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Total Return, Duration and Convexity

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Let V denote the value or price of a fixed-income security. If the cash flows of the security are fixed and certain, the (continuous) yield to maturity or force of interest, δ, is a well-defined quantity and we may consider V as a function of δ,

$$V = V(\delta).$$

Using Taylor's formula

$$V(\delta + \varepsilon) = V(\delta) + V'(\delta)\varepsilon + V''(\delta)\varepsilon^2/2 + \dots,$$

we can estimate the change in the price of the security as its yield changes. We note that Jordan [Jo, p. 56, (2.49)] had given the approximation formula

$$\overset{j}{a}_x \approx \overset{i}{a}_x - \frac{j-i}{1+i}(\overset{i}{Ia})_x.$$

In this note we consider the price, V, as a function of both yield δ and time t. Let $V(t, \delta)$ denote the value of the security at time t evaluated with the continous yield rate δ. We also assume that δ is a differentiable function of time t, $\delta = \delta(t)$. We wish to examine how the security price changes as time passes and as yield changes.

The change in the price of the security as time passes from t to t + dt is given by the differential

$$dV = V(t + dt, \delta(t + dt)) - V(t, \delta(t)).$$

Applying the chain rule in multivariate calculus, we have

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial \delta}d\delta \tag{1}$$

[For example, the value at time t of a zero coupon bond that will pay 1 at time T, $t < T$, is

$$V = e^{-(T-t)\delta}.$$

In this case formula (1) becomes $dV = V\delta dt - V(T-t)d\delta$.]

Rewrite (1) as

$$\frac{dV}{V} = \frac{\frac{\partial V}{\partial t}}{V}dt + \frac{\frac{\partial V}{\partial \delta}}{V}d\delta . \qquad (2)$$

The partial derivative $\partial V/\partial t$ gives the rate of change of the security value with respect to time, while everything else (i.e. interest rate) is held fixed. Thus the ratio

$$\frac{\frac{\partial V}{\partial t}}{V}$$

is the yield rate $\delta(t)$ of the security. The partial derivative $\partial V/\partial \delta$ gives the rate of change of the security value with respect to yield, while everything else (i.e. time) is fixed. Defining

$$D(t, \delta) = -\frac{\frac{\partial V}{\partial \delta}}{V},$$

we can rewrite equation (2) as

$$\frac{dV}{V} = \delta dt - D d\delta, \qquad (3)$$

which says that there are **two** components that make up the security's (instantaneous) **total** return — (i) yield (interest accrued) and (ii) capital gain or loss due to interest rate fluctuation. [For the zero-coupon bond, formula (3) is $dV/V = \delta dt - (T-t)d\delta$.] The quantity D is called *duration*. Formula (3) can be found in the book [Gr, p. 62, (4.3)]. Ayres and Barry [AB, (3)] also derive (3), but they use a different approach.

We now examine how the security price changes over a period of time, say, from time $t = t_0$ to time $t = t_1$. Integrating equation (3) from $t = t_0$ to $t = t_1$ yields

$$\frac{V(t_1, \delta(t_1))}{V(t_0, \delta(t_0))} = \exp\left(\int_{t_0}^{t_1} \delta(t)dt - \int_{t_0}^{t_1} D(t, \delta(t))d\delta(t) \right). \qquad (4)$$

(Formula (4) can be found in [Gr, p. 63, (4.7)].) It follows from (4) that the security price at time t_1 is the product of three terms: (i) the security price at time t_0, (ii) the interest accumulation factor $e^{\int \delta dt}$ and (iii) the capital gain/loss factor $e^{-\int D d\delta}$.

Let us evaluate the integral $\int D d\delta$. Integrating by parts yields

$$\int_{t=t_0}^{t=t_1} D \, d\delta(t) = \int_{t_0}^{t_1} D \, d[\delta(t) - \delta(t_1)]$$

$$= [\delta(t_1) - \delta(t_0)] D(t_0, \delta(t_0)) + \int_{t_0}^{t_1} [\delta(t_1) - \delta(t)] dD. \qquad (5)$$

