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LIFE CONTINGENCY FUNCTIONS AND THEIR DERIVATIVES

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ABSTRACT

Deferred term, pure endowment and endowment life contingency functions and their derivatives are discussed in general and some familiar formulas are derived in particulr.

We shall consolidate the ideas presented in [1] and [2] in a more general fashion. Formulas in the continuous case will be derived formally and some errors in those papers can be easily identified.

For our purpose, let the deferred term, pure endowment and endowment life contingency functions be denoted by

$$ {}_{k|n|}\overset{m}{\alpha}{}^{\,1}_{x}, $$

$$ {}_{k|\overset{1}{n}|}\overset{m}{\alpha}_{x}, $$

and

$$ {}_{k|n|}\overset{m}{\alpha}_{x} $$

respectively. Thus

$$ {}_{k|n|}\overset{m}{\alpha}{}^{\,1}_{x} = {}_{k}E_{x}\,\overset{m}{\alpha}{}^{\,1}_{x+k:\overline{n|}}, $$

$$ {}_{k|\overset{1}{n}|}\overset{m}{\alpha}_{x} = {}_{k+n}p_{x}\,\overset{m}{\alpha}_{n}, $$

and

$$ {}_{k|n|}\overset{m}{\alpha}_{x} = {}_{k}E_{x}\,\overset{m}{\alpha}_{x+k:\overline{n|}}, $$

where

$$\overset{m}{\alpha}{}^{\,1}_{y:\overline{n}|} = \sum_{t=1}^{n} \overset{m}{\alpha}_{t}\ {}_{t-1|}q_y \qquad (1)$$

and (α_y, α_t) is a pair of corresponding life contingency and interest function notations in the context of (1) such as $(\ddot{a}_y, \ddot{a}_{\overline{t}|})$, $(a_y, a_{\overline{t}|})$, (A_y, v^t), $((I\ddot{a})_y, (I\ddot{a})_t)$, $((Ia)_y, (Ia)_t)$ and $((IA)_y, tv^t)$.

For the continuous case, we define

$$\bar{\alpha}_x = \lim_{m\to\infty} \overset{m}{\alpha}_x .$$

Since

$$\frac{d\ {}_kE_x}{dx} = {}_kE_x(\mu_x - \mu_{x+k})$$

and since

$$\frac{d\ {}_{\frac{t}{m}}p_{x+k}}{dx} = {}_{\frac{t}{m}}p_{x+k}(\mu_{x+k} - \mu_{x+k+\frac{t}{m}}),$$

by taking y = x + k in (1) we can obtain

$$\frac{d\ {}_{k|n|}\overset{m}{\alpha}{}^{1}_{x}}{dx} = {}_{k|n|}\overset{m}{\alpha}{}^{1}_{x}\mu_x - \overset{m}{\alpha}_{\frac{1}{m}}\ {}_kE_x\mu_{x+k} + \overset{m}{\alpha}_n\ {}_np_{x+k}\ {}_kE_x\mu_{x+k+n}$$
$$+ {}_kE_x \sum_{t=1}^{mn-1} \left[\overset{m}{\alpha}_{\frac{t}{m}} - \overset{m}{\alpha}_{\frac{t+1}{m}}\right] {}_{\frac{t}{m}}p_{x+k}\mu_{x+k+\frac{t}{m}}. \quad (2)$$

We can readily obtain

$$\frac{d\ {}_{k|\overset{1}{n}|}\overset{m}{\alpha}_x}{dx} = {}_{k|\overset{1}{n}|}\overset{m}{\alpha}_x\mu_x - {}_{k+n}p_x\ \overset{m}{\alpha}_n\mu_{x+k+n}. \quad (3)$$

Combining (2) and (3), we have

$$\frac{d\ {}_{k|n|}\overset{m}{\alpha}_x}{dx} = {}_{k|n|}\overset{m}{\alpha}_x\mu_x - \overset{m}{\alpha}_{\frac{1}{m}}\ {}_kE_x\mu_{x+k} + \overset{m}{\alpha}_n\ {}_np_{x+k}\ {}_kq_x\mu_{x+k+n}$$
$$+ {}_kE_x \sum_{t=1}^{mn-1} \left[\overset{m}{\alpha}_{\frac{t}{m}} - \overset{m}{\alpha}_{\frac{t+1}{m}}\right] {}_{\frac{t}{m}}p_{x+k}\mu_{x+k+\frac{t}{m}}. (4)$$

Let us look at some examples.

