (1964) *Handbook of Mathematical Functions.* National

    Bureau of Standards, Washington, D.C. Reprinted by Dover, New York.

Bowers, N.J., Jr., Gerber, H.U., Hickman, J.C., Jones, D.A. and Nesbitt, C.J. (1986)

    Actuarial Mathematics. Society of Actuaries, Itasca, Illinois.

DeGroot, M.H. (1975) *Probability and Statistics.* Addison-Wesley, Reading, Massachusetts.

Dufresne, F. and Gerber, H.U. (1988) The surplus immediately before and at ruin, and the

    amount of the claim causing ruin. *Insurance: Mathematics and Economics* **7**, 193-199.

Dufresne, F. and Gerber, H.U. (1989) Three methods to calculate the probability of ruin.

    ASTIN Bulletin **19**, 71-90.

Feller, W. (1971) *An Introduction to Probability Theory and Its Applications,* Vol. 2 (2nd

    edn). Wiley, New York.

Panjer, H.H. and Willmot, G.E. (1984) Models for the distribution of aggregate claims in risk

    theory. *Transactions of the Society of Actuaries* **36**, 399-477.

Seal, H.L. (1969) *The Stochastic Theory of a Risk Business.* Wiley, New York.

Willmot, G.E. (1988) Further use of Shiu's approach to the evaluation of ultimate ruin

    probabilities. *Insurance: Mathematics and Economics* **7**, 275-281.

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