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SOME USEFUL THEOREMS IN ACTUARIAL MATHEMATICS

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Theorem A. Let a, c, d and e be positive numbers. Then the function

$$f(x) = (dx + e)/(ax^2 + c)^{1/2}$$

attains its maximum value $[(d^2/a) + (e^2/c)]^{1/2}$ at $x = cd/ae$.

Proof. The value of $f''(x)$ at the root $x = cd/ae$ of

$$f'(x) = (cd - aex)/(ax^2 + c)^{3/2}$$

is $-c(ae^2 + cd^2)/e[c + a(cd/ae)^2]^{5/2}$. Thus the maximum value of $f(x)$ is

$$f(cd/ae) = [d(cd/ae) + e]/[a(cd/ae)^2 + c]^{1/2}$$

$$= (\{d(cd/ae)[d(cd/ae)+e] + e[d(cd/ae)+e]\}/[a(cd/ae)^2 + c])^{1/2}$$

$$= (\{d^2[a(cd/ae)^2 + c]/a[a(cd/ae)^2 + c]\} + (e^2/c))^{1/2}$$

$$= [(d^2/a) + (e^2/c)]^{1/2}.$$

Corollary A. For an insurance organization, let S denote the random loss on a segment of its risks and let x be the retention limit that minimizes the probability $Pr((S-E[S])/Var[S]^{1/2} > f(x))$ where $f(x)$ is the ratio of the security loading $g(x) = dx + e$ and the standard deviation $h(x) = Var[S]^{1/2} = (ax^2 + c)^{1/2}$. Then $x = cd/ae$ and $f(cd/ae) = [(d^2/a) + (e^2/c)]^{1/2}$.

Corllary B. Let a, b, c, d and e be positive numbers such that $4ac > b^2$ and $2ae > bd$. Then $f(x) = (dx + c)/(ax^2 + bx + c)^{1/2}$ attains its maximum value $[(d^2/a) + (2ae - bd)^2/a(4ac - b^2)]^{1/2}$ at $x = (2cd - be)/(2ae - bd)$.

Proof. Write

$$f(x) = \{d[x + (b/2a)] + [(2ae - bd)/2a]\}/\{a[x + (b/2a)]^2 + [(4ac - b^2)/4a]\}^{1/2}$$

and use Theorem A.

Theorem B. Let $f(x) = qb[\exp(-bx)]$ and let $g(x) = -\exp(-ax)$. Then $h(d;c) = \int_0^\infty f(x)g(d - cx)dx = -qb[\exp(-ad)]/(b - ca)$.

Corollary C. Let p be the probability that a property will not be damaged in the next period and let f(x) in Theorem B be the probabilty density function of a positive random variale X with $q = 1 - p$. If the owner of the property with wealth w has a utility function g(x) in Theorem B and is offered an insurance policy that will pay $1 - c$ portion of any loss during the next period, then the maximum premium G that the property owner will pay for this insurance is

$$G = (1/a)\ln\{[p + qb/(b - a)]/[p + qb/(b - ca)]\}.$$

Proof. Equating the utilities with and without insurance, we have

$$pg(w - G) + h(w - G;c) = pg(w) + h(w;1).$$

It follows from Theorem B that

$$-p\{\exp[-a(w - G)]\} - [qb/(b - ca)]\exp[-a(w - G)]$$

$$= -p[\exp(-aw)] - [qb/(b - a)]\exp(-aw)$$

and that

$$\{p + [qb/(b - ca)]\}\exp(aG) = p + [qb/(b - a)].$$

The theorem follows.

Theorem C. Let

$$f(x) = (2/a)[1 - (x/a)], \quad 0 \leq x \leq a,$$

be the probability density function of a random variable X. Then

$$E[X^n] = a^n / \binom{n+2}{2}.$$

Corollary D. The mean and variance of the random variable X in Theorem C are $a/3$ and $a^2/18$, respectively.

Theorem D. A decision maker has wealth w, has a utility function

$$u(x) = x^r, \quad 0 < r < 1,$$

and faces a random loss X with a uniform distribution on [0,w]. Then the maximum amount this decision maker will pay for the complete insurance against the random loss is

$$G = \{1 - [1/(r + 1)]^{1/r}\}w.$$

Proof. Equating utilities with and without insurance, we have

$$(w - G)^r = \int_0^w (1/w)(w - x)^r dx.$$

It follows that

$$(w - G)^r = [1/(r + 1)]w^r.$$

The theorem follows.

Theorem E. Assume that a decision maker will retain wealth w with probabilty p and will suffer a loss c with probabilty $q = 1 - p$. Based on the utility function

$$u(x) = x - ax^2, \quad 0 < x < 1/2a \quad (a > 0),$$

the maximum insurance premium that the decision maker will pay for the complete insurance is

$$G = w - (1/2a)[1 - (1 - 4a\{pw(1-aw) + q(w-c)[1-a(w-c)]\})^{1/2}].$$

Proof. Equating utilities with and without insurance, we have

$$(w - G) - a(w - G)^2 = pw(1 - aw) + q(w - c)[1 - a(w - c)].$$

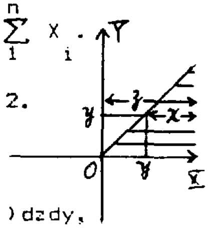
It follows that

$$w - G = (1/2a)[1 - (4a\{pw(1-aw) + q(w-c)[1 - a(w-c)]\})^{1/2}].$$

The theorem follows.

We shall conclude by providing the direct proof of

Theorem F. Let X_i, $i = 1, 2, 3, \ldots, n$, be nonnegative mutually independent random variables with the probability density function $f_i(t)$. If the moment generation function $M_{X_1}(t)$ of each X_i is finite on some open interval, then the convolution $f_1 * f_2(x)$ is the unique probabilty density function of $S = \sum_1^n X_i$.



Proof. We shall only prove the continuous case with $n = 2$.

For any t in the given interval, we have

$$M_S(t) = \int_0^\infty e^{tx} \int_0^x f_1(x-y) f_2(y)\,dy\,dx = \int_0^\infty e^{ty} f_2(y) \int_0^\infty e^{tz} f_1(z)\,dz\,dy,$$

where $z = x - y$. Hence $M_S(t) = M_{X_1}(t) M_{X_2}(t)$ and hence the theorem follows.