EXAMPLE 1. $\overset{m}{\alpha}_t = v^t$.

From (2), we have

$$\frac{d\, {}_{k|n|}\overset{m}{\bar{A}}{}^{1}_{x}}{dx} = {}_{k|n|}\overset{m}{\bar{A}}{}^{1}_{x}\mu_x - v^{\frac{1}{m}}\,{}_kE_x\mu_{x+k} + v^n\,{}_np_{x+k}\mu_{x+k+n}$$
$$+ {}_kE_x\sum_{t=1}^{mn-1}\left(v^{\frac{t}{m}} - v^{\frac{t+1}{m}}\right){}_{\frac{t}{m}}p_{x+k}\mu_{x+k+\frac{t}{m}}$$
$$= {}_{k|n|}\overset{m}{\bar{A}}{}^{1}_{x}\mu_x - v^{\frac{1}{m}}\,{}_kE_x\mu_{x+k} + v^n\,{}_np_{x+k}\mu_{x+k+n}$$
$$+ \left(1 - v^{\frac{1}{m}}\right){}_kE_x\sum_{t=1}^{mn-1} v^{\frac{t}{m}}\,{}_{\frac{t}{m}}p_{x+k}\mu_{x+k+\frac{t}{m}}. \quad (5)$$

From (3) and (4), we can also have

$$\frac{d\, {}_{k|\overset{1}{n}|}\overset{m}{\bar{A}}_x}{dx} = {}_{k|\overset{1}{n}|}\overset{m}{\bar{A}}_x\mu_x - {}_kp_x\,{}_nE_{x+k}\mu_{x+k+n}$$

and

$$\frac{d\, {}_{k|n|}\overset{m}{\bar{A}}_x}{dx} = {}_{k|n|}\overset{m}{\bar{A}}_x - v^{\frac{1}{m}}\,{}_kE_x\mu_{x+k}$$
$$+ \left(1 - v^{\frac{1}{m}}\right){}_kE_x\sum_{t=1}^{mn-1}{}_{\frac{t}{m}}E_{x+k}\mu_{x+k+\frac{t}{m}}.$$

By taking $m = 1$ and $k = 0$, these formulas become

$$\frac{dA^1_{x:\overline{n}|}}{dx} = A^1_{x:\overline{n}|}\mu_x - v\mu_x + {}_nE_x\mu_{x+n} + d\sum_{t=1}^{n-1}{}_tE_x\mu_{x+t},$$
$$\frac{dA_{x:\overset{1}{\overline{n}|}}}{dx} = A_{x:\overset{1}{\overline{n}|}}\mu_x - {}_nE_x\mu_{x+n},$$

and

$$\frac{dA_{x:\overline{n}|}}{dx} = A_{x:\overline{n}|}\mu_x - v\mu_x + d\sum_{t=1}^{n-1}{}_tE_x\mu_{x+t}.$$

Let us now look at the continuous case.

Since the last term of (5) can be written as

$${}_kE_x\,\frac{1 - v^{\frac{1}{m}}}{\frac{1}{m}}\sum_{t=1}^{mn-1} v^{\frac{t}{m}}\,{}_{\frac{t}{m}}p_{x+k}\mu_{x+k+\frac{t}{m}}\left(\frac{t+1}{m} - \frac{t}{m}\right),$$

by letting $m \to \infty$ we have

$$\frac{d\, {}_{k|n|}\bar{A}^1_x}{dx} = {}_{k|n|}\bar{A}^1_x\mu_x - {}_kE_x\mu_{x+k} + {}_{k+n}E_x\mu_{x+k+n} + {}_kE_x\,\delta\int_0^n v^t\,{}_tp_x\mu_{x+t}\,dt$$
$$= {}_{k|n|}\bar{A}^1_x(\mu_x + \delta) - {}_kE_x\mu_{x+k} + {}_{k+n}E_x\mu_{x+k+n}.$$