Since

$$dD = \frac{\partial D}{\partial \delta} d\delta + \frac{\partial D}{\partial t} dt,$$

we need to find the partial derivatives $\partial D/\partial\delta$ and $\partial D/\partial t$. The *convexity* of the security is defined by the formula

$$C(t, \delta) = \frac{\frac{\partial^2 V}{\partial \delta^2}}{V}.$$

Applying the quotient rule for differentiation gives

$$\frac{\partial D}{\partial \delta} = -(C - D^2), \qquad (6)$$

which is a well-known formula and can be found in [KBT, pp. 148], [Gr, p. 40], [S1, p. 101] and [Re, (1.18)]. With the definition

$$M(t, \delta) = C(t, \delta) - [D(t, \delta)]^2,$$

we have

$$\frac{\partial D}{\partial \delta} = -M. \qquad (7)$$

(We note that Fong and Vasicek [FV1; FV2] use M^2 to denote our M, and Bierwag [Bi] calls the quantity M *inertia*.) The partial derivative of D with respect to t, $\partial D/\partial t$, turns out to be quite simple:

$$\frac{\partial}{\partial t} D = -\frac{\partial^2}{\partial t \, \partial \delta} \log_e V = -\frac{\partial^2}{\partial \delta \, \partial t} \log_e V = -\frac{\partial}{\partial \delta} \delta = -1. \qquad (8)$$

It follows from (7) and (8) that

$$\int_{t=t_0}^{t=t_1} [\delta(t_1) - \delta(t)]\, dD = -\int_{t_0}^{t_1} [\delta(t_1) - \delta(t)]\, M\, d\delta(t) - \int_{t_0}^{t_1} [\delta(t_1) - \delta(t)]\, dt. \quad (9)$$

It also follows from (7) and (8) that $\partial M/\partial t = 0$. Hence the function M is a function of δ only, i.e.,

$$M(t, \delta) = M(\delta).$$

Thus, by the mean value theorem for integrals, there exists a point ξ between t_0 and t_1 such that $\delta(\xi)$ is between $\delta(t_0)$ and $\delta(t_1)$, and

$$-\int_{t=t_0}^{t=t_1} [\delta(t_1) - \delta(t)]\, M\, d\delta(t) = \frac{1}{2}\int_{t_0}^{t_1} M(\delta(t))\, d[\delta(t_1) - \delta(t)]^2$$

$$= -\frac{1}{2} M(\delta(\xi))\, [\delta(t_1) - \delta(t_0)]^2. \quad (10)$$

Applying formulas (4), (5), (9) and (10), we obtain the formula

$$V(t_1, \delta(t_1)) = V(t_0, \delta(t_0))\exp\{(t_1 - t_0)\delta(t_1) - D(t_0, \delta(t_0))[\delta(t_1) - \delta(t_0)] + M(\delta(\xi))[\delta(t_1) - \delta(t_0)]^2/2\}.$$

This is a valuation equation at time t_1. We can rewrite it as one at time t_0:

$$V(t_1, \delta(t_1))\exp[-(t_1 - t_0)\delta(t_1)]$$

$$= V(t_0, \delta(t_0))\exp\{-D(t_0, \delta(t_0))[\delta(t_1) - \delta(t_0)] + M(\delta(\xi))[\delta(t_1) - \delta(t_0)]^2/2\}. \quad (11)$$

Note that, for the formulas

$$\partial \log_e V/\partial t = \delta(t), \quad t \in [t_0, t_1],$$

and

$$\partial D/\partial t = -1, \quad t \in [t_0, t_1],$$

to hold, there should be no payment or cash flow occurring in the time interval $[t_0, t_1]$.

Remarks

There is another approach to obtain (11). Note that

$$V(t_1, \delta(t_1))\exp[-(t_1 - t_0)\delta(t_1)] = V(t_0, \delta(t_1)).$$

Apply the formula

$$\frac{V(t, \delta_1)}{V(t, \delta_0)} = \exp\{\int_{\delta = \delta_0}^{\delta = \delta_1} [\frac{\partial}{\partial \delta}\log_e V(t, \delta)]\, d\delta\} = \exp[-\int_{\delta = \delta_0}^{\delta = \delta_1} D(t, \delta)\, d\delta],$$

and expand $D(t, \delta)$ as a Taylor series in δ. The derivatives of D with respect to δ can be obtained in terms of cumulants; see pages 100 and 101 of [S1]. Also see [Re].

If we assume $\delta(t)$ to follow a diffusion process, we need to apply Itô's lemma to calculate the differential dV. Then there is an extra term

$$\frac{1}{2}\frac{\partial^2 V}{\partial \delta^2}(d\delta)^2$$

in formula (1). See [Bo], [Al], [Le] and [Ma].

This note can be viewed as an attempt to have a better understanding of Redington's theory of immunization. For an approach to extend Redington's theory, see [FV1], [FV2], [MP], [S2] and [S3].

One may want to generalize the above to the case in which the cash flows are not fixed and certain. However, what is the yield rate for a stream of stochastic cash flows?

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