We can also obtain

$$\frac{d\,{}_{k|}{}_{n}^{1}|\bar{A}_x}{dx} = {}_{k|}{}_{n}^{1}|\bar{A}_x \mu_x - {}_kp_x\, {}_nE_{x+k}\mu_{x+k+n}$$

and

$$\frac{d\,{}_{k|n|}\bar{A}_x}{dx} = {}_{k|n|}\bar{A}_x \mu_x + {}_{k|n|}\bar{A}_x^1 \delta - {}_k\bar{E}_x \mu_{x+k} + {}_kg_x\, {}_nE_{x+k}\mu_{x+k+n}.$$

Upon taking k = 0 in the last three formulas, we obtain

$$\frac{d\bar{A}^1_{x:\overline{n|}}}{dx} = \bar{A}^1_{x:\overline{n|}}(\mu_x+\delta) - \mu_x + {}_nE_x\mu_{x+n},$$

$$\frac{d\bar{A}_{x:\overline{n|}}^{\;\,1}}{dx} = \bar{A}_{x:\overline{n|}}^{\;\,1}\mu_x - {}_nE_x\mu_{x+n},$$

and

$$\frac{d\bar{A}_{x:\overline{n|}}}{dx} = \bar{A}_{x:\overline{n|}}\mu_x + \bar{A}^1_{x:\overline{n|}}\delta - \mu_x.$$

EXAMPLE 2. $\overset{m}{\alpha}_t = \ddot{a}^{(m)}_{\overline{t|}}$.

From (2), we have

$$\begin{aligned}\frac{d\,{}_{k|n|}\ddot{a}^{(m)1}_x}{dx} &= {}_{k|n|}\ddot{a}^{(m)1}_x\mu_x - \ddot{a}^{(m)}_{\overline{\frac{1}{m}|}}\,{}_kE_x\mu_{x+k} + \ddot{a}^{(m)}_{\overline{n|}}\,{}_np_{x+k}\mu_{x+k+n} \\ &\quad + {}_kE_x\sum_{t=1}^{mn-1}\left[\ddot{a}^{(m)}_{\overline{\frac{t}{m}|}} - \ddot{a}^{(m)}_{\overline{\frac{t+1}{m}|}}\right]{}_{\frac{t}{m}}p_{x+k}\mu_{x+k+\frac{t}{m}} \\ &= {}_{k|n|}\ddot{a}^{(m)1}_x\mu_x - \frac{1}{m}\,{}_kE_x\mu_{x+k} + \ddot{a}^{(m)}_{\overline{n|}}\,{}_np_{x+k}\,{}_kE_x\mu_{x+k+n} \\ &\quad - \frac{1}{m}\sum_{t=1}^{mn-1}{}_{k+\frac{t}{m}}E_x\mu_{x+k+\frac{t}{m}}. \end{aligned} \qquad (6)$$

From (3) and (4) we can also obtain

$$d\,{}_{k|}{}_{n}^{1}|\ddot{a}^{(m)}_x\mu_x - \ddot{a}^{(m)}_{\overline{n|}}\,{}_{k+n}p_x\mu_{x+k+n}$$

and

$$\begin{aligned}\frac{d\,{}_{k|n|}\ddot{a}^{(m)}_x}{dx} &= {}_{k|n|}\ddot{a}^{(m)}_x\mu_x - \frac{1}{m}\,{}_k\bar{E}_x\mu_{x+k} + \ddot{a}^{(m)}_{\overline{n|}}\,{}_np_{x+k}\,{}_kg_x\,{}_k\bar{E}_x\mu_{x+k+} \\ &\quad - \frac{1}{m}\,{}_k\bar{E}_x\sum_{t=1}^{mn-1}v^{\frac{t}{m}}\,{}_{\frac{t}{m}}p_{x+k}\mu_{x+k+\frac{t}{m}}.\end{aligned}$$

By taking m = 1 and k = 0, these formulas become

$$\frac{d\ddot{a}^{1}_{x:\overline{n}|}}{dx} = \ddot{a}^{1}_{x:\overline{n}|}\mu_x + \ddot{a}_{\overline{n}|}\, {}_nP_x\mu_{x+n} - \sum_{t=0}^{n-1} {}_tE_x\mu_{x+t},$$

$$\frac{d\ddot{a}_{x:\overset{1}{\overline{n}|}}}{dx} = \ddot{a}_{x:\overset{1}{\overline{n}|}}\mu_x - \ddot{a}_{\overline{n}|}\, {}_nP_x\mu_{x+n},$$

and

$$\frac{d\ddot{a}_{x:\overline{n}|}}{dx} = \ddot{a}_{x:\overline{n}|}\mu_x - \sum_{t=0}^{n-1} {}_tE_x\mu_{x+t}.$$

Let us now look at the continuous case.

Since the last term of (6) can be written as

$$-{}_kE_x \sum_{t=1}^{mn-1} V^{\frac{t}{m}}\, {}_{\frac{t}{m}}P_{x+k}\,\mu_{x+k+\frac{t}{m}}\left(\frac{t+1}{m} - \frac{t}{m}\right),$$

by letting $m \to \infty$ we obtain

$$\frac{d\,{}_{k|n|}\bar{a}^{1}_{x}}{dx} = {}_{k|n|}\bar{a}^{1}_{x}\mu_x - {}_{k|n|}\bar{A}^{1}_{x} + \bar{a}_{\overline{n}|}\, {}_nP_{x+k}\, {}_k\bar{E}_x\mu_{x+k+n}.$$

We can also obtain

$$\frac{d\,{}_{k|\overset{1}{n}|}\bar{a}_{x}}{dx} = {}_{k|\overset{1}{n}|}\bar{a}_{x}\mu_x - \bar{a}_{\overline{n}|}\, {}_{k+n}P_x\mu_{x+k+n}$$

and

$$\frac{d\,{}_{k|n|}\bar{a}_{x}}{dx} = {}_{k|n|}\bar{a}_{x}\mu_x - {}_{k|n|}\bar{A}^{1}_{x} + \bar{a}_{\overline{n}|}\, {}_nP_{x+k}\, {}_kg_x\mu_{x+k+n}.$$

Upon taking k = 0 in these formulas we have

$$\frac{d\bar{a}^{1}_{x:\overline{n}|}}{dx} = \bar{a}^{1}_{x:\overline{n}|}\mu_x - \bar{A}^{1}_{x:\overline{n}|} + \bar{a}_{\overline{n}|}\, {}_nP_x\mu_{x+n},$$

$$\frac{d\bar{a}_{x:\overset{1}{\overline{n}|}}}{dx} = \bar{a}_{x:\overset{1}{\overline{n}|}}\mu_x - \bar{a}_{\overline{n}|}\, {}_nP_x\mu_{x+n},$$

and

$$\frac{d\bar{a}_{x:\overline{n}|}}{dx} = \bar{a}_{x:\overline{n}|}\mu_x - \bar{A}^{1}_{x:\overline{n}|}.$$

Formulas for increasing functions can be derived in a similar fashion. The derivation is rather complicated and therefore omitted. However, by making use of the relationship

$$(I\bar{a})_{x:\overline{n}|} = \sum_{t=0}^{n-1} {}_{t|n-t|}\bar{a}_x$$

and

$$(I\bar{A})_{x:\overline{n}|} = \bar{a}_{x:\overline{n}|} - \delta(I\bar{a})_{x:\overline{n}|},$$

we can obtain the following formulas for the continuous case:

$$\frac{d(I\bar{a})_{x:\overline{n}|}}{dx} = (I\bar{a})_{x:\overline{n}|}\mu_x - (I\bar{A})^{1}_{x:\overline{n}|} + \sum_{t=0}^{n-1}\bar{a}_{\overline{n-t}|}\,{}_{n-t}p_{x+t}\,{}_{t}q_x\,\mu_{x+n}$$

and

$$\frac{d(I\bar{A})_{x:\overline{n}|}}{dx} = (I\bar{A})_{x:\overline{n}|}\mu_x - \bar{A}^{1}_{x:\overline{n}|} + \delta(I\bar{A})^{1}_{x:\overline{n}|} - \delta\sum_{t=0}^{n-1}\bar{a}_{\overline{n-t}|}\,{}_{n-t}p_{x+t}\,{}_{t}q_x\,\mu_{x+n}.$$

REFERENCES.

[1] Tsao, H., "A New Derivation of Life Annuity and Life Insurance Functions", ARCH, 1985.1, P. 61.

[2] Tsao, H., "On Derivatives of Life Contingency Functions", ARCH, 1987.1, P. 11.